A General System of Nonlinear Functional Equations in Non-Archimedean Spaces

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ABSTRACT. In this paper, we prove the generalized Hyers–Ulam–Rassias stability for a system of functional equations, called general system of nonlinear functional equations, in non-Archimedean normed spaces and Menger probabilistic non-Archimedean normed spaces.

1. Introduction and Preliminaries

The first stability problem concerning group homomorphisms was raised by Ulam[31] in 1940 and solved in the next year by Hyers[15]. Hyers' theorem was generalized by Aoki[2] for additive mappings and by Rassias[28] for linear mappings by considering an unbounded Cauchy difference. In 1994, a generalization of the Rassias theorem was obtained by Găvruta[11] by replacing the unbounded Cauchy difference by a general control function. In recent years many authors have investigated the stability of various functional equations in various spaces (see for instance [4, 6, 7, 8, 12, 16, 17, 21, 26, 27]).

In this paper we establish some stability result concerning a general system of nonlinear functional equations in non-Archimedean normed spaces and Menger probabilistic non-Archimedean normed spaces.

It has been turned out that non-Archimedean spaces have many useful applications in quantum physics, p -adic strings and superstrings (see [5, 18, 19, 24]). The proofs for non-Archimedean spaces are essentially different and entirely require new kind of intuition (see for instance [3, 9, 10, 23, 25, 32]).

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Definition 1.1. Let K be a field. A valuation mapping on K is a function $|\cdot| : K \rightarrow R$ such that for any $a, b \in K$ we have

- (i) $|a| \geq 0$ and equality holds if and only if $a = 0$,
- (ii) $|ab| = |a||b|$,
- (iii) $|a + b| \leq |a| + |b|$.

A field endowed with a valuation mapping will be called a valued field. If the condition (iii) in the definition of a valuation mapping is replaced with

$$(iii)' \quad |a + b| \leq \max\{|a|, |b|\}$$

then the valuation $|\cdot|$ is said to be non-Archimedean. The condition (iii)' is called the strict triangle inequality. By (ii), we have $|1| = |-1| = 1$. Thus, by induction, it follows from (iii)' that $|n| \leq 1$ for each integer n . We always assume in addition that $|\cdot|$ is non trivial, i.e., that there is an $a_0 \in K$ such that $|a_0| \notin \{0, 1\}$. The most important examples of non-Archimedean spaces are p -adic numbers.

Example 1.2. Let p be a prime number. For any non-zero rational number $a = p^r \frac{m}{n}$ such that m and n are coprime to the prime number p , define the p -adic absolute value $|a|_p = p^{-r}$. Then $|\cdot|$ is a non-Archimedean norm on $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to $|\cdot|$ is denoted by $\mathbb{Q}_p$ and is called the p -adic number field.

Definition 1.3. Let X be a linear space over a scalar field K with a non-Archimedean non-trivial valuation $|\cdot|$. A function $\|\cdot\| : X \rightarrow R$ is a non-Archimedean norm (valuation) if it satisfies the following conditions:

- (NA1) $\|x\| = 0$ if and only if $x = 0$;
- (NA2) $\|rx\| = |r|\|x\|$ for all $r \in K$ and $x \in X$;
- (NA3) the strong triangle inequality (ultrametric); namely,
 $\|x + y\| \leq \max\{\|x\|, \|y\|\} \quad (x, y \in X).$

Then $(X, \|\cdot\|)$ is called a non-Archimedean space.

It follows from (NA3) that

$$\|x_m - x_l\| \leq \max\{\|x_{j+1} - x_j\| : l \leq j \leq m-1\} \quad (m > l),$$

therefore a sequence $\{x_m\}$ is Cauchy in X if and only if $\{x_{m+1} - x_m\}$ converges to zero in a non-Archimedean space.

Probabilistic normed spaces were first defined by Šerstnev in 1962 (see [30]). Their definition was generalized by Alsina, Schewizer and Sklar [1]. We recall and apply the definition of Menger probabilistic normed spaces briefly as given in [29].

Definition 1.4. A distance distribution function (briefly, a d.d.f.) is a non-decreasing function F from $[0, +\infty]$ into $[0, 1]$ that satisfies $F(0) = 0$ and $F(+\infty) = 1$, and is left-continuous on $(0, +\infty)$. The space of d.d.f.'s will be denoted by Δ^+ , and the set of all F in Δ^+ for which $\lim_{t \rightarrow +\infty} F(t) = 1$ by D^+ . The space Δ^+ is partially ordered by the usual pointwise ordering of functions, i.e., $F \leq G$ if and only if $F(x) \leq G(x)$ for all x in $[0, +\infty]$. For any $a \geq 0$, ε_a is the d.d.f. given by

$$\varepsilon_a(t) = \begin{cases} 0, & \text{if } t \leq a, \\ 1, & \text{if } t > a. \end{cases}$$

Definition 1.5. A triangular norm (briefly t -norm) is a binary operation $T : [0, 1] \times [0, 1] \rightarrow [0, 1]$ which is commutative, associative, non-decreasing in each variable and has 1 as the unit element. Basic examples are the Łukasiewicz t -norm T_L , $T_L(a, b) = \max(a + b - 1, 0)$, the product t -norm T_P , $T_P(a, b) = ab$ and the strongest triangular norm T_M , $T_M(a, b) = \min(a, b)$.

