Some New Application of Power Increasing Sequences

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ABSTRACT. A general theorem for absolute summability using quasi f -power increasing sequence is obtained. This result generalized and improved the result of Bor and Özarslan [2].

1. Introduction

Let $\sum a_n$ be an infinitive series with the sequence of partial sums (s_n) . By (C, α) we mean the Cesàro matrix of order α , and σ_n^α denotes the n -th term of the (C, α) -transform of (s_n) , that is

$$(1.1) \quad \sigma_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=0}^n A_{n-v}^{\alpha-1} s_v.$$

A series $\sum a_n$ is said to be summable $|C, \alpha|_k$, $k \geq 1$, if

$$(1.2) \quad \sum_{n=1}^{\infty} n^{k-1} |\sigma_n^\alpha - \sigma_{n-1}^\alpha|^k < \infty,$$

and it is said to be summable $\varphi - |C, \alpha|_k$, $k \geq 1$, $\alpha > -1$ if (see Balcı [1])

$$(1.3) \quad \sum_{n=1}^{\infty} |\varphi_n (\sigma_n^\alpha - \sigma_{n-1}^\alpha)|^k < \infty,$$

or equivalently

$$(1.4) \quad \sum_{n=1}^{\infty} n^{-k} |\varphi_n t_n^\alpha|^k < \infty,$$

where

$$(1.5) \quad t_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=0}^n A_{n-v}^{\alpha-1} v a_v.$$

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In the special case when $\varphi_n = n^{1-1/k}, n^{\delta+1-1/k}$ ($n \in N, k \geq 1$), φ - $|C, \alpha|_k$ -summability reduces to $|C, \alpha|_k$ -summability, $|C, \alpha, \delta|_k$ -summability respectively. Let (p_n) be a sequence of positive real numbers such that

$$(1.6) \quad P_n = p_0 + p_1 + \dots + p_n \rightarrow \infty \text{ as } n \rightarrow \infty, \quad P_{-1} = p_{-1} = 0.$$

A series $\sum a_n$ is said to be summable $|N, p_n|_k, k \geq 1$, if

$$(1.7) \quad \sum_{n=1}^{\infty} n^{k-1} |u_n - u_{n-1}|^k < \infty,$$

where

$$(1.8) \quad u_n = \frac{1}{P_n} \sum_{v=1}^{\infty} p_{n-v} s_v.$$

For $k = 1$, $|N, p_n|_k$ -summability reduces to $|N, p_n|$ -summability, and for $p_n = A_n^\alpha$, $|N, p_n|_k$ -summability reduces to $|C, \alpha|_k$ -summability. By M we denote the set of sequences $p = (p_n)$ satisfying

$$(1.9) \quad \frac{p_{n+1}}{p_n} \leq \frac{p_{n+2}}{p_{n+1}} \leq 1, \quad p_n > 0, \quad n = 0, 1, \dots$$

It is known that (see Das [4]) for $p \in M, k = 1$, (1.7) holds if and only if

$$(1.10) \quad \sum_{n=1}^{\infty} \frac{1}{nP_n} \left| \sum_{v=1}^n p_{n-v} v a_v \right| < \infty.$$

For $p \in M$, the series $\sum a_n$ is summable $|N, p_n|_k, k \geq 1$, if (see Sulaiman [7])

$$(1.11) \quad \sum_{n=1}^{\infty} \frac{1}{nP_n^k} \left| \sum_{v=1}^n p_{n-v} v a_v \right|^k < \infty.$$

Here we give the following definition: For $p \in M$, the series $\sum a_n$ is summable $\varphi - |N, p_n|_k, k \geq 1$, if

$$(1.12) \quad \sum_{n=1}^{\infty} n^{-k} |\varphi_n \Phi_n|^k < \infty,$$

where

$$(1.13) \quad \Phi_n = \frac{1}{P_n} \sum_{v=1}^n p_{n-v} v a_v.$$

A positive sequence $a = (a_n)$ is said to be quasi β -power increasing sequence if there exists a constant $K = K(\beta, a) \geq 1$ such that (see Leindler [5])

$$Kn^\beta a_n \geq m^\beta a_m \quad (n \geq m \geq 1, n, m \in N).$$

A positive sequence (a_n) is said to be quasi f -power increasing sequence if there exists a constant $K = K(f, a) \geq 1$ such that (see Sulaiman [8])

$$Kf_n a_n \geq f_m a_m \quad (n \geq m \geq 1, n, m \in N).$$

where $f = (f_n)$, $f = n^\beta (\log n)^\gamma$, $0 < \beta < 1$, $\gamma \geq 0$. Recently, Bor and Özarslan [2] presented the following theorem

