On a Class of Meromorphic Functions defined by Certain Linear Operators

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ABSTRACT. In the present investigation, we introduce new classes of p -valent meromorphic functions defined by Liu-Srivastava linear operator and the multiplier transform and study their properties by using certain first order differential subordination and superordination.

1. Introduction

Let $\mathcal{H}$ be the class of functions analytic in $\Delta := \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{H}(a, n)$ be the subclass of $\mathcal{H}$ consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots$ and set $\mathcal{H}_1 := \mathcal{H}(1, 1)$. Let Σ_p denote the class of all analytic functions of the form

$$(1.1) \quad f(z) = \frac{1}{z^p} + \sum_{k=1-p}^{\infty} a_k z^k \quad (z \in \Delta^* := \{z \in \mathbb{C} : 0 < |z| < 1\}; p \in \mathbb{N})$$

and set $\Sigma := \Sigma_1$.

For two functions $f(z)$ given by (1.1) and $g(z) = \frac{1}{z^p} + \sum_{k=1-p}^{\infty} b_k z^k$, the *Hadamard product (or convolution)* of f and g is defined by

$$(1.2) \quad (f * g)(z) := \frac{1}{z^p} + \sum_{k=1-p}^{\infty} a_k b_k z^k =: (g * f)(z).$$

For $\alpha_j \in \mathbb{C}$ ($j = 1, 2, \dots, l$) and $\beta_j \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$ ($j = 1, 2, \dots, m$), the *generalized hypergeometric function* ${}_lF_m(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m; z)$ is defined by the

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infinite series

$${}_lF_m(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m; z) := \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_l)_n}{(\beta_1)_n \cdots (\beta_m)_n} \frac{z^n}{n!}$$

$$(l \leq m+1; l, m \in \mathbb{N}_0 := \{0, 1, 2, \dots\})$$

where $(a)_n$ is the Pochhammer symbol defined by

$$(a)_n := \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1, & (n=0); \\ a(a+1)(a+2)\cdots(a+n-1), & (n \in \mathbb{N} := \{1, 2, 3, \dots\}). \end{cases}$$

Corresponding to the function

$$h_p(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m; z) := z^{-p} {}_lF_m(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m; z),$$

the *Liu-Srivastava operator* [7] $\mathcal{H}_p^{(l,m)}(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m) : \Sigma_p \mapsto \Sigma_p$ is defined by the Hadamard product

$$\begin{aligned} \mathcal{H}_p^{(l,m)}(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m)f(z) &:= h_p(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m; z) * f(z) \\ &= \frac{1}{z^p} + \sum_{n=1-p}^{\infty} \frac{(\alpha_1)_{n+p} \cdots (\alpha_l)_{n+p}}{(\beta_1)_{n+p} \cdots (\beta_m)_{n+p}} \frac{a_n z^n}{(n+p)!}. \end{aligned}$$

To make the notation simple, we write

$$\mathcal{H}_p^{l,m}[\alpha_1]f(z) := \mathcal{H}_p^{(l,m)}(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m)f(z).$$

Special cases of the Liu-Srivastava linear operator includes the meromorphic analogue of the Carlson-Shaffer linear operator

$$(1.3) \quad \mathcal{L}_p(a, c) := \mathcal{H}_p^{(2,1)}(a, 1; c)$$

considered by Liu [6] and the operator

$$(1.4) \quad \mathcal{D}^{n+p-1} := \mathcal{L}_p(n+p, 1)$$

investigated by Yang [14]. When $p=1$, the operator was first introduced by Ganigi and Uralegaddi [5] and then generalized by Yang [13] to an operator analogous to the Ruscheweyh derivative operator, and the operator $\mathcal{J}_{c,p} = \mathcal{L}_p(c, c+1)$ was studied, for example, by Uralegaddi and Somanatha [12]. Note that

$$\mathcal{J}_{c,p}f = \frac{c}{z^{c+p}} \int_0^z t^{c+p-1} f(t) dt \quad (c > 0).$$

Motivated by the operator studied by Aouf and Hossen [1] (see also [4], [6], [10]), we define the operator $\mathcal{I}_p(n, \lambda)$ on Σ_p by the following infinite series

$$(1.5) \quad \mathcal{I}_p(n, \lambda)f(z) := \frac{1}{z^p} + \sum_{k=1-p}^{\infty} \left(\frac{k+\lambda}{\lambda-p} \right)^n a_k z^k \quad (\lambda > p, n \in \mathbb{Z}).$$

A function $f(z) \in \Sigma_p$ is said to be in the class $\Omega_p^{l,m}(\alpha_1; A, B)$ if it satisfies the following subordination:

$$\frac{(\mathcal{H}_p^{l,m}[\alpha_1 + 1]f(z))'}{(\mathcal{H}_p^{l,m}[\alpha_1]f(z))'} \prec 1 - \frac{p(A - B)z}{\alpha_1(1 + Bz)}$$

$$(z \in \Delta; \alpha_1 \in \mathbb{C}; -1 < B < A \leq 1; p, l, m \in \mathbb{N})$$

This class $\Omega_p^{l,m}(\alpha_1; A, B)$ was introduced by Liu and Srivastava [7] and they have proved the following:

Theorem 1.1[7, Theorem 1, p.23]. *Let α_1 be real number. If*

$$\alpha_1 \geq \frac{p(A - B)}{1 + B} \quad (-1 < B < A \leq 1; p \in \mathbb{N}),$$

then

$$(1.6) \quad \Omega_p^{l,m}(\alpha_1 + 1; A, B) \subset \Omega_p^{l,m}(\alpha_1; A, B).$$

Theorem 1.2[7, Theorem 2, p.25]. *Let λ be a complex number such that*

$$\Re(\lambda) > \frac{p(A - B)}{1 + B} \quad (-1 < B < A \leq 1; p \in \mathbb{N}).$$

If $f \in \Omega_p^{l,m}(\alpha_1; A, B)$, then the function

$$F(z) = \frac{\lambda}{z^{\lambda+p}} \int_0^z t^{\lambda+p-1} f(t) dt$$

also belongs to the class $\Omega_p^{l,m}(\alpha_1; A, B)$.

