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On Generators in the Category of Actions of Pomonoids on Posets and its Slices

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Abstract. Where S is a pomonoid, let **Pos**-S be the category of S-posets and S-poset maps. We start off by characterizing the pomonoids S for which all projectives in this category are either generators or free. We then study the notions of regular injectivity and weakly regularly d-injectivity in this category. This leads to homological classification results for pomonoids. Among other things, we get find relationships between regular injectivity in the slice category **Pos**-S/B_S, for any S-poset B_S, and generators and cyclic projectives in **Pos**-S.

1. Introduction and Preliminaries

General ordered algebraic structures play a key role in a wide range of areas, including analysis, logic, theoretical computer science, and physics [2]. One of these structures, which is of interest to mathematicians, is the category of representations of a pomonoid by order preserving maps of partially ordered sets (see for example [3, 4, 5, 6, 7, 8, 9, 14, 16, 18, 19]). Although there exist many papers which investigate various properties of generator acts over a fixed monoid (see [10, 11, 12, 17] for example), among them there seems to be very little known on generator S-posets, where S is a pomonoid. In [14], V. Laan investigated some properties of generator S-posets. Furthermore, in [9] some homoligical characterizations of pomonoids by properties of generators were presented. Continuing this study, in this paper, after some introductory results in Section 1, we attempt in Section 2 to collect new results on generators in **Pos**-S to apply to the question of the homological classification of pomonoids.

$\mathcal{M}$-injective objects in the slice category $\mathcal{C}/B$, for any B in $\mathcal{C}$, form the right part of a weak factorization system that has morphisms of $\mathcal{M}$ as the left part (see [1]). Here, we consider the same case in the slice category **Pos**-S/B_S of right S-poset

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maps over B_S, where B_S is an arbitrary S-poset. In Section 3, we first find conditions for when all generators are regular d-injective or weakly regularly injective. Then, we prove that every $\mathcal{M}$-injective object in **Pos**-S/B_S is a split epimorphism, where $\mathcal{M} = \mathrm{Emb}$ is the class of all order-embeddings of S-posets. Also, we investigate the relationship between regular injectivity in **Pos**-S and **Pos**-S/B_S and generators and cyclic projectives which becomes evident when passing to acts over their endomorphism monoids.

For the rest of this section, we give some preliminaries about S-acts, S-posets and slice category which we will need in the sequel. The reader is referred to [13] and [3], respectively, for information on general properties of S-acts and S-posets that are not fully explained here.

Let S be a monoid with identity 1. Recall that a (right) *S-act* is a set A equipped with a map $\mu : A \times S \to A$ called its action, such that, denoting $\mu(a, s)$ by as, we have $a1 = a$ and $a(st) = (as)t$, for all $a \in A$, and $s, t \in S$. The category of all S-acts, with action-preserving (S-act) maps ($f : A \to B$ with $f(as) = f(a)s$, for $s \in S, a \in A$), is denoted by **Act**-S. For instance, take any monoid S and a non-empty set A. Then A becomes a right S-act by defining $as = a$ for all $a \in A$, $s \in S$; we call that A an S-act with trivial action. Clearly S itself is an S-act with its operation as the action.

On a monoid S we define the following relations: for every $s, t \in S$

1. $s\mathcal{R}t$ iff $sS = tS$.
2. $s\mathcal{J}t$ iff $SsS = StS$.
3. $s\mathcal{D}t$ iff there exists $u \in S$ with $sS = uS$ and $St = Su$.

These relations are called Green's relations on S (see [13]). Here, we consider these notions for a pomonoid S and supply some suitable results. A monoid S is said to be a *partially ordered monoid* (briefly a *pomonoid*) if it is also a poset whose partial order $\leq$ is compatible with the binary operation, i.e., $s \leq t, s' \leq t'$ imply $ss' \leq tt'$ (see [2]). In this paper S denotes a pomonoid with an arbitrary order, unless otherwise stated.

