A New Approach to the Lebesgue-Radon-Nikodym Theorem with respect to Weighted p -adic Invariant Integral on $\mathbb{Z}_p$

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ABSTRACT. We will give a new proof of the Lebesgue-Radon-Nikodym theorem with respect to weighted p -adic q -measure on $\mathbb{Z}_p$, using Mahler expansion of continuous functions, studied by the authors in 2012. In the special case, $q = 1$, we can derive the same result as in Kim, 2012, Kim et al, 2011.

1. Introduction

Let p be a fixed odd prime number. Throughout this paper, the symbols $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ denote the ring of p -adic integers, the field of p -adic rational numbers, and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. The p -adic norm $|\cdot|_p$ is defined by $|x|_p = p^{-v_p(x)} = p^{-r}$ for $x = p^r \frac{s}{t}$ with $(p, s) = (p, t) = 1$ and $r \in \mathbb{Q}$ (see [1–11]).

When one speaks of q -extension, q can be regarded as an indeterminate, a complex $q \in \mathbb{C}$, or a p -adic number $q \in \mathbb{C}_p$. In this paper we assume that $q \in \mathbb{C}_p$ with $|q - 1|_p < p^{-\frac{1}{p-1}}$ and we use the notations of q -numbers as follows:

$$[x]_q = [x : q] = \frac{1 - q^x}{1 - q}, \quad \text{and} \quad [x]_{-q} = \frac{1 - (-q)^x}{1 + q}.$$

Let $C(\mathbb{Z}_p)$ be the space of continuous functions on $\mathbb{Z}_p$. The fermionic invariant measure on $\mathbb{Z}_p$ is defined by Kim as follows:

$$(1.1) \quad \mu_{-1}(a + p^n \mathbb{Z}_p) = (-1)^a,$$

where

$$a + p^n \mathbb{Z}_p = \{x \in \mathbb{Z}_p \mid x \equiv a \pmod{p^n}\},$$

and $a \in \mathbb{Z}$ with $0 \leq a < p^n$ (see [3, 5, 6, 9]).

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From (1.1), the fermionic p -adic invariant integral on Z_p is defined by Kim as follows:

$$(1.2) \quad I(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) \lim_{N \rightarrow \infty} \sum_{x=0}^{p^N-1} f(x) (-1)^x,$$

where $f \in C(Z_p)$ (see [3,5,6,8,9,10,11]).

The idea for generalizing the fermionic integral is replacing the fermionic Haar measure with weakly (strongly) fermionic measure Z_p satisfying

$$(1.3) \quad |\mu_{-1}(a + p^n Z_p) - \mu_{-1}(a + p^{n+1} Z_p)|_p \leq \delta_n, \quad (\text{see [3, 10]}),$$

where $\delta_n \rightarrow 0$, a is an element of Z_p , and δ_n is independent of a (for strongly fermionic measure, δ_n is replaced by Cp^{-n} , where C is a positive constant).

Let $f(x)$ be a function defined on Z_p . The fermionic integral of f with respect to a weakly fermionic measure μ_{-1} is

$$\int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = \lim_{n \rightarrow \infty} \sum_{x=0}^{p^n-1} f(x) \mu_{-1}(x + p^n Z_p),$$

if the limit exists.

If μ_{-1} is a weakly fermionic measure on Z_p , then we can define Radon-Nikodym derivative of μ_{-1} with respect to the Haar measure on Z_p as follows:

$$(1.4) \quad f_{\mu_{-1}}(x) = \lim_{n \rightarrow \infty} \mu_{-1}(x + p^n Z_p), \quad (\text{see [6, 13]}).$$

Note that $f_{\mu_{-1}}$ is only a continuous function on Z_p . Let $UD(Z_p)$ be the space of uniformly differentiable functions on Z_p . For $f \in UD(Z_p)$, let us define $\mu_{-1,f}$ as follows:

$$(1.5) \quad \mu_{-1,f}(x + p^n Z_p) = \int_{x+p^n Z_p} f(x) d\mu_{-1}(x), \quad (\text{see [6, 13]}),$$

where the integral is the fermionic p -adic invariant integral. From (1.5), we can easily note that $\mu_{-1,f}$ is a strongly fermionic measure on Z_p . Since

$$\begin{aligned} |\mu_{-1,f}(x + p^n Z_p) - \mu_{-1,f}(x + p^{n+1} Z_p)|_p &= \left| \sum_{x=0}^{p^n-1} f(x) (-1)^x - \sum_{x=0}^{p^n} f(x) (-1)^x \right|_p \\ &= \left| \frac{f(p^n)}{p^n} \right| |p^n| \leq Cp^{-n}, \end{aligned}$$

where C is positive constant.