For two analytic functions f and F , we say that F is *superordinate* to f if f is subordinate to F . Recently Miller and Mocanu [9] considered certain second order differential subordinations. Using the results of Miller and Mocanu [9], Bulboaca [3] has considered certain classes of first order differential subordinations and Bulboaca [2] considered certain superordination-preserving integral operators.

In recent years, many results of various interesting subclasses of the class Σ_p of meromorphically p -valent functions were investigated extensively by (among others) Aouf *et al.* [1], Liu and Srivasava [7], Ravichandran *et al.* [11], Uralegaddi and Somanatha [12] and Yang [14]. In this paper, we generalize the above-stated classes of Liu and Srivasava [7] to a more general classes of meromorphic p -valent functions which we define below using differential subordination and superordination.

Definition 1.1. A function $f(z) \in \Sigma_p$ is said to be in $\Omega_p(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m; \varphi)$ if it satisfies the following subordination:

$$(1.7) \quad \frac{(\mathcal{H}_p^{l,m}[\alpha_1 + 1]f(z))'}{(\mathcal{H}_p^{l,m}[\alpha_1]f(z))'} \prec \varphi(z) \quad (z \in \Delta),$$

and is said to be in $\tilde{\Omega}_p(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m; \varphi)$ if f satisfies the following superordination:

$$(1.8) \quad \varphi(z) \prec \frac{(\mathcal{H}_p^{l,m}[\alpha_1 + 1]f(z))'}{(\mathcal{H}_p^{l,m}[\alpha_1]f(z))'} \quad (z \in \Delta),$$

where $\varphi(z)$ is analytic in Δ and $\varphi(0) = 1$.

To make the notation simple, we write

$$\Omega_p(\alpha_1; \varphi) := \Omega_p(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m; \varphi)$$

and

$$\tilde{\Omega}_p(\alpha_1; \varphi) := \tilde{\Omega}_p(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m; \varphi).$$

Also we define the class $\Omega_p(\alpha_1; \varphi_1, \varphi_2)$ by the following:

$$\Omega_p(\alpha_1; \varphi_1, \varphi_2) := \tilde{\Omega}_p(\alpha_1; \varphi_1) \cap \Omega_p(\alpha_1; \varphi_2).$$

For

$$\varphi(z) = 1 - \frac{p(A - B)z}{\alpha_1(1 + Bz)} \quad (z \in \Delta; \alpha_1 \in \mathbb{C}; -1 < B < A \leq 1; p \in \mathbb{N}),$$

the class $\Omega_p(\alpha_1; \varphi)$ reduces to the class $\Omega_p^{l,m}(\alpha_1; A, B)$, introduced and studied by Liu and Srivastava [7].

Definition 1.2. A function $f(z) \in \Sigma_p$ is said to be in the class $\mathcal{M}_p(n, \lambda; \varphi)$ if it satisfies the following subordination:

$$(1.9) \quad \frac{(\mathcal{I}_p(n+1, \lambda)f(z))'}{(\mathcal{I}_p(n, \lambda)f(z))'} \prec \varphi(z) \quad (f(z) \in \Sigma_p),$$

and is said to be in $\tilde{\mathcal{M}}_p(n, \lambda; \varphi)$ if f satisfies the following superordination:

$$(1.10) \quad \varphi(z) \prec \frac{(\mathcal{I}_p(n+1, \lambda)f(z))'}{(\mathcal{I}_p(n, \lambda)f(z))'} \quad (f(z) \in \Sigma_p),$$

where $\varphi(z)$ is analytic in Δ and $\varphi(0) = 1$. Also we define the class $\mathcal{M}_p(n, \lambda; \varphi_1, \varphi_2)$ by the following:

$$\mathcal{M}_p(n, \lambda; \varphi_1, \varphi_2) := \tilde{\mathcal{M}}_p(n, \lambda; \varphi_1) \cap \mathcal{M}_p(n, \lambda; \varphi_2).$$

2. Preliminaries

In order to prove our main results we will need to use the next definition and lemmas.

Definition 2.1([9, Definition 2, p.817]). Denote by $\mathcal{Q}$, the set of all functions $f(z)$ that are analytic and injective on $\bar{\Delta} - E(f)$, where

$$E(f) = \{\zeta \in \partial\Delta : \lim_{z \rightarrow \zeta} f(z) = \infty\},$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial\Delta - E(f)$.

Lemma 2.1([8]). Let $q(z)$ be univalent in the unit disk Δ and ϑ and φ be analytic in a domain $\mathbb{D}$ containing $q(\Delta)$ with $\varphi(w) \neq 0$ when $w \in q(\Delta)$. Set $Q(z) := zq'(z)\varphi(q(z))$, $h(z) := \vartheta(q(z)) + Q(z)$. Suppose that either (i) $h(z)$ is convex, or (ii) $Q(z)$ is starlike univalent in Δ . In addition, assume that

$$\Re \frac{zh'(z)}{Q(z)} > 0 \quad (z \in \Delta).$$

If $p(z)$ is analytic in Δ , with $p(0) = q(0)$, $p(\Delta) \subset \mathbb{D}$ and

$$(2.1) \quad \vartheta(p(z)) + zp'(z)\varphi(p(z)) \prec \vartheta(q(z)) + zq'(z)\varphi(q(z)) = h(z),$$

then $p(z) \prec q(z)$ and $q(z)$ is the best dominant.

