A New Hilbert-type Integral Inequality with Some Parameters and Its Reverse

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ABSTRACT. In this paper, by introducing some parameters and estimating the weight function, we give a new Hilbert-type integral inequality with a best constant factor. The equivalent inequality and the reverse forms are considered.

1. Introduction

If $f(x), g(x) \geq 0$, such that $0 < \int_0^\infty f^2(x)dx < \infty$ and $0 < \int_0^\infty g^2(x)dx < \infty$, then (see [1])

$$(1.1) \quad \int_0^\infty \int_0^\infty \frac{f(x)g(x)}{x+y} dx dy < \pi \left\{ \int_0^\infty f^2(x)dx \int_0^\infty g^2(x)dx \right\}^{1/2},$$

where the constant factor π is the best possible. Inequality (1.1) is well-known as Hilbert's integral inequality, which had been extended by Hardy-Riesz as (see [2]):

If $p > 1, \frac{1}{p} + \frac{1}{q} = 1, f(x), g(x) \geq 0$, such that $0 < \int_0^\infty f^p(x)dx < \infty$ and $0 < \int_0^\infty g^q(x)dx < \infty$, then we have the following Hardy-Hilbert's integral inequality:

$$(1.2) \quad \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dx dy < \frac{\pi}{\sin(\pi/p)} \left\{ \int_0^\infty f^p(x)dx \right\}^{1/p} \left\{ \int_0^\infty g^q(x)dx \right\}^{1/q},$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible.

Hardy-Hilbert's integral inequality is important in analysis and its applications (see [3]). In recent years, Yang [4], [5] gave two different best generalizations of (1.2) by introducing a parameter $\lambda > 0$, and Yang et al. [6] gave an extension of the above

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results by introducing the index of conjugate parameter (r, s) ($r > 1, \frac{1}{r} + \frac{1}{s} = 1$) as follows (see [6], (21) for $n=2$):

If $f(x), g(x) \geq 0$, and $0 < \int_0^\infty x^{p(1-\frac{\lambda}{r})-1} f^p(x) dx < \infty, 0 < \int_0^\infty x^{q(1-\frac{\lambda}{s})-1} g^q(x) dx < \infty$, then

$$(1.3) \quad \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda} dx dy < B\left(\frac{\lambda}{r}, \frac{\lambda}{s}\right) \left\{ \int_0^\infty x^{p(1-\frac{\lambda}{r})-1} f^p(x) dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty x^{q(1-\frac{\lambda}{s})-1} g^q(x) dx \right\}^{\frac{1}{q}},$$

where the constant factor $B\left(\frac{\lambda}{r}, \frac{\lambda}{s}\right)$ is the best possible ($B(u, v)$ is the Beta function). In particular, for $\lambda = 1, r = q, s = p$, inequality (1.3) reduces to (1.2); for $\lambda = 4, r = s = 2$, (1.3) reduces to:

$$(1.4) \quad \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^4} dx dy < \frac{1}{6} \left\{ \int_0^\infty \frac{1}{x^{p+1}} f^p(x) dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty \frac{1}{x^{q+1}} g^q(x) dx \right\}^{\frac{1}{q}}.$$

In this paper, by introducing some parameters and estimating the weight function, we prove a new Hilbert-type integral inequality with a best constant factor similar to (1.4). The equivalent inequality and the reverse forms are considered.

2. Some lemmas

Lemma 2.1. *If $a, b > 0, a \neq b$, define the weight functions $\varpi(x)$ and $\omega(y)$ as*

$$(2.1) \quad \varpi(x) = \int_0^\infty \frac{x^2 y}{(x+ay)^2(x+by)^2} dy \quad (x \in (0, \infty));$$

$$(2.2) \quad \omega(y) = \int_0^\infty \frac{xy^2}{(x+ay)^2(x+by)^2} dx \quad (y \in (0, \infty)),$$

then we have

$$(2.3) \quad \varpi(x) = \omega(y) = K := \frac{a+b}{(b-a)^2} \left[\frac{\ln(b/a)}{b-a} - \frac{2}{a+b} \right].$$

Proof. For fixed x , setting $u = y/x$ in (2.1), we obtain

$$\begin{aligned} \varpi(x) &= \int_0^\infty \frac{u}{(1+au)^2(1+bu)^2} du \\ &= \frac{1}{(b-a)^2} \left[\frac{1}{1+au} + \frac{1}{1+bu} \right]_0^\infty - \left[\frac{a+b}{(b-a)^3} \ln\left(\frac{1+au}{1+bu}\right) \right]_0^\infty \\ &= \frac{1}{(b-a)^2} \left[-2 + (a+b) \frac{\ln(b/a)}{b-a} \right]. \end{aligned}$$