Definition 1.6. A *Menger Probabilistic Normed space* is a triple (X, ν, T) , where X is a real vector space, T is continuous t -norm and ν is a mapping (the *probabilistic norm*) from X into Δ^+ , such that for every choice of p and q in X and a, s, t in $(0, +\infty)$, the following hold:

(PN1) $\nu(p) = \varepsilon_0$, if and only if, $p = \theta$ (θ is the null vector in X);

(PN2) $\nu(ap)(t) = \nu(p)(\frac{t}{|a|})$;

(PN3) $\nu(p + q)(s + t) \geq T(\nu(p)(s), \nu(q)(t))$.

Now we introduce definition of a Menger probabilistic non-Archimedean normed space by the definition of a non-Archimedean fuzzy normed space which is given in [20] and [22].

Definition 1.7. Let X be a vector space over a non-Archimedean field K and T be a continuous t -norm. A triple (X, ν, T) is said to be a Menger probabilistic non-Archimedean normed space if (PN1) and (PN2) (in Definition??) and

$$(PNA3) \nu(x + y)(\max\{s, t\}) \geq T(\nu(x)(s), \nu(y)(t)),$$

for all $x, y \in X$ and all $s, t > 0$, are satisfied.

It follows from $\nu(x) \in \Delta^+$ that $\nu(x)$ is non-decreasing for every $x \in X$. So one can show that the condition (PNA3) is equivalent to the following condition:

$$\nu(x + y)(t) \geq T(\nu(x)(t), \nu(y)(t)).$$

Definition 1.8. Let (X, ν, T) be a Menger probabilistic non-Archimedean normed space. Let $\{x_n\}$ be a sequence in X . Then $\{x_n\}$ is said to be convergent if there exists $x \in X$ such that $\lim_{n \rightarrow \infty} \nu(x_n - x)(t) = 1$, for all $t > 0$. In that case, x is called the limit of the sequence $\{x_n\}$. A sequence $\{x_n\}$ in X is called Cauchy if for each $\varepsilon > 0$ and each $t > 0$ there exists n_0 such that for all $n \geq n_0$ and all $p > 0$ we have $\nu(x_{n+p} - x_n)(t) > 1 - \varepsilon$.

Let T be a given t-norm. Then (by associativity) a family of mappings $T^n : [0, 1] \rightarrow [0, 1]$, $n \in \mathbb{N}$, is defined as follows:

$$T^1(x) = T(x, x), \quad T^n(x) = T(T^{n-1}(x), x), \quad x \in [0, 1].$$

For three important t-norms T_M , T_P and T_L we have

$$T_M^n(x) = x, \quad T_P^n(x) = x^n, \quad T_L^n(x) = \max\{(n+1)x - n, 0\}, \quad n \in \mathbb{N}.$$

Definition 1.9(Hadzić[13]). A t-norm T is said to be of H-type if a family of functions $\{T^n(t)\}$; $n \in \mathbb{N}$, is equicontinuous at $t = 1$, that is,

$$\forall \varepsilon \in (0, 1) \exists \delta \in (0, 1) : t > 1 - \delta \Rightarrow T^n(t) > 1 - \varepsilon \quad (n \geq 1).$$

The t-norm T_M is a trivial example of t-norm of H-type, but there are t-norms of H-type with $T \neq T_M$ (see e.g., Hadzić[14]).

Lemma 1.10. We consider the notations of the Definition(1.8.). Also assume that T is a t-norm of H-type. Then the sequence $\{x_n\}$ is Cauchy if for each $\varepsilon > 0$ and each $t > 0$ there exists n_0 such that for all $n \geq n_0$ we have $\nu(x_{n+1} - x_n)(t) > 1 - \varepsilon$.

Proof. Due to

$$\begin{aligned} \nu(x_{n+p} - x_n)(t) &\geq T\left(\nu(x_{n+p} - x_{n+p-1})(t), \nu(x_{n+p-1} - x_n)(t)\right) \geq \\ &T\left(\nu(x_{n+p} - x_{n+p-1})(t), T(\nu(x_{n+p-1} - x_{n+p-2})(t), \nu(x_{n+p-2} - x_n)(t))\right) \geq \\ &\vdots \\ &\geq T\left(\nu(x_{n+p} - x_{n+p-1})(t), T(\nu(x_{n+p-1} - x_{n+p-2})(t), \dots, \right. \\ &\left. T(\nu(x_{n+2} - x_{n+1})(t), \nu(x_{n+1} - x_n)(t)) \dots)\right), \end{aligned}$$

and by the assumption of T , which is an H-type t-norm, the sequence $\{x_n\}$ is Cauchy if for each $\varepsilon > 0$ and each $t > 0$ there exists n_0 such that for all $n \geq n_0$ we have $\nu(x_{n+1} - x_n)(t) > 1 - \varepsilon$. We will use this criterion in this paper. $\square$

It is easy to see that every convergent sequence in a (Menger probabilistic) non-Archimedean normed space is Cauchy. If each Cauchy sequence is convergent, then

the (Menger probabilistic) non-Archimedean normed space is said to be complete and is called (Menger probabilistic) non-Archimedean Banach space.