Lemma 2.2([3]). Let $q(z)$ be univalent in the unit disk Δ and ϑ and φ be analytic in a domain $\mathbb{D}$ containing $q(\Delta)$. Suppose that

- (1) $\Re [\vartheta'(q(z))/\varphi(q(z))] > 0$ for $z \in \Delta$,
- (2) $Q(z) := zq'(z)\varphi(q(z))$ is starlike univalent in Δ .

If $p(z) \in \mathcal{H}[q(0), 1] \cap \mathcal{Q}$, with $p(\Delta) \subset \mathbb{D}$, and $\vartheta(p(z)) + zp'(z)\varphi(p(z))$ is univalent in Δ , then

$$(2.2) \quad \vartheta(q(z)) + zq'(z)\varphi(q(z)) \prec \vartheta(p(z)) + zp'(z)\varphi(p(z)),$$

implies $q(z) \prec p(z)$ and $q(z)$ is the best subdominant.

3. The classes $\Omega_p(\alpha_1; \varphi)$ and $\tilde{\Omega}_p(\alpha_1; \varphi)$

By making use of Lemma 2.1, we first prove the following result:

Theorem 3.1. Let $\psi(z)$ be univalent in Δ , $\psi(0) = 1$ and $\psi(z) \neq 0$. Assume that $z\psi'/\psi$ is starlike in Δ and $\Re \{\alpha_1\psi(z)\} > 0$. Let $\chi(z)$ be defined by

$$(3.1) \quad \chi(z) := \frac{1}{\alpha_1 + 1} \left[\alpha_1\psi(z) + 1 + \frac{z\psi'(z)}{\psi(z)} \right].$$

If $f(z) \in \Omega_p(\alpha_1 + 1; \chi)$, then $f(z) \in \Omega_p(\alpha_1; \psi)$. If $f(z) \in \tilde{\Omega}_p(\alpha_1 + 1; \chi)$,

$$(3.2) \quad 0 \neq \frac{(\mathcal{H}_p^{l,m}[\alpha_1 + 1]f(z))'}{(\mathcal{H}_p^{l,m}[\alpha_1]f(z))'} \in \mathcal{H}_1 \cap \mathcal{Q} \text{ and } \frac{(\mathcal{H}_p^{l,m}[\alpha_1 + 2]f(z))'}{(\mathcal{H}_p^{l,m}[\alpha_1 + 1]f(z))'} \text{ is univalent in } \Delta,$$

then $f(z) \in \tilde{\Omega}_p(\alpha_1; \psi)$.

Proof. First of all consider the following identity

$$(3.3) \quad z(\mathcal{H}_p^{l,m}[\alpha_1]f(z))' = \alpha_1 \mathcal{H}_p^{l,m}[\alpha_1 + 1]f(z) - (\alpha_1 + p)\mathcal{H}_p^{l,m}[\alpha_1]f(z),$$

which upon differentiation, yields

$$(3.4) \quad z(\mathcal{H}_p^{l,m}[\alpha_1]f(z))'' = \alpha_1(\mathcal{H}_p^{l,m}[\alpha_1 + 1]f(z))' - (\alpha_1 + p + 1)(\mathcal{H}_p^{l,m}[\alpha_1]f(z))'.$$

Now define the function $q(z)$ by

$$(3.5) \quad q(z) := \frac{(\mathcal{H}_p^{l,m}[\alpha_1 + 1]f(z))'}{(\mathcal{H}_p^{l,m}[\alpha_1]f(z))'}.$$

Then, clearly, $q(z)$ is analytic in Δ .

By logarithmic differentiation of (3.5) with respect to z and using (3.4), we obtain

$$(3.6) \quad \frac{(\mathcal{H}_p^{l,m}[\alpha_1 + 2]f(z))'}{(\mathcal{H}_p^{l,m}[\alpha_1 + 1]f(z))'} = \frac{1}{\alpha_1 + 1} \left(\alpha_1 q(z) + 1 + \frac{zq'(z)}{q(z)} \right).$$

Since $f(z) \in \Omega_p(\alpha_1 + 1; \chi)$, we have from (3.6) that

$$\alpha_1 q(z) + \frac{zq'(z)}{q(z)} \prec \alpha_1 \psi(z) + \frac{z\psi'(z)}{\psi(z)}$$

and this can be written as (2.1), by defining

$$\vartheta(w) := \alpha_1 w \text{ and } \varphi(w) := \frac{1}{w}.$$

Note that $\varphi(w) \neq 0$ and $\vartheta(w)$, $\varphi(w)$ are analytic in $\mathbb{C} \setminus \{0\}$. Set

$$(3.7) \quad Q(z) := \frac{z\psi'(z)}{\psi(z)}$$

$$(3.8) \quad h(z) := \vartheta(\psi(z)) + Q(z) = \alpha_1 \psi(z) + \frac{z\psi'(z)}{\psi(z)}.$$

In light of the hypothesis of our Theorem 3.1, we see that $Q(z)$ is starlike and

$$\Re \left\{ \frac{zh'(z)}{Q(z)} \right\} = \Re \left\{ \alpha_1 \psi(z) + 1 + \frac{z\psi''(z)}{\psi'(z)} - \frac{z\psi'(z)}{\psi(z)} \right\} > 0.$$

By an application of Lemma 2.1, we obtain that $q(z) \prec \psi(z)$ or

$$\frac{(\mathcal{H}_p^{l,m}[\alpha_1 + 1]f(z))'}{(\mathcal{H}_p^{l,m}[\alpha_1]f(z))'} \prec \psi(z),$$

which shows that $f(z) \in \Omega_p(\alpha_1; \psi)$.