Hence we obtain $\varpi(x) = K$. In the same way, we obtain $\omega(y) = K$. The lemma is proved. $\square$

Lemma 2.2. *If $a, b > 0$ and $a \neq b$, then for $0 < \varepsilon < p$, we have*

$$(2.4) \quad \int_0^\infty \frac{u^{1-\varepsilon/p}}{(1+au)^2(1+bu)^2} du = K + o(1) \quad (\varepsilon \rightarrow 0^+).$$

Proof. Since for $a, b > 0$ and $a \neq b$, we find

$$\begin{aligned} & \left| \int_0^\infty \frac{u^{1-\varepsilon/p}}{(1+au)^2(1+bu)^2} du - K \right| = \left| \int_0^\infty \frac{u^{1-\varepsilon/p} - u}{(1+au)^2(1+bu)^2} du \right| \\ & \leq \int_0^1 \frac{|u^{1-\varepsilon/p} - u|}{(1+au)^2(1+bu)^2} du + \int_1^\infty \frac{|u^{1-\varepsilon/p} - u|}{(1+au)^2(1+bu)^2} du \\ & \leq \int_0^1 (u^{1-\varepsilon/p} - u) du + \frac{1}{(ab)^2} \int_1^\infty \frac{u - u^{1-\varepsilon/p}}{u^4} du \\ & = \left(\frac{1}{2-\varepsilon/p} - \frac{1}{2} \right) + \frac{1}{(ab)^2} \left(\frac{1}{2} - \frac{1}{2+\varepsilon/p} \right) \rightarrow 0, \quad \text{for } \varepsilon \rightarrow 0^+. \end{aligned}$$

Then (2.4) is valid, and the lemma is proved. $\square$

Lemma 2.3. *If $p > 1$ (or $0 < p < 1$), $\frac{1}{p} + \frac{1}{q} = 1$, $a, b > 0, a \neq b$, and $0 < \varepsilon < p$, setting*

$$I := \int_1^\infty \left[\int_1^\infty \frac{y^{1-\varepsilon/p}}{(x+ay)^2(x+by)^2} dy \right] x^{1-\varepsilon/q} dx,$$

then we have

$$(2.5) \quad \frac{1}{\varepsilon}(K + o(1)) - O(1) \leq I \leq \frac{1}{\varepsilon}(K + o(1)), \quad \varepsilon \rightarrow 0^+.$$

Proof. For fixed x , setting $y = xu$, then by (2.4), we obtain

$$\begin{aligned} I &= \int_1^\infty x^{-1-\varepsilon} \left[\int_{x^{-1}}^\infty \frac{u^{1-\varepsilon/p}}{(1+au)^2(1+bu)^2} du \right] dx \\ &= \int_1^\infty x^{-1-\varepsilon} \left[\int_0^\infty \frac{u^{1-\varepsilon/p}}{(1+au)^2(1+bu)^2} du \right] dx \\ &\quad - \int_1^\infty x^{-1-\varepsilon} \left[\int_0^{x^{-1}} \frac{u^{1-\varepsilon/p}}{(1+au)^2(1+bu)^2} du \right] dx \\ &\geq \frac{1}{\varepsilon}(K + o(1)) - \frac{1}{a+b} \int_1^\infty x^{-1} \left(\int_0^{x^{-1}} u^{-\frac{\varepsilon}{p}} du \right) dx \quad ((1+au)^2(1+bu)^2 > (a+b)u) \\ &= \frac{1}{\varepsilon}(K + o(1)) - \frac{1}{a+b} \left(1 - \frac{\varepsilon}{p}\right)^{-2} = \frac{1}{\varepsilon}(K + o(1)) - O(1). \end{aligned}$$

By the same way, we have

$$I \leq \int_1^\infty \left[\int_0^\infty \frac{y^{1-\varepsilon/p}}{(x+ay)^2(x+by)^2} dy \right] x^{1-\varepsilon/q} dx = \frac{1}{\varepsilon} (K + o(1)).$$

The lemma is proved. $\square$

3. Main results

Theorem 3.1. *If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $a, b > 0, a \neq b, f(x), g(x) \geq 0$, such that $0 < \int_0^\infty \frac{1}{x^{p+1}} f^p(x) dx < \infty$ and $0 < \int_0^\infty \frac{1}{x^{q+1}} g^q(x) dx < \infty$, then*

$$(3.1) \quad \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+ay)^2(x+by)^2} dx dy \\ < K \left\{ \int_0^\infty \frac{1}{x^{p+1}} f^p(x) dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty \frac{1}{x^{q+1}} g^q(x) dx \right\}^{\frac{1}{q}},$$

where the constant factor K is the best possible and K is defined by (2.3).