Theorem 1.1. *Let (X_n) be a quasi- β -power increasing sequence for some β ($0 < \beta < 1$). Suppose also that there exist sequences (β_n) and $(\lambda_n) \in BV_0$ such that*

$$(1.14) \quad |\Delta \lambda_n| \leq \beta_n,$$

$$(1.15) \quad \beta_n \rightarrow \infty, \quad (n \rightarrow \infty),$$

$$(1.16) \quad \sum_{n=1}^{\infty} n |\Delta \beta_n| X_n < \infty,$$

and

$$(1.17) \quad |\lambda_n| X_n = O(1). \quad (n \rightarrow \infty)$$

If there exists an $\epsilon > 0$ such that the sequence $(n^{\epsilon-k} |\varphi_n|^k)$ is non-increasing and if the sequence (ω_n^α) is defined by

$$(1.18) \quad \omega_n^\alpha = \begin{cases} |t_n^\alpha|, & \alpha = 1 \\ \max_{1 \leq v \leq n} \{|t_v^\alpha|\}, & 0 < \alpha < 1 \end{cases}$$

satisfying the following condition:

$$(1.19) \quad \sum_{n=1}^m n^{-k} (|\varphi_n| \omega_n^\alpha)^k = O(X_m), \quad (m \rightarrow \infty),$$

then the series $\sum a_n \lambda_n$ is summable $\varphi - |C, \alpha|_k$, $k \geq 1$, $0 < \alpha \leq 1$, $k\alpha + \epsilon > 1$.

2. Lemmas

In this section, the following lemmas are needed for our aim.

Lemma 2.1. *If $0 \leq m \leq n$, p_{n-v}/p_{m-v} is decreasing with respect to v , then*

$$(2.1) \quad \left| \sum_{v=0}^m p_{n-v} s_v \right| \leq \max_{0 \leq \mu \leq m} \left| \sum_{v=0}^{\mu} p_{\mu-v} s_v \right|.$$

Proof. The proof is similar to that given in [3]. □

Lemma 2.2. *Condition (1.14) – (1.16) implies*

$$(2.2) \quad n\beta_n X_n = O(1), \quad n \rightarrow \infty,$$

$$(2.3) \quad \sum_{n=1}^{\infty} \beta_n X_n < \infty.$$

Proof. The proof is similar to that given in [8]. □

3. Results

The following is our main result.

Theorem 3.1. *Let (X_n) be a quasi- f -power increasing sequence, $f = (f_n)$, $f_n = n^{\beta}(\log n)^{\gamma}$, $0 < \beta < 1$, $\gamma \geq 0$, such that the sequences (X_n) and (λ_n) satisfying conditions (1.14) – (1.17), and*

$$(3.1) \quad \sum_{n=1}^m \frac{1}{n^k X_n^{k-1}} (|\varphi_n| \omega_n)^k = O(X_m), \quad (m \rightarrow \infty),$$

$$(3.2) \quad \sum_{n=v+1}^{m+1} \frac{|\varphi_n|^k}{n^k P_n^k} = O\left(\frac{|\varphi_v|^k}{v^{k-1} P_v^k}\right)$$

where

$$(3.3) \quad \omega_n = \begin{cases} \Phi_n, & v = n \\ \max \{\Phi_v\}, & 1 \leq v \leq n. \end{cases}$$

Then the series $\sum a_n \lambda_n$ is summable $\varphi - |N, p_n|_k$, $k \geq 1$.

Proof. Let T_n be n -th (N, p_n) -mean of the sequence $(na_n \lambda_n)$. Then, we have

$$T_n = \frac{1}{P_n} \sum_{v=1}^n v a_v p_{n-v} \lambda_v$$

We have, via Abel's transformation

$$\begin{aligned}
|T_n| &= \frac{1}{P_n} \left| \sum_{v=1}^n v a_v p_{n-v} \lambda_v \right| \\
&= \frac{1}{P_n} \left| \sum_{v=1}^{n-1} \left(\sum_{r=1}^v r p_{n-r} a_r \right) \Delta \lambda_v + \lambda_n \sum_{v=1}^n v p_{n-v} a_v \right| \\
&\leq \frac{1}{P_n} \sum_{v=0}^{n-1} P_v |\omega_v| |\Delta \lambda_v| + |\lambda_n| |\omega_n| \\
&= T_{n1} + T_{n2}.
\end{aligned}$$