The second half of the Theorem 3.1 follows by a similar application of Lemma 2.2.

□

Remark 3.1. The subordination result of Theorem 3.1 also holds if we replace the condition $\Re \{\alpha_1 \psi(z)\} > 0$ by

$$\Re \left\{ \alpha_1 \psi(z) + 1 + \frac{z\psi''(z)}{\psi'(z)} - \frac{z\psi'(z)}{\psi(z)} \right\} > 0$$

and hence we obtain as its special case the following results using (1.3) and (1.4) respectively:

- (1) Let $\psi(z)$ be univalent in Δ , $\psi(0) = 1$ and $\psi(z) \neq 0$. Assume that $z\psi'/\psi$ is starlike in Δ and

$$\Re \left\{ a\psi(z) + 1 + \frac{z\psi''(z)}{\psi'(z)} - \frac{z\psi'(z)}{\psi(z)} \right\} > 0.$$

Let $\chi(z)$ be defined by

$$\chi(z) := \frac{1}{a+1} \left[a\psi(z) + 1 + \frac{z\psi'(z)}{\psi(z)} \right].$$

If $f(z) \in \Sigma_p$ and

$$\frac{[\mathcal{L}_p(a+2, c)f(z)]'}{[\mathcal{L}_p(a+1, c)f(z)]'} \prec \chi(z)$$

then

$$\frac{[\mathcal{L}_p(a+1, c)f(z)]'}{[\mathcal{L}_p(a, c)f(z)]'} \prec \psi(z)$$

and $\psi(z)$ is the best dominant.

- (2) Let $\psi(z)$ be univalent in Δ , $\psi(0) = 1$ and $\psi(z) \neq 0$. Assume that $z\psi'/\psi$ is starlike in Δ and

$$\Re \left\{ (n+p)\psi(z) + 1 + \frac{z\psi''(z)}{\psi'(z)} - \frac{z\psi'(z)}{\psi(z)} \right\} > 0.$$

Let $\chi(z)$ be defined by

$$\chi(z) := \frac{1}{n+p+1} \left[(n+p)\psi(z) + 1 + \frac{z\psi'(z)}{\psi(z)} \right].$$

If $f(z) \in \Sigma_p$ and

$$\frac{[\mathcal{D}^{n+p+1}f(z)]'}{[\mathcal{D}^{n+p}f(z)]'} \prec \chi(z)$$

then

$$\frac{[\mathcal{D}^{n+p}f(z)]'}{[\mathcal{D}^{n+p-1}f(z)]'} \prec \psi(z)$$

and $\psi(z)$ is the best dominant.

Using Theorem 3.1, we obtain the following “sandwich result”:

Corollary 3.2. *Let $\psi_i(z) \neq 0$ ($i = 1, 2$) be univalent in Δ . Further assume that $z\psi'_i(z)/\psi_i(z)$ ($i = 1, 2$) is starlike univalent in Δ and $\Re\{\alpha_1\psi_i(z)\} > 0$. If $f(z) \in \Omega_p(\alpha_1 + 1; \chi_1, \chi_2)$ satisfies (3.2), then $f(z) \in \Omega_p(\alpha_1; \psi_1, \psi_2)$, where*

$$\chi_i(z) := \frac{1}{\alpha_1 + 1} \left[\alpha_1 \psi_i(z) + 1 + \frac{z\psi'_i(z)}{\psi_i(z)} \right] \quad (i = 1, 2).$$

Theorem 3.3. *Let ψ be univalent in Δ , $\psi(0) = 1$, and λ be a complex number. Assume that $z\psi'/(\lambda - p - \alpha_1 + \alpha_1\psi)$ is starlike in Δ and $\Re\{\lambda - p - \alpha_1 + \alpha_1\psi(z)\} > 0$. Define the functions $\mathcal{F}$ and h by*

$$(3.9) \quad \mathcal{F}(z) := \frac{\lambda - p}{z^\lambda} \int_0^z t^{\lambda-1} f(t) dt,$$

$$h(z) := \psi(z) + \frac{z\psi'(z)}{\lambda - p - \alpha_1 + \alpha_1\psi(z)}.$$

If $f(z) \in \Omega_p(\alpha_1; h)$, then $\mathcal{F} \in \Omega_p(\alpha_1; \psi)$. If $f(z) \in \tilde{\Omega}_p(\alpha_1; h)$,

$$(3.10) \quad 0 \neq \frac{\mathcal{H}_p^{l,m}[\alpha_1 + 1]\mathcal{F}(z)}{\mathcal{H}_p^{l,m}[\alpha_1]\mathcal{F}(z)} \in \mathcal{H}_1 \cap \mathcal{Q} \text{ and } \frac{(\mathcal{H}_p^{l,m}[\alpha_1 + 1]f(z))'}{(\mathcal{H}_p^{l,m}[\alpha_1]f(z))'} \text{ is univalent in } \Delta,$$

then the function $\mathcal{F} \in \tilde{\Omega}_p(\alpha_1; \psi)$.