The purpose of this paper is to give a new proof of a Lebesgue-Radon-Nikodym type theorem with respect to the fermionic weighted p -adic q -measure on Z_p (see

[13]), by using Mahler expansion of continuous functions in [12].

2. The Lebesgue-Radon-Nikodym theorem with respect to the weighted p -adic q -measure

For any positive integer a and n with $a < p^n$ and $f \in UD(Z_p)$, we define $\tilde{\mu}_{f,-q}$, weighted fermionic measure on Z_p as follows:

$$(2.1) \quad \tilde{\mu}_{f,-q}(a + p^n Z_p) = \int_{a+p^n Z_p} q^x f(x) d\mu_{-1}(x),$$

where the integral is the fermionic p -adic invariant integral on Z_p .

From (2.1), we note that

$$(2.2) \quad \begin{aligned} \tilde{\mu}_{f,-q}(a + p^n Z_p) &= \lim_{m \rightarrow \infty} \sum_{x=0}^{p^m-1} f(a + p^n x) (-1)^{a+p^n x} q^{a+p^n x} \\ &= (-1)^a q^a \lim_{m \rightarrow \infty} \sum_{x=0}^{p^{m-n}-1} f(a + p^n x) (-1)^x q^{p^n x} \\ &= (-1)^a \int_{\mathbb{Z}_p} f(a + p^n x) q^{a+p^n x} d\mu_{-1}(x). \end{aligned}$$

By (2.2), we get

$$(2.3) \quad \tilde{\mu}_{f,-q}(a + p^n Z_p) = (-1)^a \int_{\mathbb{Z}_p} f(a + p^n x) q^{a+p^n x} d\mu_{-1}(x).$$

Thus, by (2.3), we have

$$(2.4) \quad \tilde{\mu}_{\alpha f + \beta g, -q} = \alpha \tilde{\mu}_{f, -q} + \beta \tilde{\mu}_{g, -q},$$

where $f, g \in UD(Z_p)$ and α, β are positive constants.

By (2.1), (2.2), (2.3) and (2.4), we get

$$(2.5) \quad |\tilde{\mu}_{f,-q}(a + p^n Z_p)|_p \leq \|f_q\|_\infty,$$

where $\|f_q\|_\infty = \sup_{x \in Z_p} |f(x)q^x|_p$.

Let $P(x) \in C_p[x]$ be an arbitrary polynomial. Now we show $\tilde{\mu}_{P,-q}$ is a strongly weighted fermionic p -adic invariant measure on Z_p . Without a loss of generality, it is enough to prove the statement for $P(x) = x^k$.

For $a \in Z$ with $0 \leq a < p^n$, we have

$$(2.6) \quad \tilde{\mu}_{P,-q}(a + p^n Z_p) = \lim_{m \rightarrow \infty} (-q)^a \sum_{i=0}^{p^{m-n}-1} q^{p^n i} (a + ip^n)^k (-1)^i.$$

Note that

$$(2.7) \quad (a + ip^n)^k = \sum_{\ell=0}^k \binom{k}{\ell} a^{k-\ell} (ip^n)^\ell \equiv a^k \pmod{p^n}.$$

By (2.6) and (2.7), we easily get

$$(2.8) \quad \begin{aligned} \tilde{\mu}_{P,-q}(a + p^n Z_p) &\equiv (-1)^a q^a a^k \frac{2}{[2]_{q^{p^n}}} \pmod{p^n} \\ &\equiv (-1)^a P(a) q^a \frac{2}{[2]_{q^{p^n}}} \pmod{p^n}. \end{aligned}$$

For $x \in Z_p$, let $x \equiv x_n \pmod{p^n}$ and $x \equiv x_{n+1} \pmod{p^{n+1}}$, where $x_n, x_{n+1} \in Z$ with $0 \leq x_n < p^n$ and $0 \leq x_{n+1} < p^{n+1}$. Then we have

$$(2.9) \quad |\tilde{\mu}_{P,-q}(a + p^n Z_p) - \tilde{\mu}_{P,-q}(a + p^{n+1} Z_p)|_p \leq Cp^{-n},$$

where C is positive constant and $n \gg 0$.

Let

$$f_{\tilde{\mu}_{P,-q}}(a) = \lim_{n \rightarrow \infty} \tilde{\mu}_{P,-q}(a + p^n Z_p).$$

Then, by (2.8) and (2.9), we see that

$$(2.10) \quad f_{\tilde{\mu}_{P,-q}}(a) = (-1)^a q^a a^k = (-1)^a q^a P(a).$$

We can approach this procedure (2.10) following the idea of Kim, 2012 [4] and using the q -Euler numbers in [2].