Since $|T_{n1} + T_{n2}|^k \leq 2^k (|T_{n1}|^k + |T_{n2}|^k)$, then in order to complete the proof, it is sufficient to show that

$$\sum_{n=1}^{\infty} n^{-k} |\varphi_n T_{nj}|^k < \infty, \quad j = 1, 2.$$

Applying Hölder's inequality,

$$\begin{aligned}
\sum_{n=2}^{m+1} n^{-k} |\varphi_n T_{n1}|^k &= \sum_{n=2}^{m+1} \frac{|\varphi_n|^k}{n^k P_n^k} \left| \sum_{v=1}^{n-1} P_v |\omega_v| |\Delta \lambda_v| \right|^k \\
&\leq \sum_{n=2}^{m+1} \frac{|\varphi_n|^k}{n^k P_n^k} \sum_{v=1}^{n-1} P_v^k |\omega_v|^k \beta_v X_v^{1-k} \left(\sum_{v=1}^{n-1} X_v \beta_v \right)^{k-1} \\
&= O(1) \sum_{n=2}^{m+1} \frac{|\varphi_n|^k}{n^k P_n^k} \sum_{v=1}^{n-1} P_v^k |\omega_v|^k \beta_v X_v^{1-k} \\
&= O(1) \sum_{v=1}^m P_v^k |\omega_v|^k \beta_v X_v^{1-k} \sum_{n=v+1}^{m+1} \frac{|\varphi_n|^k}{n^k P_n^k} \\
&= O(1) \sum_{v=1}^m \frac{v}{v^k X_v^{k-1}} |\varphi_v|^k |\omega_v|^k \beta_v \\
&= O(1) \sum_{v=1}^m \left(\sum_{r=1}^v \frac{|\varphi_r|^k |\omega_r|^k}{r^k X_r^{k-1}} \right) \Delta(v \beta_v) \\
&\quad + O(1) m \beta_m \sum_{v=1}^m \frac{1}{v^k X_v^{k-1}} |\varphi_v|^k |\omega_v|^k \\
&= O(1) \sum_{v=1}^m \beta_v X_v + O(1) \sum_{v=1}^m v |\Delta \beta_v| X_v + O(1) m \beta_m X_m \\
&= O(1).
\end{aligned}$$

$$\begin{aligned}
\sum_{n=1}^m n^{-k} |\varphi_n T_{n2}|^k &= \sum_{n=1}^m \frac{|\varphi_n|^k}{n^k} |\lambda_n|^k |\omega_n|^k \\
&= \sum_{n=1}^m \frac{|\varphi_n|^k}{n^k X_n^{k-1}} |\lambda_n| |\omega_n|^k (|\lambda_n| X_n)^{k-1} \\
&= O(1) \sum_{n=1}^m \frac{|\varphi_n|^k |\omega_n|^k}{n^k X_n^{k-1}} |\lambda_n| \\
&= O(1) \sum_{n=1}^m \left(\sum_{v=1}^n \frac{|\varphi_v|^k |\omega_v|^k}{v^k X_v^{k-1}} \right) \Delta |\lambda_n| + O(1) |\lambda_m| \sum_{n=1}^m \frac{|\varphi_n|^k |\omega_n|^k}{n^k X_n^{k-1}} \\
&= O(1) \sum_{n=1}^m |\Delta \lambda_n| X_n + O(1) |\lambda_m| X_m \\
&= O(1) \sum_{n=1}^m \beta_n X_n + O(1) |\lambda_m| X_m \\
&= O(1). \quad \square
\end{aligned}$$

Remark 1. (1) It may be mentioned that an improvement to Theorem 1.1 follows from Theorem 3.1 by putting $p_n = A_n^{\alpha-1}$, provided from some $\epsilon > 0$, $(n^{\epsilon-k} |\varphi_n|^k)$ is non-increasing, and $k\alpha + \epsilon > 1$.

(2) One advantage of condition (3.1) is that we have no loss of power of $|\lambda_n|$ through estimation such as by having $|\lambda_n|^{k-1} = O(1)$.

Remark 2. Condition (3.1) does not imply condition (1.19).

Proof.

$$\begin{aligned}
\sum_{n=1}^m n^{-k} (|\varphi_n| \omega_n^\alpha)^k &= \sum_{n=1}^m \frac{1}{n^k X_n^{k-1}} (|\varphi_n| \omega_n^\alpha)^k X_n^{k-1} \\
&= O(X_m^{k-1}) \sum_{n=1}^m \frac{1}{n^k X_n^{k-1}} (|\varphi_n| \omega_n^\alpha)^k \\
&= O(X_m^k) \\
&\neq O(X_m), \text{ for } k > 1. \quad \square
\end{aligned}$$

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