Proof. From the definition of $\mathcal{F}(z)$ and (3.4), we obtain that

$$(3.11) \quad \begin{aligned} (\lambda - p)\mathcal{H}_p^{l,m}[\alpha_1]f(z) &= \lambda\mathcal{H}_p^{l,m}[\alpha_1]\mathcal{F}(z) + z(\mathcal{H}_p^{l,m}[\alpha_1]\mathcal{F}(z))' \\ &= \alpha_1\mathcal{H}_p^{l,m}[\alpha_1 + 1]\mathcal{F}(z) + (\lambda - p - \alpha_1)\mathcal{H}_p^{l,m}[\alpha_1]\mathcal{F}(z). \end{aligned}$$

Define the function $q(z)$ by

$$(3.12) \quad q(z) := \frac{(\mathcal{H}_p^{l,m}[\alpha_1 + 1]\mathcal{F}(z))'}{(\mathcal{H}_p^{l,m}[\alpha_1]\mathcal{F}(z))'}.$$

Then, clearly, $q(z)$ is analytic in Δ . Using (3.11) and (3.12), we have

$$(3.13) \quad (\lambda - p) \frac{(\mathcal{H}_p^{l,m}[\alpha_1]f(z))'}{(\mathcal{H}_p^{l,m}[\alpha_1]\mathcal{F}(z))'} = \lambda - p - \alpha_1 + \alpha_1 q(z).$$

Upon logarithmic differentiation of (3.13) and using (3.4), and (3.12), we get

$$(3.14) \quad \frac{(\mathcal{H}_p^{l,m}[\alpha_1 + 1]f(z))'}{(\mathcal{H}_p^{l,m}[\alpha_1]f(z))'} = q(z) + \frac{zq'(z)}{\lambda - p - \alpha_1 + \alpha_1 q(z)}.$$

Since $f(z) \in \Omega_p(\alpha_1; h)$, we have, from (3.14),

$$q(z) + \frac{zq'(z)}{\lambda - p - \alpha_1 + \alpha_1 q(z)} \prec \psi(z) + \frac{z\psi'(z)}{\lambda - p - \alpha_1 + \alpha_1 \psi(z)}$$

and this can be written as (2.1), by defining

$$\vartheta(w) := w \text{ and } \varphi(w) := \frac{1}{\lambda - p - \alpha_1 + \alpha_1 w}.$$

The first half of Theorem 3.3 now follows by an application of Lemma 2.1 and the second half follows by a similar application of Lemma 2.2. $\square$

Remark 3.2. The subordination result of Theorem 3.3 also holds if we replace the condition $\Re\{\lambda - p - \alpha_1 + \alpha_1 \psi(z)\} > 0$ by

$$\Re\left\{\lambda - p - \alpha_1 + \alpha_1 \psi(z) + 1 + \frac{z\psi''(z)}{\psi'(z)} - \frac{\alpha_1 z\psi'(z)}{\lambda - p - \alpha_1 + \alpha_1 \psi(z)}\right\} > 0$$

and hence we obtain as its special case the following results using (1.3) and (1.4) respectively:

- (1) Let ψ be univalent in Δ , $\psi(0) = 1$, and λ be a complex number. Assume that $z\psi'/(\lambda - p - a + a\psi)$ is starlike in Δ and

$$\Re\left\{\lambda - p - a + a\psi(z) + 1 + \frac{z\psi''(z)}{\psi'(z)} - \frac{az\psi'(z)}{\lambda - p - a + a\psi(z)}\right\} > 0.$$

Let F be defined as in (3.9) and $h(z)$ defined by

$$h(z) := \psi(z) + \frac{z\psi'(z)}{\lambda - p - a + a\psi(z)}.$$

If $f(z) \in \Sigma_p$ and

$$\frac{[\mathcal{L}_p(a+1, c)f(z)]'}{[\mathcal{L}_p(a, c)f(z)]'} \prec h(z)$$

then

$$\frac{[\mathcal{L}_p(a+1, c)F(z)]'}{[\mathcal{L}_p(a, c)F(z)]'} \prec \psi(z)$$

and $\psi(z)$ is the best dominant.

- (2) Let ψ be univalent in Δ , $\psi(0) = 1$, and λ be a complex number. Assume that $z\psi'/(\lambda - n + (n+p)\psi)$ is starlike in Δ and

$$\Re\left\{\lambda - n + (n+p)\psi(z) + 1 + \frac{z\psi''(z)}{\psi'(z)} - \frac{(n+p)z\psi'(z)}{\lambda - n + (n+p)\psi(z)}\right\} > 0.$$

Let F be defined as in (3.9) and $h(z)$ defined by

$$h(z) := \psi(z) + \frac{z\psi'(z)}{\lambda - n + (n+p)\psi(z)}.$$

If $f(z) \in \Sigma_p$ and

$$\frac{[\mathcal{D}^{n+p}f(z)]'}{[\mathcal{D}^{n+p-1}f(z)]'} \prec h(z)$$

then

$$\frac{[\mathcal{D}^{n+p}F(z)]'}{[\mathcal{D}^{n+p-1}F(z)]'} \prec \psi(z)$$

and $\psi(z)$ is the best dominant.

Using Theorem 3.3, we have the following result:

Corollary 3.4. *Let ψ_i be univalent in Δ ($i = 1, 2$) and λ be a complex number. Assume that $\frac{z\psi'_i}{\lambda - p - \alpha_1 + \alpha_1\psi_i}$ is starlike in Δ and $\Re\{\lambda - p - \alpha_1 + \alpha_1\psi_i(z)\} > 0$ for $i = 1, 2$. If $f(z) \in \Omega_p(\alpha_1; h_1, h_2)$ satisfies (3.10), then the function $\mathcal{F}$ defined by (3.9) belongs to $\Omega_p(\alpha_1; \psi_1, \psi_2)$ where*

$$h_i(z) := \psi_i(z) + \frac{z\psi'_i(z)}{\lambda - p - \alpha_1 + \alpha_1\psi_i(z)} \quad (i = 1, 2).$$

Theorem 3.5. *Let $f(z) \in \Sigma_p$ and $\alpha_1 \neq -1$. Define $\mathcal{F}$ by*

$$(3.15) \quad \mathcal{F}(z) := \frac{\alpha_1}{z^{\alpha_1+p}} \int_0^z t^{\alpha_1+p-1} f(t) dt.$$

Then $f(z) \in \Omega_p(\alpha_1; \varphi)$ if and only if $\mathcal{F} \in \Omega\left(\alpha_1 + 1; \frac{1 + \alpha_1\varphi}{1 + \alpha_1}\right)$. Also $f(z) \in \tilde{\Omega}_p(\alpha_1; \varphi)$ if and only if $\mathcal{F} \in \tilde{\Omega}\left(\alpha_1 + 1; \frac{1 + \alpha_1\varphi}{1 + \alpha_1}\right)$.