Theorem 3.2. *If $0 < p < 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $a, b > 0, a \neq b, f(x), g(x) \geq 0$, such that $0 < \int_0^\infty \frac{1}{x^{p+1}} f^p(x) dx < \infty$ and $0 < \int_0^\infty \frac{1}{x^{q+1}} g^q(x) dx < \infty$, then*

$$(3.2) \quad \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+ay)^2(x+by)^2} dx dy \\ > K \left\{ \int_0^\infty \frac{1}{x^{p+1}} f^p(x) dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty \frac{1}{x^{q+1}} g^q(x) dx \right\}^{\frac{1}{q}},$$

where the constant factor K is the best possible.

Proof of Theorem 3.1. By Hölder's inequality with weight (see [7]) and (2.1)-(2.3), we have,

$$(3.3) \quad J : = \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+ay)^2(x+by)^2} dx dy \\ = \int_0^\infty \int_0^\infty \frac{1}{(x+ay)^2(x+by)^2} \left[\left(\frac{y^{1/p}}{x^{1/q}} \right) f(x) \right] \left[\left(\frac{x^{1/q}}{y^{1/p}} \right) g(y) \right] dx dy \\ \leq \left\{ \int_0^\infty \int_0^\infty \frac{1}{(x+ay)^2(x+by)^2} \left(\frac{y}{x^{p-1}} \right) f^p(x) dy dx \right\}^{\frac{1}{p}} \\ \quad \times \left\{ \int_0^\infty \int_0^\infty \frac{1}{(x+ay)^2(x+by)^2} \left(\frac{x}{y^{q-1}} \right) g^q(y) dx dy \right\}^{\frac{1}{q}} \\ = \left\{ \int_0^\infty \varpi(x) \frac{1}{x^{p+1}} f^p(x) dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty \omega(y) \frac{1}{y^{q+1}} g^q(y) dy \right\}^{\frac{1}{q}} \\ = K \left\{ \int_0^\infty \frac{1}{x^{p+1}} f^p(x) dx \right\}^{1/p} \left\{ \int_0^\infty \frac{1}{x^{q+1}} g^q(x) dx \right\}^{1/q}.$$

If (3.3) takes the form of equality, then there exists constants M and N , such that they are not all zero, and (see [7])

$$M\left(\frac{y}{x^{p-1}}\right)f^p(x) = N\left(\frac{x}{y^{q-1}}\right)g^q(y) \quad \text{a.e. in } (0, \infty) \times (0, \infty).$$

Hence, there exists a constant C , such that

$$Mx^{-p}f^p(x) = Ny^{-q}g^q(y) = C \quad \text{a.e. in } (0, \infty).$$

We claim that $M = 0$. In fact, if $M \neq 0$, then $x^{-1-p}f^p(x) = C/(Mx)$ a.e. in $(0, \infty)$, which contradicts the fact that $0 < \int_0^\infty x^{-1-p}f^p(x)dx < \infty$. In the same way, we claim that $N = 0$. This is a contradiction. Hence by (3.3), we have (3.1).

If the constant factor K in (3.1) is not the best possible, then there exists a positive constant H (with $H < K$), such that (3.1) is still valid if we replace K by H . For $0 < \varepsilon < p$ small enough, setting f_ε and g_ε as $f_\varepsilon(x) = g_\varepsilon(x) = 0$, for $x \in (0, 1)$; $f_\varepsilon(x) = x^{1-\varepsilon/p}$; $g_\varepsilon(x) = x^{1-\varepsilon/q}$, for $x \in [1, \infty)$, then we obtain

$$\begin{aligned} & H \left\{ \int_0^\infty \frac{1}{x^{p+1}} f_\varepsilon^p(x) dx \right\}^{1/p} \left\{ \int_0^\infty \frac{1}{x^{q+1}} g_\varepsilon^q(x) dx \right\}^{1/q} \\ &= H \left\{ \int_1^\infty x^{-\varepsilon-1} dx \right\}^{1/p} \left\{ \int_1^\infty x^{-\varepsilon-1} dx \right\}^{1/q} = \frac{H}{\varepsilon}. \end{aligned}$$