Let S be a pomonoid and A be a poset. Then $A \times S$ becomes a poset with componentwise order. A poset A is said to be a *(right) S-poset* over a pomonoid S if it is an S-act and the action is monotone ($(a_1, s_1) \leq (a_2, s_2)$ implies, $a_1 s_1 \leq a_2 s_2$, where $a_1, a_2 \in A$ and $s_1, s_2 \in S$). We denote it by A_S. The category of all S-posets with action preserving monotone maps is denoted by **Pos**-S. Clearly S itself is an S-poset with its operation as the action. A left S-poset A can be defined analogously (see [3]) and denoted by ${}_SA$. Also, we denote the category of all left S-posets with action preserving monotone maps by S-**Pos**. As in the unordered case, the coproduct in **Pos**-S is simply the disjoint union, with S-action and order given componentwise, and as usual the coproduct of a family $\{A_i \mid i \in I\}$ will be denoted by $\coprod_{i \in I} A_i$. Let T and S be pomonoids. Then a poset A is called a *T-S-biposet* if it is a left T-poset and a right S-poset and $(ta)s = t(sa)$ for every $s \in S, t \in T$ and $a \in A$. We denote it by ${}_TA_S$.

We recall the following results from [14]:

(i) For every A_S in **Pos**-S, consider the set $\mathrm{End}(A_S) = \mathbf{Pos}_S(A, A)$ as a pomonoid with respect to composition and pointwise order. We define the left $\mathrm{End}(A_S)$-action on A by $f \cdot a = f(a)$, for every $f \in \mathrm{End}(A_S)$, $a \in A$. Note that this action is monotone because if $f, g \in \mathrm{End}(A_S)$ and $a, b \in A$ are such that $f \leq g$ and $a \leq b$ then we have $f \cdot a = f(a) \leq f(b) \leq g(b) = g \cdot b$. Thus one has ${}_{\mathrm{End}(A_S)}A_S$.

(ii) The following two mappings are pomonoid homomorphisms:

$$\rho : S \to \mathrm{End}(A_S); \quad s \mapsto \rho_s,$$

$$\lambda : T \to \mathrm{End}({}_TA); \quad t \mapsto \lambda_t. \tag{1.1}$$

Here, $\rho_s : A_S \to A_S,\ a \mapsto as$ and $\lambda_t : {}_TA \to {}_TA,\ a \mapsto ta$ are morphisms in **Pos**-S and T-**Pos**, respectively.

(iii) For every T-S-biposet ${}_TA_S$ recall that if $B \in$ **Pos**-S then the set $\mathbf{Pos}_S(B, A)$ of all S-poset maps from B_S to A_S is an object in T-**Pos** with the action defined by $t \cdot f = \lambda_t f$ for every $t \in T, f \in \mathbf{Pos}_S(B, A)$. Consequently, we have a functor

$$\mathbf{Pos}_S(-, A) : \mathbf{Pos}\text{-}S \to T\text{-}\mathbf{Pos}$$

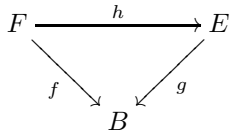
by taking

$$\mathbf{Pos}_S(-, A)(B) = \mathbf{Pos}_S(B, A)$$

for every $B \in$ **Pos**-S.

An S-poset G_S is a generator in the category **Pos**-S if for any distinct S-poset maps $\alpha, \beta : X_S \to Y_S$ there exists an S-poset map $f : G_S \to X_S$ such that $\alpha f \neq \beta f$.

For any category $\mathcal{C}$ and an object B of $\mathcal{C}$, there is a *slice category* (also called *comma category*) $\mathcal{C}/B$. The objects of $\mathcal{C}/B$ are morphisms of $\mathcal{C}$ with codomain B, and morphisms in $\mathcal{C}/B$ from one such object $f : F \to B$ to another $g : E \to B$ are commutative triangles in $\mathcal{C}$:



i.e, $gh = f$. We write $h : f \to g$. The composition in $\mathcal{C}/B$ is defined from the composition in $\mathcal{C}$, in the obvious way– the triangles are pasted together (for more details see [15]).