Proof. From (3.15), we have

$$(3.16) \quad \alpha_1 f(z) = (\alpha_1 + p)\mathcal{F}(z) + z\mathcal{F}'(z).$$

By convoluting (3.16) with $h_p(\alpha_1, \dots, \alpha_i; \beta_1, \dots, \beta_m; z)$ and using the fact that $z(f * g)'(z) = f(z) * zg'(z)$, we obtain

$$\alpha_1 \mathcal{H}_p^{l,m}[\alpha_1]f(z) = (\alpha_1 + p)\mathcal{H}_p^{l,m}[\alpha_1]\mathcal{F}(z) + z(\mathcal{H}_p^{l,m}[\alpha_1]\mathcal{F}(z))'$$

and by using (3.4), we get

$$(3.17) \quad \mathcal{H}_p^{l,m}[\alpha_1]f(z) = \mathcal{H}_p^{l,m}[\alpha_1 + 1]\mathcal{F}(z)$$

and

$$\begin{aligned}
 \alpha_1 \mathcal{H}_p^{l,m}[\alpha_1 + 1]f(z) &= z(\mathcal{H}_p^{l,m}[\alpha_1]f(z))' + (\alpha_1 + p)\mathcal{H}_p^{l,m}[\alpha_1]f(z) \\
 &= z(\mathcal{H}_p^{l,m}[\alpha_1 + 1]\mathcal{F}(z))' + (\alpha_1 + p)\mathcal{H}_p^{l,m}[\alpha_1 + 1]\mathcal{F}(z) \\
 &= (\alpha_1 + 1)\mathcal{H}_p^{l,m}[\alpha_1 + 2]\mathcal{F}(z) - (\alpha_1 + p + 1)\mathcal{H}_p^{l,m}[\alpha_1 + 1]\mathcal{F}(z) \\
 &\quad + (\alpha_1 + p)\mathcal{H}_p^{l,m}[\alpha_1 + 1]\mathcal{F}(z) \\
 (3.18) \quad &= (\alpha_1 + 1)\mathcal{H}_p^{l,m}[\alpha_1 + 2]\mathcal{F}(z) - \mathcal{H}_p^{l,m}[\alpha_1 + 1]\mathcal{F}(z).
 \end{aligned}$$

Therefore, from (3.17) and (3.18), we have

$$\frac{(\mathcal{H}_p^{l,m}[\alpha_1 + 2]\mathcal{F}(z))'}{(\mathcal{H}_p^{l,m}[\alpha_1 + 1]\mathcal{F}(z))'} = \frac{1}{\alpha_1 + 1} \left[1 + \alpha_1 \frac{(\mathcal{H}_p^{l,m}[\alpha_1 + 1]f(z))'}{(\mathcal{H}_p^{l,m}[\alpha_1]f(z))'} \right],$$

and the desired results follow at once. $\square$

Using (1.3), we have the following result:

Example 3.1. Let $f(z) \in \mathcal{A}_p$ and $a \neq -1$. Define F as in (3.15). Then

$$\frac{[\mathcal{L}_p(a+1, c)f(z)]'}{[\mathcal{L}_p(a, c)f(z)]'} \prec \varphi(z) \quad \text{if and only if} \quad \frac{[\mathcal{L}_p(a+2, c)F(z)]'}{[\mathcal{L}_p(a+1, c)F(z)]'} \prec \frac{1+a\varphi}{1+a}.$$

Also

$$\varphi(z) \prec \frac{[\mathcal{L}_p(a+1, c)f(z)]'}{[\mathcal{L}_p(a, c)f(z)]'} \quad \text{if and only if} \quad \frac{1+a\varphi}{1+a} \prec \frac{[\mathcal{L}_p(a+2, c)F(z)]'}{[\mathcal{L}_p(a+1, c)F(z)]'}.$$

Using Theorem 3.5, we have

Corollary 3.6. Let $f(z) \in \Sigma_p$ and $\alpha_1 \neq -1$. Then $f(z) \in \Omega_p(\alpha_1; \varphi_1, \varphi_2)$ if and only if $\mathcal{F}$ given by (3.15) is in $\Omega\left(\alpha_1 + 1; \frac{1 + \alpha_1\varphi_1}{1 + \alpha_1}, \frac{1 + \alpha_1\varphi_2}{1 + \alpha_1}\right)$.

4. The classes $\mathcal{M}_p(n, \lambda; \varphi)$ and $\widetilde{\mathcal{M}}_p(n, \lambda; \varphi)$

By making use of Lemma 2.1, we prove the following result:

Theorem 4.1. Let $\psi(z)$ be univalent in Δ , $\psi(0) = 1$, $\Re\psi(z) > 0$ and $z\psi'/\psi$ be starlike in Δ . Let $\chi(z)$ be defined by

$$\chi(z) := \frac{1}{\lambda - p} \left[(\lambda - p)\psi(z) + \frac{z\psi'(z)}{\psi(z)} \right].$$

If $f(z) \in \mathcal{M}_p(n+1, \lambda; \chi)$, then $f(z) \in \mathcal{M}_p(n, \lambda; \psi)$. If $f(z) \in \widetilde{\mathcal{M}}_p(n+1, \lambda; \chi)$,

$$(4.1) \quad 0 \neq \frac{(\mathcal{I}_p(n+1, \lambda)f(z))'}{(\mathcal{I}_p(n, \lambda)f(z))'} \in \mathcal{H}_1 \cap \mathcal{Q} \quad \text{and} \quad \frac{(\mathcal{I}_p(n+2, \lambda)f(z))'}{(\mathcal{I}_p(n+1, \lambda)f(z))'} \text{ is univalent in } \Delta,$$

then $f(z) \in \widetilde{\mathcal{M}}_p(n, \lambda; \psi)$.