In (2.6), we apply (2.7) to obtain

$$\begin{aligned} \tilde{\mu}_{p,-q}(a + p^n Z_p) &= (-q)^a \lim_{m \rightarrow \infty} \left\{ a^k \sum_{i=0}^{p^{m-n}-1} q^{p^n i} (-1)^i + ka^{k-1} \sum_{i=0}^{p^{m-n}-1} (ip^n) (-1)^i + \right. \\ &\quad \left. \cdots + \sum_{i=0}^{p^{m-n}-1} q^{p^n i} (ip^n)^k (-1)^i \right\} \\ &= (-q)^a \left\{ a^k \mathcal{E}_{0,q^{p^n}} + ka^{k-1} p^n \mathcal{E}_{1,q^{p^n}} + \cdots + p^{nk} \mathcal{E}_{n,q^{p^n}} \right\}. \end{aligned}$$

Therefore

$$\begin{aligned} f_{\tilde{\mu}_{p,-q}}(a) &= (-q)^a \lim_{n \rightarrow \infty} \left\{ a^k \mathcal{E}_{0,q^{p^n}} + ka^{k-1} p^n \mathcal{E}_{1,q^{p^n}} + \cdots + p^{nk} \mathcal{E}_{n,q^{p^n}} \right\} \\ &= \lim_{n \rightarrow \infty} (-q)^a a^k \mathcal{E}_{0,q^{p^n}} \\ &= \lim_{n \rightarrow \infty} (-q)^a a^k \frac{2}{1 + q^{p^n}} \\ &= (-q)^a a^k. \end{aligned}$$

Since $f_{\tilde{\mu}_{P,-q}}(x)$ is continuous function on Z_p . For $x \in Z_p$, we have

$$(2.11) \quad f_{\tilde{\mu}_{P,-q}}(x) = (-1)^x q^x x^k, (k \in Z_+).$$

Let $g \in UD(Z_p)$. Then, by (2.9), (2.10) and (2.11), we get

$$(2.12) \quad \begin{aligned} \int_{\mathbb{Z}_p} g(x) d\tilde{\mu}_{P,-q}(x) &= \lim_{n \rightarrow \infty} \sum_{x=0}^{p^n-1} g(x) \tilde{\mu}_{P,-q}(x + p^n Z_p) \\ &= \lim_{n \rightarrow \infty} \sum_{x=0}^{p^n-1} g(x) q^x [2]_{q^{p^n}} x^k (-1)^x \\ &= \int_{\mathbb{Z}_p} g(x) q^x x^k d\mu_{-1}(x). \end{aligned}$$

Therefore, by (2.12), we obtain the following theorem.

Theorem 2.1. *Let $P(x) \in C_p[x]$ be an arbitrary polynomial. Then $\tilde{\mu}_{P,-q}$ is a strongly weighted fermionic p -adic invariant measure on Z_p . That is,*

$$f_{\tilde{\mu}_{P,-q}}(x) = (-1)^x q^x P(x) \quad \text{for all } x \in Z_p.$$

Furthermore, for any $g \in UD(Z_p)$,

$$\int_{\mathbb{Z}_p} g(x) d\tilde{\mu}_{P,-q}(x) = \int_{\mathbb{Z}_p} g(x) P(x) q^x d\mu_{-1}(x),$$

where the second integral is weighted fermionic p -adic invariant integral on Z_p .

Let $f(x) = \sum_{n=0}^{\infty} a_n \binom{x}{n}$ be the Mahler expansion of continuous function on Z_p . Then we note that $\lim_{n \rightarrow \infty} n|a_n|_p = 0$.

Let

$$f_m(x) = \sum_{i=0}^m a_i \binom{x}{i} \in C_p[x].$$

Then

$$(2.13) \quad \|f - f_m\|_{\infty} \leq \sup_{n \leq m} n|a_n|_p.$$

The function $f(x)$ can be rewritten as $f = f_m + f - f_m$. Thus, by (2.4) and (2.13), we get

$$(2.14) \quad \begin{aligned} &|\tilde{\mu}_{f,-q}(a + p^n Z_p) - \tilde{\mu}_{f,-q}(a + p^{n+1} Z_p)| \\ &\leq \max \{ |\tilde{\mu}_{f_m,-q}(a + p^n Z_p) - \tilde{\mu}_{f_m,-q}(a + p^{n+1} Z_p)|, \\ &\quad |\tilde{\mu}_{f-f_m,-q}(a + p^n Z_p) - \tilde{\mu}_{f-f_m,-q}(a + p^{n+1} Z_p)| \}. \end{aligned}$$

From Theorem 2.1, we note that

$$(2.15) \quad \left| \tilde{\mu}_{f-f_m, -q}(a + p^n Z_p) \right|_p \leq C^* \|f - f_m\|_\infty \leq C_1 p^{-n},$$

where C^* and C_1 are positive constants. For $m \gg 0$, we have $\|f\|_\infty = \|f_m\|_\infty$. So, we see that

$$(2.16) \quad \begin{aligned} & \left| \tilde{\mu}_{f_m, -q}(a + p^n Z_p) - \tilde{\mu}_{f_m, -q}(a + p^{n+1} Z_p) \right|_p \\ &= \left| \frac{f_m(p^n) q^{p^n}}{p^n} \right|_p |p^n|_p \leq \|f_m q^x\|_\infty |p^n|_p \leq C_2 p^{-n}, \end{aligned}$$

where C_2 is a positive constant.