We assume that $f : X^n \rightarrow Y$ and $\lambda_i : \mathbb{K} \rightarrow \mathbb{K}$ be mappings and introduce the following system:

$$(1.1) \quad \begin{cases} f(a_1 x_1, x_2, \dots, x_n) = \lambda_1(a_1) f(x_1, \dots, x_n); \\ f(x_1, a_2 x_2, \dots, x_n) = \lambda_2(a_2) f(x_1, \dots, x_n); \\ \vdots \\ f(x_1, \dots, a_i x_i, \dots, x_n) = \lambda_i(a_i) f(x_1, \dots, x_n); \\ \vdots \\ f(x_1, \dots, x_{n-1}, a_n x_n) = \lambda_n(a_n) f(x_1, \dots, x_n); \end{cases}$$

for all $x_i \in X$ and $a_i \in \mathbb{K} \setminus \{0\}$, $i = 1, \dots, n$. We call the above system the general system of nonlinear functional equations. One can show that the following mappings satisfying system(1.1).

$$\begin{cases} \lambda_i(a) = a^i \implies f(x_1, x_2, \dots, x_n) = x_1 x_2^2 \cdots x_n^n \\ \lambda_i(a) = b_i^{a_i} \text{ for } b_i \neq 0, \pm 1 \implies f(x_1, x_2, \dots, x_n) = b_1^{x_1} b_2^{x_2^2} \cdots b_n^{x_n^n} \end{cases}$$

In the section(2), we establish the generalized Hyers-Ulam-Rassias stability of system(1.1) in non-Archimedean Banach spaces. In the section(3), we establish the generalized Hyers-Ulam-Rassias stability of system(1.1) in Menger probabilistic non-Archimedean Banach spaces.

2. System(1.1) Stability in non-Archimedean Banach Spaces

In this section, we prove the generalized Hyers-Ulam-Rassias stability of system(1.1) in non-Archimedean Banach spaces. Throughout this section, we assume that $i, m, n, p \in \mathbb{N} \cup \{0\}$, K is a non-Archimedean field, Y is a non-Archimedean Banach space over K and X is a vector space over K . Also assume that $f : X^n \rightarrow Y$ and $\lambda_i : \mathbb{K} \rightarrow \mathbb{K}$ are mappings.

Theorem 2.1. *Let $\varphi_i : X^n \rightarrow [0, \infty)$ for $i \in \{1, \dots, n\}$ be a function such that*

$$(2.1) \quad \lim_{m \rightarrow \infty} \max \left\{ \frac{1}{|(\lambda_1(a_1))^{m+1} \cdots (\lambda_i(a_i))^{m+1} (\lambda_{i+1}(a_{i+1}))^m \cdots (\lambda_n(a_n))^m|} \right. \\ \left. \varphi_i(a_1^{m+1} x_1, \dots, a_{i-1}^{m+1} x_{i-1}, a_i^m x_i, \dots, a_n^m x_n) : i = 1, \dots, n \right\} = 0,$$

and

$$(2.2) \quad \Phi = \Phi(x_1, \dots, x_n) = \lim_{p \rightarrow \infty} \max \left\{ \frac{1}{\max \left\{ \frac{1}{|(\lambda_1(a_1))^{m+1} \dots (\lambda_i(a_i))^{m+1} (\lambda_{i+1}(a_{i+1}))^m \dots (\lambda_n(a_n))^m|} \right.} \right. \\ \left. \varphi_i(a_1^{m+1}x_1, \dots, a_{i-1}^{m+1}x_{i-1}, a_i^m x_i, \dots, a_n^m x_n) \right. \\ \left. : i = 1, \dots, n \right\} : m = 0, 1, \dots, p \Big\} < \infty,$$

and

$$(2.3) \quad \lim_{m \rightarrow \infty} \frac{1}{|(\lambda_1(a_1))^m \dots (\lambda_n(a_n))^m|} \varphi_i(a_1^m x_1, \dots, a_n^m x_n) = 0,$$

for all $x_i \in X$ and $a_i \in \mathbb{K} \setminus \{0\}$, $i = 1, \dots, n$. Let $f : X^n \rightarrow Y$ be a mapping satisfying

$$\begin{cases} \|f(a_1 x_1, x_2, \dots, x_n) - \lambda_1(a_1)f(x_1, \dots, x_n)\| \leq \varphi_1(x_1, \dots, x_n); \\ \vdots \\ \|f(x_1, \dots, a_i x_i, \dots, x_n) - \lambda_i(a_i)f(x_1, \dots, x_n)\| \leq \varphi_i(x_1, \dots, x_n); \\ \vdots \\ \|f(x_1, \dots, x_{n-1}, a_n x_n) - \lambda_n(a_n)f(x_1, \dots, x_n)\| \leq \varphi_n(x_1, \dots, x_n); \end{cases}$$

for all $x_i \in X$ and $a_i \in \mathbb{K} \setminus \{0\}$, $i = 1, \dots, n$. Then there exists a unique mapping $T : X^n \rightarrow Y$ satisfying system (1.1) and

$$(2.4) \quad \|f(x_1, \dots, x_n) - T(x_1, \dots, x_n)\| \leq \Phi,$$

for all $x_i \in X$, $i = 1, \dots, n$.