Proof. First of all consider the following identity

$$(4.2) \quad (\lambda - p)\mathcal{I}_p(n+1, \lambda)f(z) = z[\mathcal{I}_p(n, \lambda)f(z)]' + \lambda\mathcal{I}_p(n, \lambda)f(z),$$

which yields up on differentiation

$$(4.3) \quad z[\mathcal{I}_p(n, \lambda)f(z)]'' = (\lambda - p)[\mathcal{I}_p(n+1, \lambda)f(z)]' - (\lambda + 1)[\mathcal{I}_p(n, \lambda)f(z)]'.$$

Now define the function $q(z)$ by

$$(4.4) \quad q(z) := \frac{(\mathcal{I}_p(n+1, \lambda)f(z))'}{(\mathcal{I}_p(n, \lambda)f(z))'}.$$

Then, clearly, $q(z)$ is analytic in Δ . By logarithmic differentiation of (4.4) and using (4.3), we obtain

$$(4.5) \quad \frac{(\mathcal{I}_p(n+2, \lambda)f(z))'}{(\mathcal{I}_p(n+1, \lambda)f(z))'} = \frac{1}{\lambda - p} \left((\lambda - p)q(z) + \frac{zq'(z)}{q(z)} \right).$$

Since $f(z) \in \mathcal{M}_p(n+1, \lambda; \chi)$ and in view of (4.5), we have

$$(\lambda - p)q(z) + \frac{zq'(z)}{q(z)} \prec (\lambda - p)\psi(z) + \frac{z\psi'(z)}{\psi(z)}.$$

The first result follows by an application of Lemma 2.1. Similarly the second result follows from Lemma 2.2. $\square$

Remark 4.1. The subordination result of Theorem 4.1 also holds if we replace the condition $\Re\psi(z) > 0$ by

$$\Re \left\{ (\lambda - p)\psi(z) + 1 + \frac{z\psi''(z)}{\psi'(z)} - \frac{z\psi'(z)}{\psi(z)} \right\} > 0.$$

Using Theorem 4.1, we obtain the following “sandwich result”:

Corollary 4.2. Let $\psi_i(z)$ be univalent in Δ , $\Re\psi_i(z) > 0$ and $z\psi'_i(z)/\psi_i(z)$ be starlike univalent in Δ for $i = 1, 2$. Define

$$\chi_i(z) := \frac{1}{\lambda - p} \left[(\lambda - p)\psi_i(z) + \frac{z\psi'_i(z)}{\psi_i(z)} \right] \quad (i = 1, 2).$$

If $f(z) \in \mathcal{M}_p(n+1, \lambda; \chi_1, \chi_2)$ satisfies (4.1), then $f(z) \in \mathcal{M}_p(n, \lambda; \psi_1, \psi_2)$.

Theorem 4.3. Let ψ be univalent in Δ , $\psi(0) = 1$, δ be a complex number, $z\psi'/(\delta -$

$\lambda + (\lambda - p)\psi$ be starlike in Δ and $\Re\{\delta - \lambda + (\lambda - p)\psi(z)\} > 0$. Define the function $\mathcal{F}$ by

$$(4.6) \quad \begin{aligned} \mathcal{F}(z) &:= \frac{\delta - p}{z^\delta} \int_0^z t^{\delta-1} f(t) dt, \\ h(z) &:= \psi(z) + \frac{z\psi'(z)}{\delta - \lambda + (\lambda - p)\psi(z)}. \end{aligned}$$

If $f(z) \in \mathcal{M}_p(n, \lambda; h)$, then $\mathcal{F} \in \mathcal{M}_p(n, \lambda; \psi)$. If $f(z) \in \widetilde{\mathcal{M}}_p(n, \lambda; h)$,

$$(4.7) \quad 0 \neq \frac{(\mathcal{I}_p(n+1, \lambda)\mathcal{F}(z))'}{(\mathcal{I}_p(n, \lambda)\mathcal{F}(z))'}(z) \in \mathcal{H}_1 \cap \mathcal{Q} \text{ and } \frac{(\mathcal{I}_p(n+1, \lambda)f(z))'}{(\mathcal{I}_p(n, \lambda)f(z))'}$$

is univalent in Δ , then $\mathcal{F} \in \widetilde{\mathcal{M}}_p(n, \lambda; \psi)$.