By (2.5), we have

$$\begin{aligned} & \int_0^\infty \int_0^\infty \frac{f_\varepsilon(x)g_\varepsilon(y)}{(x+ay)^2(x+by)^2} dx dy \\ &= \int_1^\infty \left[\int_1^\infty \frac{y^{1-\frac{\varepsilon}{q}}}{(x+ay)^2(x+by)^2} dy \right] x^{1-\frac{\varepsilon}{p}} dx = I \\ &\geq \frac{1}{\varepsilon}(K + o(1)) - O(1). \end{aligned}$$

Hence we find

$$\frac{1}{\varepsilon}(K + o(1)) - O(1) < \frac{H}{\varepsilon} \quad \text{or} \quad (K + o(1)) - \varepsilon O(1) < H.$$

For $\varepsilon \rightarrow 0^+$, it follows that $K \leq H$. This contradicts the fact that $H < K$. Hence the constant factor K in (3.1) is the best possible. The theorem is proved. $\square$

Proof of Theorem 3.2. By the reverse Hölder's inequality with weight (see [7]) and the same way of giving (3.3), we obtain (3.2).

If the constant factor K in (3.2) is not the best possible, then there exists a positive constant H (with $H > K$), such that (3.2) is still valid if we replace K by H . For $0 < \varepsilon < p$ small enough, setting f_ε and g_ε as: $f_\varepsilon(x) = g_\varepsilon(x) = 0$, for

$x \in (0, 1)$; $f_\varepsilon(x) = x^{1-\varepsilon/p}$; $g_\varepsilon(x) = x^{1-\varepsilon/q}$, for $x \in [1, \infty)$, then we obtain

$$(3.4) \quad \begin{aligned} & H \left\{ \int_0^\infty \frac{1}{x^{p+1}} f_\varepsilon^p(x) dx \right\}^{1/p} \left\{ \int_0^\infty \frac{1}{x^{q+1}} g_\varepsilon^q(x) dx \right\}^{1/q} \\ &= H \left\{ \int_1^\infty x^{-\varepsilon-1} dx \right\}^{1/p} \left\{ \int_1^\infty x^{-\varepsilon-1} dx \right\}^{1/q} = \frac{H}{\varepsilon}. \end{aligned}$$

By (2.5), we have

$$\begin{aligned} & \int_0^\infty \int_0^\infty \frac{f_\varepsilon(x) g_\varepsilon(y)}{(x+ay)^2(x+by)^2} dx dy \\ &= \int_1^\infty \left[\int_1^\infty \frac{y^{1-\frac{\varepsilon}{p}}}{(x+ay)^2(x+by)^2} dy \right] x^{1-\frac{\varepsilon}{q}} dx = I \\ &\leq \frac{1}{\varepsilon} (K + o(1)). \end{aligned}$$

Hence we find

$$\frac{1}{\varepsilon} (K + o(1)) > \frac{H}{\varepsilon} \quad \text{or} \quad (K + o(1)) > H.$$

For $\varepsilon \rightarrow 0^+$, it follows that $K \geq H$. This contradicts the fact that $H > K$. Hence the constant K in (3.2) is the best possible. The theorem is proved. $\square$

Theorem 3.3. *Under the same assumption of Theorem 3.1. we have*

$$(3.5) \quad \int_0^\infty y^{2p-1} \left[\int_0^\infty \frac{f(x)}{(x+ay)^2(x+by)^2} dx \right]^p dy < K^p \int_0^\infty \frac{1}{x^{p+1}} f^p(x) dx,$$

where the constant factor K^p is the best possible. Inequalities (3.5) and (3.1) are equivalent.

Theorem 3.4. *Under the same assumption of Theorem 3.2. we have*

$$(3.6) \quad \int_0^\infty y^{2p-1} \left[\int_0^\infty \frac{f(x)}{(x+ay)^2(x+by)^2} dx \right]^p dy > K^p \int_0^\infty \frac{1}{x^{p+1}} f^p(x) dx,$$

where the constant factor K^p is the best possible. Inequalities (3.6) and (3.2) are equivalent.