Proof. Fix $i \in \{1, 2, \dots, n\}$ and consider the following inequality.

$$(2.5) \quad \|f(x_1, \dots, a_i x_i, \dots, x_n) - \lambda_i(a_i)f(x_1, \dots, x_n)\| \leq \varphi_i(x_1, \dots, x_n).$$

From (2.5) we get

$$\|f(x_1, \dots, x_n) - \frac{1}{\lambda_i(a_i)}f(x_1, \dots, a_i x_i, \dots, x_n)\| \leq \frac{1}{|\lambda_i(a_i)|} \varphi_i(x_1, \dots, x_n).$$

Therefore one can obtain

$$(2.6) \quad \begin{aligned} & \left\| \frac{1}{\lambda_1(a_1) \dots \lambda_{i-1}(a_{i-1})} f(a_1 x_1, \dots, a_{i-1} x_{i-1}, x_i, \dots, x_n) - \right. \\ & \left. \frac{1}{\lambda_1(a_1) \dots \lambda_i(a_i)} f(a_1 x_1, \dots, a_i x_i, x_{i+1}, \dots, x_n) \right\| \leq \\ & \frac{1}{|\lambda_1(a_1) \dots \lambda_i(a_i)|} \varphi_i(a_1 x_1, \dots, a_{i-1} x_{i-1}, x_i, \dots, x_n). \end{aligned}$$

So by induction and (2.6), we conclude

$$\|f(x_1, \dots, x_n) - \frac{1}{\lambda_1(a_1) \dots \lambda_n(a_n)} f(a_1 x_1, \dots, a_n x_n)\| \leq \max\left\{ \frac{1}{|\lambda_1(a_1) \dots \lambda_i(a_i)|} \varphi_i(a_1 x_1, \dots, a_{i-1} x_{i-1}, x_i, \dots, x_n) : i = 1, \dots, n \right\}.$$

Therefore we get

$$(2.7) \quad \begin{aligned} & \left\| \frac{1}{(\lambda_1(a_1))^m \dots (\lambda_n(a_n))^m} f(a_1^m x_1, \dots, a_n^m x_n) - \right. \\ & \left. \frac{1}{(\lambda_1(a_1))^{m+1} \dots (\lambda_n(a_n))^{m+1}} f(a_1^{m+1} x_1, \dots, a_n^{m+1} x_n) \right\| \leq \\ & \max\left\{ \frac{1}{|(\lambda_1(a_1))^{m+1} \dots (\lambda_i(a_i))^{m+1} (\lambda_{i+1}(a_{i+1}))^m \dots (\lambda_n(a_n))^m|} \right. \\ & \left. \varphi_i(a_1^{m+1} x_1, \dots, a_{i-1}^{m+1} x_{i-1}, a_i^m x_i, \dots, a_n^m x_n) : i = 1, \dots, n \right\}, \end{aligned}$$

for all $m \in \mathbb{N} \cup \{0\}$. It follows from (2.7) and (2.1) that the sequence

$$\left\{ \frac{1}{(\lambda_1(a_1))^m \dots (\lambda_n(a_n))^m} f(a_1^m x_1, \dots, a_n^m x_n) \right\}$$

is Cauchy. Since the space Y is complete, this sequence is convergent. Therefore we can define

$T : X^n \rightarrow Y$ by

$$(2.8) \quad T(x_1, \dots, x_n) := \lim_{m \rightarrow \infty} \frac{1}{(\lambda_1(a_1))^m \dots (\lambda_n(a_n))^m} f(a_1^m x_1, \dots, a_n^m x_n),$$

for all $x_i \in X$ and $a_i \in \mathbb{K} \setminus \{0\}$, $i = 1, \dots, n$. Using induction with (2.7) one can show that

$$(2.9) \quad \begin{aligned} & \left\| f(x_1, \dots, x_n) - \frac{1}{(\lambda_1(a_1))^p \dots (\lambda_n(a_n))^p} f(a_1^p x_1, \dots, a_n^p x_n) \right\| \leq \\ & \max \left\{ \max \left\{ \frac{1}{|(\lambda_1(a_1))^{m+1} \dots (\lambda_i(a_i))^{m+1} (\lambda_{i+1}(a_{i+1}))^m \dots (\lambda_n(a_n))^m|} \right. \right. \\ & \left. \varphi_i(a_1^{m+1} x_1, \dots, a_{i-1}^{m+1} x_{i-1}, a_i^m x_i, \dots, a_n^m x_n) \right. \\ & \left. : i = 1, \dots, n \right\} : m = 0, 1, \dots, p \left. \right\}. \end{aligned}$$

for all $x_i \in X$, $i = 1, \dots, n$ and $p \in \mathbb{N} \cup \{0\}$. By taking p to approach infinity in (2.9) and using (2.2) one obtains (2.4).