A poset is said to be *complete* if each of its subsets has an infimum and a supremum, in particular, a complete poset is bounded, that is, it has a least (bottom) element $\perp$ and a greatest (top) element $\top$.

2. Some Homological Classifications for Pomonoids by Generators in **Pos-**S

In this section, we discuss the properties of generators and projective generators in **Pos**-S. Recall that a projective S-poset A_S which is also a generator is called a projective generator S-poset. A cyclic S-poset is an S-poset A for which there exists an element $a \in A$ such that $A = aS$. By a cyclic projective S-poset we mean a cyclic S-poset which is also projective.

As we mentioned in the introduction, generators for the category **Pos**-S were characterized in [14] with the following two propositions.

Proposition 2.1. *Cyclic projectives in* **Pos**-S *are precisely retracts of* S_S.

Proposition 2.2. *An* S*-poset* A_S *is a cyclic projective generator in* **Pos**-S *if and only if* $A_S \cong eS_S$ *for an idempotent* $e \in S$ *with* $e\mathcal{J}1$.

The following is immediate from Proposition 2.2:

Proposition 2.3. *Let* S *be a commutative pomonoid. Then all cyclic projective generators in* **Pos**-S *are isomorphic to* S_S.

We will also need the following characterization of cyclic projective S-posets from [19, Proposition 4.2].

Proposition 2.4. *Let* A_S *be an* S*-poset and* $a \in A$. *Then the following statements are equivalent:*
(i) aS_S *is projective.*
(ii) $aS_S \cong eS_S$ *for some idempotent* $e \in S$.

We state the following two facts about projectives and generators from [19] and [14] respectively. They will be used throughout the paper.

Theorem 2.5. *An* S*-poset* P_S *is projective if and only if* $P_S \cong \coprod_{i\in I} e_iS$ *where* $e_i^2 = e_i \in S, i \in I$.

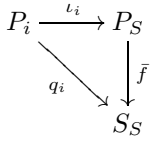
Theorem 2.6. *The following assertions are equivalent for a right* S*-poset* A_S.

1. *For all* $X_S, Y_S \in$ **Pos**-S *and* $f, g \in \mathbf{Pos}_S(X, Y), f \leq g$ *whenever* $fk \leq gk$ *for all* $k \in \mathbf{Pos}_S(A; X)$.
2. A_S *is a generator.*
3. *For every* $X_S \in$ **Pos**-S *there exists a set* I *and an epimorphism* $h : \coprod_I A \to X$ *in* **Pos**-S.
4. *There exists an epimorphism* $\pi : A \to S$ *in* **Pos**-S.
5. S_S *is a retract of* A_S.

Now we can prove the following result.

Theorem 2.7. *Every S-poset P_S is projective generator if and only if $P_S = \coprod_{i\in I} P_i$ where $P_i \cong e_iS$ for every $i \in I$, and at least one P_j, $j \in I$ is a generator with $e_j\mathcal{J}1$.*

Proof. On the one hand, let the S-poset P_S be a projective generator. By Theorem 2.5 we have $P_S \cong \coprod_{i\in I} e_iS$ where $e_i^2 = e_i \in S, i \in I$. And by Theorem 2.6 there exists a surjective S-poset epimorphism $\pi : P_S \longrightarrow S_S$, so $1 = \pi(a)$ for some $a \in e_jS, j \in I$. Now $\pi|_{e_jS} : e_jS \longrightarrow S_S$ is also an epimorphism in **Pos**-S, because for any $s \in S$ we have $s = 1s = \pi(a)s = \pi(as)$ and $as \in e_jS$. Hence, e_jS is a generator and by Proposition 2.2, $e_j\mathcal{J}1$.
On the other hand, assume that P_S has the factorisation in the statement of the theorem. By Theorem 2.5, P_S is projective. That P_j is generator, implies that there exists an S-poset epimorphism $\pi_j : P_j \longrightarrow S_S$. Now, for the following diagram



take $q_j = \pi_j$ and q_i the composite S-poset map $P_i \cong e_iS \hookrightarrow S$ for every $i \in I$, $i \neq j$. By the property of the coproduct S-poset $P_S = \coprod_{i\in I} P_i$, corresponding to the S-poset epimorphisms $\{q_i \mid i \in I\}$, there exists a unique S-poset map $\pi : P \longrightarrow S_S$ such that $\pi|_{P_i} = q_i$ for all $i \in I$. In particular, $\pi|_{P_j} = \pi_j$ and π_j is an S-poset epimorphism, so π is also an S-poset epimorphism. Hence, P_S is generator. □