We prove only Theorem 3.3, since the proof of Theorem 3.4 is the similar.

Proof. Setting $g(y) = y^{2p-1} \left[\int_0^\infty \frac{f(x)}{(x+ay)^2(x+by)^2} dx \right]^{p-1}$, by (3.1), we have

$$(3.7) \quad \begin{aligned} & \int_0^\infty y^{-1-q} g^q(y) dy = \int_0^\infty y^{2p-1} \left[\int_0^\infty \frac{f(x)}{(x+ay)^2(x+by)^2} dx \right]^p dy \\ &= J \leq K \left\{ \int_0^\infty \frac{1}{x^{p+1}} f^p(x) dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty \frac{1}{x^{q+1}} g^q(x) dx \right\}^{\frac{1}{q}}; \end{aligned}$$

$$(3.8) \quad 0 < \left\{ \int_0^\infty \frac{1}{y^{q+1}} g^q(y) dy \right\}^{1/p} \leq K \left\{ \int_0^\infty \frac{1}{x^{p+1}} f^p(x) dx \right\}^{1/p} < \infty.$$

Hence by (3.1), both (3.7) and (3.8) preserve the form of strict inequalities, and we have (3.5).

By Hölder's inequality, we have

$$(3.9) \quad \begin{aligned} J &= \int_0^\infty \left[y^{1+1/q} \int_0^\infty \frac{f(x)}{(x+ay)^2(x+by)^2} dx \right] \left(y^{-1-1/q} g(y) \right) dy \\ &\leq \left\{ \int_0^\infty y^{2p-1} \left[\int_0^\infty \frac{f(x)}{(x+ay)^2(x+by)^2} dx \right]^p dy \right\}^{\frac{1}{p}} \\ &\quad \times \left(\int_0^\infty \frac{1}{y^{q+1}} g^q(y) dy \right)^{\frac{1}{q}}. \end{aligned}$$

Then by (3.5), we have (3.1). Hence inequalities (3.1) and (3.5) are equivalent.

If the constant factor in (3.5) is not the best possible, then by (3.9), we can get a contradiction that the constant factor in (3.1) is not the best possible. The theorem is proved. $\square$

Remark. Since we obtain

$$(3.10) \quad \lim_{x \rightarrow a} \left\{ \frac{x+a}{(x-a)^2} \left[\frac{\ln(\frac{x}{a})}{x-a} - \frac{2}{x+a} \right] \right\} = \lim_{x \rightarrow a} \left[\frac{(x+a)\ln(\frac{x}{a}) - 2(x-a)}{(x-a)^3} \right] = \frac{1}{6a^2},$$

then by (3.1), setting $b \rightarrow a$, we have the following new inequality:

$$(3.11) \quad \begin{aligned} &\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+ay)^4} dx dy \\ &< \frac{1}{6a^2} \left\{ \int_0^\infty \frac{1}{x^{p+1}} f^p(x) dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty \frac{1}{x^{q+1}} g^q(x) dx \right\}^{\frac{1}{q}}. \end{aligned}$$

In particular, for $a = 1$, (3.11) reduces to (1.4). Hence we can get a conclusion that inequality (3.1) is a best extension of (1.4).

References

- [1] G. H. Hardy, J. E. Littlewood and G. Pólya, *Inequalities*, Cambridge University Press, Cambridge, 1952.
- [2] G. H. Hardy, *Note on a theorem of Hilbert concerning series of positive terms*, Proceedings London Math. Soc., Records of Proc. XLV-XLIV, **23(2)**(1925).
- [3] D. S. Mintrinić, J. E. Pečarić and A. M. Fink, *Inequalities involving functions and their integrals and Derivatives*, Kluwer Academic Publishers, Boston, 1991.

- [4] Bicheng Yang, *On Hardy-Hilbert's integral inequality*, J. Math. Anal. Appl., **261**(2001), 295-306.
- [5] Bicheng Yang, *On the extended Hilbert's integral inequality*, Journal of Inequalities in Pure and Applied Mathematics, **5(4)**(2004), Article 96.
- [6] Bicheng Yang, Ilko Bnaetić, Mario Krnic and Josip Pečarić, *Generalization of Hilbert and Hardy-Hilbert integral inequalities*, *Mathematical Inequalities and Applications*, **8(2)**(2005), 259-272.
- [7] Jichang Kang, *Applied Inequalities*, Shangdong Science and Technology Press, Jinan, 2004.