For fixed $i \in \{1, 2, \dots, n\}$ and by (2.5) and (2.8), we get

$$\begin{aligned}
 & \|T(x_1, \dots, a_i x_i, \dots, x_n) - \lambda_i(a_i)T(x_1, \dots, x_n)\| = \\
 & \lim_{m \rightarrow \infty} \left\| \frac{1}{(\lambda_1(a_1))^m \dots (\lambda_n(a_n))^m} f(a_1^m x_1, \dots, a_i^{m+1} x_i, \dots, a_n^m x_n) - \right. \\
 (2.10) \quad & \left. \frac{\lambda_i(a_i)}{(\lambda_1(a_1))^m \dots (\lambda_n(a_n))^m} f(a_1^m x_1, \dots, a_n^m x_n) \right\| \\
 & \leq \lim_{m \rightarrow \infty} \frac{1}{|(\lambda_1(a_1))^m \dots (\lambda_n(a_n))^m|} \varphi_i(a_1^m x_1, \dots, a_n^m x_n).
 \end{aligned}$$

By (2.10) and (2.3), we conclude that T satisfies system(1.1).

Suppose that there exists another mapping $T' : X^n \rightarrow Y$ which satisfies system(1.1) and (2.4). So we have

$$\begin{aligned}
 & \|T(x_1, x_2, \dots, x_n) - T'(x_1, x_2, \dots, x_n)\| \leq \\
 & \frac{1}{|(\lambda_1(a_1))^m \dots (\lambda_n(a_n))^m|} \max \left\{ \|T(a_1^m x_1, \dots, a_n^m x_n) - f(a_1^m x_1, \dots, a_n^m x_n)\|, \right. \\
 & \left. \|f(a_1^m x_1, \dots, a_n^m x_n) - T'(a_1^m x_1, \dots, a_n^m x_n)\| \right\} \leq \\
 & \frac{1}{|(\lambda_1(a_1))^m \dots (\lambda_n(a_n))^m|} \max \left\{ \Phi(a_1^m x_1, \dots, a_n^m x_n), \Phi(a_1^m x_1, \dots, a_n^m x_n) \right\},
 \end{aligned}$$

which tends to zero as $m \rightarrow \infty$ by (2.2). Therefore $T = T'$. This completes the proof. $\square$

We obtain the following corollary if we assume $\lambda_i(a) = a^i$ in Theorem(2.1).

Corollary 2.2. *Let $\varphi_i : X^n \rightarrow [0, \infty)$ for $i \in \{1, 2, \dots, n\}$ be a function such that*

$$\begin{aligned}
 & \lim_{m \rightarrow \infty} \max \left\{ \frac{1}{|a_1^{m+1} a_2^{2(m+1)} \dots a_i^{i(m+1)} a_{i+1}^{(i+1)m} \dots a_n^{nm}|} \varphi_i(a_1^{m+1} x_1, \dots, a_{i-1}^{m+1} x_{i-1}, \right. \\
 & \left. a_i^m x_i, \dots, a_n^m x_n) : i = 1, \dots, n \right\} = 0,
 \end{aligned}$$

and

$$\begin{aligned}
 & \Phi = \Phi(x_1, \dots, x_n) = \\
 & \lim_{p \rightarrow \infty} \max \left\{ \max \left\{ \frac{1}{|a_1^{m+1} a_2^{2(m+1)} \dots a_i^{i(m+1)} a_{i+1}^{(i+1)m} \dots a_n^{nm}|} \varphi_i(a_1^{m+1} x_1, \dots, a_{i-1}^{m+1} x_{i-1}, \right. \right. \\
 & \left. \left. a_i^m x_i, \dots, a_n^m x_n) : i = 1, \dots, n \right\} : m = 0, 1, \dots, p \right\} < \infty,
 \end{aligned}$$

and

$$\lim_{m \rightarrow \infty} \frac{1}{|a_1^m a_2^{2m} \dots a_n^{nm}|} \varphi_i(a_1^m x_1, \dots, a_n^m x_n) = 0,$$

for all $x_i \in X$ and $a_i \in \mathbb{K} \setminus \{0\}$, $i = 1, \dots, n$. Let $f : X^n \rightarrow Y$ be a mapping satisfying

$$\begin{cases} \|f(a_1x_1, x_2, \dots, x_n) - a_1f(x_1, \dots, x_n)\| \leq \varphi_1(x_1, \dots, x_n); \\ \|f(x_1, a_2x_2, \dots, x_n) - a_2^2f(x_1, \dots, x_n)\| \leq \varphi_2(x_1, \dots, x_n); \\ \vdots \\ \|f(x_1, \dots, a_ix_i, \dots, x_n) - a_i^if(x_1, \dots, x_n)\| \leq \varphi_i(x_1, \dots, x_n); \\ \vdots \\ \|f(x_1, \dots, x_{n-1}, a_nx_n) - a_n^n f(x_1, \dots, x_n)\| \leq \varphi_n(x_1, \dots, x_n); \end{cases}$$

for all $x_i \in X$ and $a_i \in \mathbb{K} \setminus \{0\}$, $i = 1, \dots, n$. Then there exists a unique mapping $T : X^n \rightarrow Y$ satisfying system(1.1) for $\lambda_i(a_i) = a_i^i$ and

$$\|f(x_1, \dots, x_n) - T(x_1, \dots, x_n)\| \leq \Phi,$$

for all $x_i \in X$, $i = 1, \dots, n$.