Proof. From the definition of $\mathcal{F}(z)$ and

$$(4.8) \quad z(\mathcal{I}_p(n, \lambda)\mathcal{F}(z))' = (\lambda - p)\mathcal{I}_p(n+1, \lambda)\mathcal{F}(z) - \lambda\mathcal{I}_p(n, \lambda)\mathcal{F}(z),$$

we have

$$(4.9) \quad \begin{aligned} (\delta - p)\mathcal{I}_p(n, \lambda)f(z) &= \delta\mathcal{I}_p(n, \lambda)\mathcal{F}(z) + z(\mathcal{I}_p(n, \lambda)\mathcal{F}(z))' \\ &= (\lambda - p)\mathcal{I}_p(n+1, \lambda)\mathcal{F}(z) + (\delta - \lambda)\mathcal{I}_p(n, \lambda)\mathcal{F}(z). \end{aligned}$$

Define the function $q(z)$ by

$$(4.10) \quad q(z) := \frac{(\mathcal{I}_p(n+1, \lambda)\mathcal{F}(z))'}{(\mathcal{I}_p(n, \lambda)\mathcal{F}(z))'}.$$

Then, clearly, $q(z)$ is analytic in Δ . Using (4.9) and (4.10), we have

$$(4.11) \quad (\delta - p) \frac{(\mathcal{I}_p(n, \lambda)f(z))'}{(\mathcal{I}_p(n, \lambda)\mathcal{F}(z))'} = \delta - \lambda + (\lambda - p)q(z).$$

Upon logarithmic differentiation of (4.11) and using (4.3), (4.8) and (4.10), we get

$$(4.12) \quad \frac{(\mathcal{I}_p(n+1, \lambda)f(z))'}{(\mathcal{I}_p(n, \lambda)f(z))'} = q(z) + \frac{zq'(z)}{\delta - \lambda + (\lambda - p)q(z)}.$$

For $f(z) \in \mathcal{M}_p(n, \lambda, h)$, we have from (4.11),

$$q(z) + \frac{zq'(z)}{\delta - \lambda + (\lambda - p)q(z)} \prec \psi(z) + \frac{z\psi'(z)}{\delta - \lambda + (\lambda - p)\psi(z)}.$$

The first part of our result now follows by an application of Lemma 2.1. Similarly the second part follows from Lemma 2.2. $\square$

Remark 4.2. The subordination result of Theorem 4.3 also holds if we replace the condition $\Re \{\delta - \lambda + (\lambda - p)\psi(z)\} > 0$ by

$$\Re \left\{ \delta - \lambda + (\lambda - p)\psi(z) + 1 + \frac{z\psi''(z)}{\psi'(z)} - \frac{(\lambda - p)z\psi'(z)}{\delta - \lambda + (\lambda - p)\psi(z)} \right\} > 0.$$

Using Theorem 4.3, we have the following result:

Corollary 4.4. Let ψ_i be univalent in Δ and δ be a complex number. Assume that $z\psi'_i/(\delta - \lambda + (\lambda - p)\psi_i)$ is starlike in Δ and $\Re \{\delta - \lambda + (\lambda - p)\psi_i(z)\} > 0$ for $i = 1, 2$. Define the functions h_i by

$$h_i(z) := \psi_i(z) + \frac{z\psi'_i(z)}{\delta - \lambda + (\lambda - p)\psi_i(z)} \quad (i = 1, 2).$$

If $f(z) \in \mathcal{M}_p(n, \lambda; h_1, h_2)$, then $\mathcal{F}$ defined by (4.6) belongs to $\mathcal{M}_p(n, \lambda; \psi_1, \psi_2)$.

Theorem 4.5. Let $f(z) \in \Sigma_p$. Then $f(z) \in \mathcal{M}_p(n, \lambda; \varphi)$ if and only if

$$(4.13) \quad \mathcal{F}(z) := \frac{\lambda - p}{z^\lambda} \int_0^z t^{\lambda-1} f(t) dt \in \mathcal{M}_p(n+1, \lambda; \varphi).$$

Also $f(z) \in \widetilde{\mathcal{M}}_p(n, \lambda; \varphi)$ if and only if $\mathcal{F} \in \widetilde{\mathcal{M}}_p(n+1, \lambda; \varphi)$.

Proof. From (4.13), we have

$$(4.14) \quad (\lambda - p)f(z) = \lambda \mathcal{F}(z) + z\mathcal{F}'(z).$$

By convoluting (4.14) with

$$\phi_p(k, \lambda; z) := \frac{1}{z^p} + \sum_{k=1-p}^{\infty} \left(\frac{k+\lambda}{\lambda-p} \right)^r z^k$$

and using the fact that $z(f * g)'(z) = f(z) * zg'(z)$, we obtain

$$(\lambda - p)\mathcal{I}_p(n, \lambda)f(z) = \lambda \mathcal{I}_p(n, \lambda)\mathcal{F}(z) + z(\mathcal{I}_p(n, \lambda)\mathcal{F}(z))'$$

and by using (4.3), we get

$$(4.15) \quad \mathcal{I}_p(n, \lambda)f(z) = \mathcal{I}_p(n+1, \lambda)\mathcal{F}(z)$$

and

$$(4.16) \quad \begin{aligned} (\lambda - p)\mathcal{I}_p(n+1, \lambda)f(z) &= z(\mathcal{I}_p(n, \lambda)f(z))' + \lambda \mathcal{I}_p(n, \lambda)f(z) \\ &= z(\mathcal{I}_p(n+1, \lambda)\mathcal{F}(z))' + \lambda \mathcal{I}_p(n+1, \lambda)\mathcal{F}(z) \\ &= (\lambda - p)\mathcal{I}_p(n+2, \lambda)\mathcal{F}(z). \end{aligned}$$

Therefore, from (4.15) and (4.16), we have

$$\frac{(\mathcal{I}_p(n+2, \lambda)\mathcal{F}(z))'}{(\mathcal{I}_p(n+1, \lambda)\mathcal{F}(z))'} = \frac{(\mathcal{I}_p(n+1, \lambda)f(z))'}{(\mathcal{I}_p(n, \lambda)f(z))'},$$

and the desired result follows at once. $\square$

Using Theorem 4.5, we have

Corollary 4.6. *Let $f(z) \in \Sigma_p$. Then $f(z) \in \mathcal{M}_p(n, \lambda; \varphi_1, \varphi_2)$ if and only if $\mathcal{F} \in \mathcal{M}_p(n+1, \lambda; \varphi_1, \varphi_2)$.*

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