By (2.15), we get

$$\begin{aligned} & \left| (-1)^a f(a) q^a - \tilde{\mu}_{f, -q}(a + p^n Z_p) \right|_p \\ & \leq \max \{ |q^a f(a) - f_m(a) q^a|_p, |q^a f_m(a) - \tilde{\mu}_{f_m, -q}(a + p^n Z_p)|_p, |\tilde{\mu}_{f-f_m, -q}(a + p^n Z_p)|_p \} \\ & \leq \max \{ |f(a) - f_m(a)|_p, |f_m(a) - \tilde{\mu}_{f_m, -q}(a + p^n Z_p)|_p, \|f - f_m\|_\infty \}. \end{aligned}$$

Let us assume that fix $\epsilon > 0$, and fix m such that $\|f - f_m\| < \epsilon$. Then we have

$$(2.17) \quad \left| (-q)^a f(a) - \tilde{\mu}_{f, -q}(a + p^n Z_p) \right|_p \leq \epsilon \quad \text{for } n \gg 0.$$

Thus, by (2.17), we have

$$(2.18) \quad f_{\tilde{\mu}_{f, -q}}(a) = \lim_{n \rightarrow \infty} \tilde{\mu}_{f, -q}(a + p^n Z_p) = (-1)^a q^a f(a).$$

Let m be the sufficiently large number such that $\|f - f_m\|_\infty \leq p^{-n}$. Then we get

$$\begin{aligned} \tilde{\mu}_{f, -q}(a + p^n Z_p) &= \tilde{\mu}_{f_m, -q}(a + p^n Z_p) + \tilde{\mu}_{f-f_m, -q}(a + p^n Z_p) \\ &= (-1)^a q^a f(a) \pmod{p^n}. \end{aligned}$$

For $g \in UD(Z_p)$, we have

$$\int_{\mathbb{Z}_p} g(x) d\tilde{\mu}_{f, -q}(x) = \int_{\mathbb{Z}_p} f(x) g(x) q^x d\mu_{-1}(x).$$

Let f be the function from $UD(Z_p)$ to $Lip(Z_p)$. We easily see that $q^x \mu_{-1}(x + p^n Z_p)$ is a strongly weighted p -adic invariant measure on Z_p and

$$\left| (f_q)_{\mu_{-1}}(a) - q^a \mu_{-1}(a + p^n Z_p) \right|_p \leq C_3 p^{-n},$$

where $f_q(x) = f(x) q^x$ and C_3 is positive constant and $n \in \mathbb{Z}_+$.

If $\tilde{\mu}_{1,-q}$ is associated with strongly weighted fermionic invariant measure on Z_p , then we have

$$|\tilde{\mu}_{1,-q}(a + p^n Z_p) - (f_q)_{\mu_{-1}}(a)|_p \leq C_4 p^{-n},$$

where $n > 0$ and C_4 is positive constant.

For $n \gg 0$, we have

$$\begin{aligned} (2.19) \quad & |q^a \mu_{-1}(a + p^n Z_p) - \tilde{\mu}_{1,-q}(a + p^n Z_p)|_p \\ & \leq |q^a \mu_{-1}(a + p^n Z_p) - (f_q)_{\tilde{\mu}_{-1}}(a)|_p \\ & \quad + |(f_q)_{\tilde{\mu}_{-1}}(a) - \tilde{\mu}_{1,-q}(a + p^n Z_p)|_p \\ & \leq K, \end{aligned}$$

where K is positive constant.

Hence, $q\mu_{-1} - \tilde{\mu}_{1,-q}$ is a weighted measure on Z_p . Therefore, we obtain the following theorem.

Theorem 2.2. *Let $q\mu_{-1}$ be a strongly weighted p -adic invariant measure on Z_p , and assume that the fermionic weighted Radon-Nikodym derivative $(f_q)_{\mu_{-1}}$ on Z_p is uniformly differentiable function. Suppose that $\tilde{\mu}_{1,-q}$ is the strongly weighted fermionic p -adic invariant measure associated with $(f_q)_{\mu_{-1}}$. Then there exists a weighted measure $\tilde{\mu}_{2,-q}$ on Z_p such that*

$$q^x \mu_{-1}(x + p^n Z_p) = \tilde{\mu}_{1,-q}(x + p^n Z_p) + \tilde{\mu}_{2,-q}(x + p^n Z_p).$$

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