3. System (1.1) Stability in Menger Probabilistic non-Archimedean Banach Spaces

In this section, we prove the generalized Hyers–Ulam–Rassias stability of system(1.1) in Menger probabilistic non-Archimedean Banach spaces. Throughout this section, we assume that $u \in \mathbb{R}$, $i, m, n \in \mathbb{N} \cup \{0\}$, K is a non-Archimedean field, T is a continuous t-norm of H-type, (Y, ν, T) is a Menger probabilistic non-Archimedean Banach space over K , (Z, ω, T) is a Menger probabilistic non-Archimedean normed space over K and X is a vector space over K . Also assume that $f : X^n \rightarrow Y$ and $\lambda_i : \mathbb{K} \rightarrow \mathbb{K}$ are mappings.

Theorem 3.1. Let $\varphi_i : X^n \rightarrow Z$ for $i \in \{1, \dots, n\}$ be a mapping such that

$$(3.1) \quad \begin{cases} \tilde{\varphi}_i = \tilde{\varphi}_i(x_1, \dots, x_n, u) = \\ \omega\left(\frac{1}{|\lambda_1(a_1)\dots\lambda_i(a_i)|} \varphi_i(a_1x_1, \dots, a_{i-1}x_{i-1}, x_i, \dots, x_n)\right)(u); \\ \Phi_1 = \Phi_1(x_1, \dots, x_n, u) = \tilde{\varphi}_1(x_1, \dots, x_n, u); \\ \Phi_i = \Phi_i(x_1, \dots, x_n, u) = T\left(\tilde{\varphi}_i(x_1, \dots, x_n, u), \Phi_{i-1}(x_1, \dots, x_n, u)\right); \\ \lim_{m \rightarrow \infty} \Phi_n(a_1^m x_1, \dots, a_n^m x_n, |(\lambda_1(a_1))^m \dots (\lambda_n(a_n))^m|u) = 1; \end{cases}$$

and

$$(3.2) \quad \lim_{m \rightarrow \infty} \omega\left(\frac{1}{|(\lambda_1(a_1))^m \dots (\lambda_n(a_n))^m|} \varphi_i(a_1^m x_1, \dots, a_n^m x_n)\right)(u) = 1;$$

and

$$(3.3) \quad \begin{cases} \Phi_m^* = \Phi_m^*(x_1, \dots, x_n, u) = \\ \Phi_n(a_1^m x_1, \dots, a_n^m x_n, |(\lambda_1(a_1))^m \dots (\lambda_n(a_n))^m| u); \\ \Psi_0 = \Phi_0^*(x_1, \dots, x_n, u) = \Phi_n(x_1, \dots, x_n, u); \\ \Psi_m = \Psi_m(x_1, \dots, x_n, u) = \\ T\left(\Phi_m^*(x_1, \dots, x_n, u), \Psi_{m-1}(x_1, \dots, x_n, u)\right); \\ \Psi = \Psi(x_1, \dots, x_n, u) = \lim_{m \rightarrow \infty} \Psi_m = 1; \end{cases}$$

for all $u > 0$, $x_i \in X$ and $a_i \in \mathbb{K} \setminus \{0\}$, $i = 1, \dots, n$. Let $f : X^n \rightarrow Y$ be a mapping satisfying

$$\begin{cases} \nu\left(f(a_1 x_1, x_2, \dots, x_n) - \lambda_1(a_1)f(x_1, \dots, x_n)\right)(u) \geq \omega\left(\varphi_1(x_1, \dots, x_n)\right)(u); \\ \vdots \\ \nu\left(f(x_1, \dots, a_i x_i, \dots, x_n) - \lambda_i(a_i)f(x_1, \dots, x_n)\right)(u) \geq \omega\left(\varphi_i(x_1, \dots, x_n)\right)(u); \\ \vdots \\ \nu\left(f(x_1, \dots, x_{n-1}, a_n x_n) - \lambda_n(a_n)f(x_1, \dots, x_n)\right)(u) \geq \omega\left(\varphi_n(x_1, \dots, x_n)\right)(u); \end{cases}$$

for all $u > 0$, $x_i \in X$ and $a_i \in \mathbb{K} \setminus \{0\}$, $i = 1, \dots, n$. Then there exists a unique mapping $F : X^n \rightarrow Y$ satisfying system (1.1) and

$$(3.4) \quad \nu\left(f(x_1, \dots, x_n) - F(x_1, \dots, x_n)\right)(u) \geq \Psi,$$

for all $u > 0$ and $x_i \in X$, $i = 1, \dots, n$.