Notice that for every pomonoid S and idempotent $e \in S$, the sub S-poset eS_S of S_S is projective according to Proposition 2.4, but it is not a generator because $e\mathcal{J}1$ does not necessarily hold. For example, if we take a periodic monoid S endowed it with discrete order then we have a pomonoid. Now if we take an idempotent $1 \neq e \in S$, then $e\mathcal{J}1$ does not hold (see [13, Proposition I.3.26 on page 32] for more details).

Next, we have the following result.

Theorem 2.8. *For any pomonoid S the following statements are equivalent:*
(i) *All projective right S-posets are generators in* **Pos**-S.
(ii) *All cyclic projective right S-posets are generators in* **Pos**-S.
(iii) $e\mathcal{J}1$ *for every idempotent $e \in S$.*

Proof. That (i) implies (ii) is clear. To see that (ii) implies (iii) observe that for any idempotent $e \in S$, the right S-poset eS_S is cyclic, hence it is a genreator by assumption. The result thus follows by Proposition 2.2. .

For the implication (iii) ⇒ (i), let P_S be an S-poset. By Theorem 2.5 we have $P_S \cong \coprod_{i\in I} e_iS$ where $e_i^2 = e_i \in S, i \in I$. By the assumption we have $e_i\mathcal{J}1$ for every $i \in I$ and so P_S is a generator by Theorem 2.7. □

Recall [4] that a right *poideal* of a pomonoid S is a (possibly empty) subset I of S if it is both a monoid right ideal ($IS \subseteq I$) and a down set ($a \leq b, b \in I$ imply that $a \in I$). It is *principal* if it is generated (as a monoid right ideal of S) by a single element. For example

$$\downarrow rS = \{t \in S \ : \ \exists s \in S, \ t \leq rs\}$$

is a principal poideal of S, for every $r \in S$.

In the following we shall characterize pomonoids for which all principal right poideals are generators.

Proposition 2.9. *Let S be a pomonoid and $e \in S$ satisfy $e^2 = e$. If the cyclic projective sub S-poset eS_S of S_S is a generator in* **Pos**-S*, then $\downarrow eS$ is also a generator.*

Proof. By assumption there exists an S-poset epimorphism $f\colon eS_S \to S_S$. Define the mapping $g\colon \downarrow eS \to S_S$ by $g(x) := f(ex)$ for every $x \in \downarrow eS$. It is easy to see that g is an S-poset map. Also, for every $s \in S$ there exists $u \in S$ such that $f(eu) = s$. Then we have

$$g(eu) = f(eeu) = f(eu) = s.$$

This means that g is an epimorphism. By Theorem 2.6 we conclude that $\downarrow eS$ is a generator, as required. □

Lemma 2.10. *Let S be a pomonoid and $z \in S$. If the principal right poideal $\downarrow zS$ is a generator in* **Pos**-S*, then there exist $x, y \in S$ such that $1 \leq yx$, and $za \leq zb$, $a, b \in S$ implies $ya \leq yb$.*

Proof. Since $\downarrow zS$ is a generator in **Pos**-S, by Theorem 2.6, there exists an epimorphism $g : \downarrow zS \to S_S$. Hence, there are elements $u \in \downarrow zS$ and $t \in S$ such that $u \leq zt$ and $g(u) = 1$. Let $y = g(z)$ and $x = t$. Then $yx = g(z)x = g(zx)$. Since $u \leq zx$, the monotonicity of g implies that $g(u) \leq g(zx)$. Consequently, $1 = g(u) \leq g(zx) = yx$. Now, suppose $za \leq zb$, $a, b \in S$. Then $ya = g(z)a = g(za) \leq g(zb) = g(z)b = yb$. □

Next we answer the question about the conditions under which the assumptions of Proposition 2.9 are satisfied.