Proof. Fix $i \in \{1, 2, \dots, n\}$ and consider the following inequality.

$$(3.5) \quad \begin{aligned} & \nu\left(f(x_1, \dots, a_i x_i, \dots, x_n) - \lambda_i(a_i)f(x_1, \dots, x_n)\right)(u) \geq \\ & \omega\left(\varphi_i(x_1, \dots, x_n)\right)(u). \end{aligned}$$

From (3.5) we get

$$\nu\left(f(x_1, \dots, x_n) - \frac{1}{\lambda_i(a_i)}f(x_1, \dots, a_i x_i, \dots, x_n)\right)(u) \geq \omega\left(\frac{1}{|\lambda_i(a_i)|}\varphi_i(x_1, \dots, x_n)\right)(u).$$

Therefore one can obtain

$$(3.6) \quad \begin{aligned} & \nu\left(\frac{1}{\lambda_1(a_1) \dots \lambda_{i-1}(a_{i-1})}f(a_1 x_1, \dots, a_{i-1} x_{i-1}, x_i, \dots, x_n) - \right. \\ & \left. \frac{1}{\lambda_1(a_1) \dots \lambda_i(a_i)}f(a_1 x_1, \dots, a_i x_i, x_{i+1}, \dots, x_n)\right)(u) \geq \\ & \omega\left(\frac{1}{|\lambda_1(a_1) \dots \lambda_i(a_i)|}\varphi_i(a_1 x_1, \dots, a_{i-1} x_{i-1}, x_i, \dots, x_n)\right)(u) = \tilde{\varphi}_i. \end{aligned}$$

So by induction and by (3.1) and (3.6), we have

$$(3.7) \quad \nu \left(f(x_1, \dots, x_n) - \frac{1}{\lambda_1(a_1) \dots \lambda_n(a_n)} f(a_1 x_1, \dots, a_n x_n) \right)(u) \geq \Phi_n.$$

Therefore we get

$$(3.8) \quad \begin{aligned} & \nu \left(\frac{1}{(\lambda_1(a_1))^m \dots (\lambda_n(a_n))^m} f(a_1^m x_1, \dots, a_n^m x_n) - \right. \\ & \left. \frac{1}{(\lambda_1(a_1))^{m+1} \dots (\lambda_n(a_n))^{m+1}} f(a_1^{m+1} x_1, \dots, a_n^{m+1} x_n) \right)(u) \geq \\ & \Phi_n(a_1^m x_1, \dots, a_n^m x_n, |(\lambda_1(a_1))^m \dots (\lambda_n(a_n))^m| u) \end{aligned}$$

for all $m \in \mathbb{N} \cup \{0\}$. So by (3.1) and (3.8), the sequence

$$\left\{ \frac{1}{(\lambda_1(a_1))^m \dots (\lambda_n(a_n))^m} f(a_1^m x_1, \dots, a_n^m x_n) \right\}$$

is Cauchy. By completeness of Y , we conclude that it is convergent. Therefore we can define $F : X^n \rightarrow Y$ by

$$(3.9) \quad \lim_{m \rightarrow \infty} \nu \left(F(x_1, \dots, x_n) - \frac{1}{(\lambda_1(a_1))^m \dots (\lambda_n(a_n))^m} f(a_1^m x_1, \dots, a_n^m x_n) \right)(u) = 1,$$

for all $u > 0$, $x_i \in X$ and $a_i \in \mathbb{K} \setminus \{0\}$, $i = 1, \dots, n$. Using induction with (3.8) one can show that

$$(3.10) \quad \nu \left(f(x_1, \dots, x_n) - \frac{1}{(\lambda_1(a_1))^m \dots (\lambda_n(a_n))^m} f(a_1^m x_1, \dots, a_n^m x_n) \right)(u) \geq \Psi_m.$$

By taking m to approach infinity in (3.10) and using (3.3) one obtains (3.4).

For $i \in \{1, 2, \dots, n\}$ and by (3.5) and (3.9), we get

$$(3.11) \quad \begin{aligned} & \nu \left(F(x_1, \dots, a_i x_i, \dots, x_n) - \lambda_i(a_i) F(x_1, \dots, x_n) \right)(u) = \\ & \lim_{m \rightarrow \infty} \nu \left(\frac{1}{(\lambda_1(a_1))^m \dots (\lambda_n(a_n))^m} f(a_1^m x_1, \dots, a_i^{m+1} x_i, \dots, a_n^m x_n) - \right. \\ & \left. \frac{\lambda_i(a_i)}{(\lambda_1(a_1))^m \dots (\lambda_n(a_n))^m} f(a_1^m x_1, \dots, a_n^m x_n) \right)(u) \geq \\ & \lim_{m \rightarrow \infty} \omega \left(\frac{1}{|(\lambda_1(a_1))^m \dots (\lambda_n(a_n))^m|} \varphi_i(a_1^m x_1, \dots, a_n^m x_n) \right)(u). \end{aligned}$$

By (3.2) and (3.11), we conclude that F satisfies system(1.1).

Suppose that there exists another mapping $F' : X^n \rightarrow X$ which satisfies system(1.1) and (3.4). So we have

$$\begin{aligned}
& \nu \left(F(x_1, \dots, x_n) - F'(x_1, \dots, x_n) \right) (u) \\
&= \nu \left(\frac{1}{(\lambda_1(a_1))^m \dots (\lambda_n(a_n))^m} F(a_1^m x_1, \dots, a_n^m x_n) \right. \\
&\quad \left. - \frac{1}{(\lambda_1(a_1))^m \dots (\lambda_n(a_n))^m} f(a_1^m x_1, \dots, a_n^m x_n) \right. \\
&\quad \left. + \frac{1}{(\lambda_1(a_1))^m \dots (\lambda_n(a_n))^m} f(a_1^m x_1, \dots, a_n^m x_n) \right. \\
&\quad \left. - \frac{1}{(\lambda_1(a_1))^m \dots (\lambda_n(a_n))^m} F'(a_1^m x_1, \dots, a_n^m x_n) \right) (u) \\
&\geq T \left\{ \Psi_m \left(a_1^m x_1, \dots, a_n^m x_n, |(\lambda_1(a_1))^m \dots (\lambda_n(a_n))^m| u \right), \right. \\
&\quad \left. \Psi_m \left(a_1^m x_1, \dots, a_n^m x_n, |(\lambda_1(a_1))^m \dots (\lambda_n(a_n))^m| u \right) \right\},
\end{aligned}$$

which tends to 1 as $m \rightarrow \infty$ by (3.3). Therefore $F = F'$.

This completes the proof. $\square$

We conclude the following corollary if we assume $\lambda_i(a) = a^i$ in Theorem(3.1).

Corollary 3.2. *Let $\varphi_i : X^n \rightarrow Z$ for $i \in \{1, \dots, n\}$ be a mapping such that*

$$\begin{cases} \tilde{\varphi}_i = \tilde{\varphi}_i(x_1, \dots, x_n, u) = \omega \left(\frac{1}{|a_1 a_2^2 \dots a_i^i|} \varphi_i(a_1 x_1, \dots, a_{i-1} x_{i-1}, x_i, \dots, x_n) \right) (u); \\ \Phi_1 = \Phi_1(x_1, \dots, x_n, u) = \tilde{\varphi}_1(x_1, \dots, x_n, u); \\ \Phi_i = \Phi_i(x_1, \dots, x_n, u) = T \left(\tilde{\varphi}_i(x_1, \dots, x_n, u), \Phi_{i-1}(x_1, \dots, x_n, u) \right); \\ \lim_{m \rightarrow \infty} \Phi_n(a_1^m x_1, \dots, a_n^m x_n, |a_1^m a_2^{2m} \dots a_n^{nm}| u) = 1; \end{cases}$$

and

$$\lim_{m \rightarrow \infty} \omega \left(\frac{1}{|a_1^m a_2^{2m} \dots a_n^{nm}|} \varphi_i(a_1^m x_1, \dots, a_n^m x_n) \right) (u) = 1;$$

and

$$\begin{cases} \Phi_m^* = \Phi_m^*(x_1, \dots, x_n, u) = \Phi_n(a_1^m x_1, \dots, a_n^m x_n, |a_1^m a_2^{2m} \dots a_n^{nm}| u); \\ \Psi_0 = \Phi_0^*(x_1, \dots, x_n, u) = \Phi_n(x_1, \dots, x_n, u); \\ \Psi_m = \Psi_m(x_1, \dots, x_n, u) = T \left(\Phi_m^*(x_1, \dots, x_n, u), \Psi_{m-1}(x_1, \dots, x_n, u) \right); \\ \Psi = \Psi(x_1, \dots, x_n, u) = \lim_{m \rightarrow \infty} \Psi_m = 1. \end{cases}$$

for all $u > 0$, $x_i \in X$ and $a_i \in \mathbb{K} \setminus \{0\}$, $i = 1, \dots, n$. Let $f : X^n \rightarrow Y$ be a mapping

satisfying

$$\left\{ \begin{array}{l} \nu \left(f(a_1 x_1, x_2, \dots, x_n) - a_1 f(x_1, \dots, x_n) \right) (u) \geq \omega \left(\varphi_1(x_1, \dots, x_n) \right) (u); \\ \nu \left(f(x_1, a_2 x_2, \dots, x_n) - a_2^2 f(x_1, \dots, x_n) \right) (u) \geq \omega \left(\varphi_2(x_1, \dots, x_n) \right) (u); \\ \vdots \\ \nu \left(f(x_1, \dots, a_i x_i, \dots, x_n) - a_i^i f(x_1, \dots, x_n) \right) (u) \geq \omega \left(\varphi_i(x_1, \dots, x_n) \right) (u); \\ \vdots \\ \nu \left(f(x_1, \dots, x_{n-1}, a_n x_n) - a_n^n f(x_1, \dots, x_n) \right) (u) \geq \omega \left(\varphi_n(x_1, \dots, x_n) \right) (u); \end{array} \right.$$

for all $u > 0$, $x_i \in X$ and $a_i \in \mathbb{K} \setminus \{0\}$, $i = 1, \dots, n$. Then there exists a unique mapping $F : X^n \rightarrow Y$ satisfying system (1.1) for $\lambda_i(a_i) = a_i^i$ and

$$\nu \left(f(x_1, \dots, x_n) - F(x_1, \dots, x_n) \right) (u) \geq \Psi$$

for all $u > 0$ and $x_i \in X$, $i = 1, \dots, n$.

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