Proposition 2.11. *Let S be a pomonoid in which the identity element is the top element. If all poideals of S are generators in* **Pos**-S*, then the sub S-poset eS_S of S_S is a generator in* **Pos**-S*, for every idempotent $e \in S$.*

Proof. Assume that all poideals of S are generators in **Pos**-S. Then for every idempotent $e \in S$, $\downarrow eS$ is a generator in **Pos**-S. By Lemma 2.10, there exist $x, y \in S$ such that $1 \leq yx$, and $ea \leq eb$, $a, b \in S$, always implies $ya \leq yb$. In particular, since $e1 \leq ee$ we have $y \leq ye$, so $1 \leq yx \leq yex$. As we have $yex \leq 1$ by the hypothesis, we get $yex = 1$, which means that $e\mathcal{J}1$. So eS_S is a projective generator by Proposition 2.2, as needed. □

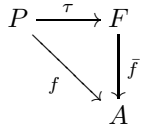
Theorem 2.12. *Let S be a pomonoid in which the identity element is the top element. The following statements are equivalent:*
(i) *All projective right S-posets are generators in* **Pos**-S.
(ii) *All cyclic projective right S-posets are generators in* **Pos**-S.
(iii) $e\mathcal{J}1$ *for every idempotent $e \in S$.*
(iv) *All principal right poideals of S which are generated by an idempotent, are generators in* **Pos**-S.

Proof. The equivalence of the first three statements is Theorem 2.8.

That (iii) implies (iv) is easy. Indeed, by Proposition 2.2 we get that eS_S is a cyclic projective generator, and Proposition 2.9 shows that $\downarrow eS$ is a generator in **Pos**-S.

To finish off, we show that (iv) implies (iii). Consider the principal right poideal $\downarrow eS$ for every idempotent $e \in S$ which is a generator in **Pos**-S. By a proof similar to that of Proposition 2.11, the cyclic projective sub S-poset eS_S of S_S is a generator. Using Proposition 2.2, we conclude that $e\mathcal{J}1$. □

By a *free S-poset on a poset P* we mean an S-poset F together with a poset map $\tau : P \to F$ with the universal property that given any S-poset A and any poset map $f : P \to A$ there exists a unique S-poset map $\bar{f} : F \to A$ such that $\bar{f} \circ \tau = f$, i.e, the diagram



commutes. The S-poset F (up to isomorphism) is given by $F = P \times S$ with componentwise order and the action $(x, s)t = (x, st)$, for $x \in P$ and $s, t \in S$ (see [3] for example). Furthermore, by a *free S-poset* we mean an S-poset which is free on some poset.

Example 2.13. Let S be a pomonoid generated by the elements e, k, k' and with discrete order such that $kk' = 1, e^2 = e$ and $ek = k'$. Then eS_S is a cyclic projective generator in **Pos**-S. But eS_S is not free (see Lemma 2.14 below). □

Now, we present some condition under which the sub S-posets eS_S of S_S are free for idempotent elements $e \in S$. The proof of the following result is similar to the proof for the unordered case in [13, Proposition 3.17.17], so we omit it. Moreover, we conclude when projectivity (or cyclic projectivity) implies freeness in **Pos**-S.

Lemma 2.14. *Let e be an idempotent of a pomonoid S. Then the sub S-poset eS_S of S_S is a free right S-poset if and only if $e\mathcal{D}1$.* □

This allows us to prove the following.

Theorem 2.15. *For any pomonoid S the following statements are equivalent:*
(i) *All projective right S-posets are free.*

(ii) *All projective generators in* **Pos**-S *are free.*
(iii) *All cyclic projective right* S*-posets are free.*
(iv) $e\mathcal{D}1$ *for every idempotent* $e \in S$.

Proof. The implication (i) $\Rightarrow$ (ii) is trivial.

To see (ii) $\Rightarrow$ (iii), observe that by Proposition 2.4, all cyclic projective S-posets are isomorphic to eS_S for some idempotent $e \in S$. Let $A = S_S \coprod eS_S$. By Proposition 2.7, A_S is a projective generator in **Pos**-S. By hypothesis A_S is free which implies that eS_S is free.

Now we show that (iii) $\Rightarrow$ (i). By decomposition theorem in [19], every projective S-poset is isomorphic to a coproduct of cyclic projective S-posets which are free by assumption. Now since the coproducts of free S-posets being free we get the result.

By the characterization of cyclic projective S-posets in Proposition 2.4 and Lemma 2.14 we get the equivalence of (iii) and (iv), which completes the proof. □

3. Regular Injectivity in Pos-S and Pos-S/B_S and Generators

Let $\mathcal{C}$ be a category and $\mathcal{M}$ a class of its morphisms. An object I of $\mathcal{C}$ is called $\mathcal{M}$-injective if for each $\mathcal{M}$-morphism $h : U \to V$ and morphism $u : U \to I$ there exists a morphism $s : V \to I$ such that $sh = u$. That is, the following diagram is commutative:

$$\begin{array}{ccc} U & \xrightarrow{u} & I \\ {\scriptstyle h}\downarrow & \nearrow_{s} & \\ V & & \end{array}$$

In particular, this means that, in the slice category $\mathcal{C}/B$, an object $f : X \to B$ is $\mathcal{M}$-injective if, for any commutative diagram in $\mathcal{C}$

$$\begin{array}{ccc} U & \xrightarrow{u} & X \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f} \\ V & \xrightarrow[v]{} & B \end{array}$$

with $h \in \mathcal{M}$, there exists an arrow $s : V \to X$ such that $sh = u$ and $fs = v$.

$$\begin{array}{ccc} U & \xrightarrow{u} & X \\ {\scriptstyle h}\downarrow & \nearrow_{s} & \downarrow{\scriptstyle f} \\ V & \xrightarrow[v]{} & B \end{array}$$

Recall that regular monomorphisms (morphisms which are equalizers) in **Pos**-S (and also in **Pos**-S/B_S) are exactly order-embeddings (see [3] and [6]). By Emb-injectivity in **Pos**-S we mean $\mathcal{M}$-injectivity in **Pos**-S, where $\mathcal{M} = \mathrm{Emb}$ is the class

of all order-embeddings of S-posets. In the following we shall deal with Emb-injectivity in **Pos**-S and **Pos**-S/B_S, where Emb is the class of all order-embeddings of S-posets.

Theorem 3.1. *All generators in* **Pos**-S *are Emb-injective if and only if all S-posets are Emb-injective.*

Proof. Clearly it is enough to show the forward implication. Let A_S be an S-poset. Consider the product S-poset $A_S \times S_S$ which is a generator in **Pos**-S by Theorem 2.6 and so is Emb-injective. By a general category-theoretic result which states that a product of a family of injective objects in a category is injective if and only if each component of the product is injective, we get that A_S is Emb-injective in **Pos**-S. □

Note that the class of all embeddings of right poideals into S_S is a subclass of all down-closed embeddings in **Pos**-S, i.e. all embeddings $g : B_S \to C_S$ with the property that $g(B)$ is down-closed in C, and hence is a subclass of all embeddings.

Definition 3.2. An S-poset A_S is called (*principally*) *weakly regularly d-injective* if it is injective with respect to all embeddings of (principal) right poideals into S_S.

Proposition 3.3. *If all generators in* **Pos**-S *are weakly regularly d-injective then all S-posets are weakly regularly d-injective.*

Proof. Let A_S be an S-poset. Since $A_S \times S_S$ is a generator in **Pos**-S it is a weakly regularly d-injective. To show that A_S is weakly regularly d-injective consider the following diagram

$$\begin{array}{ccc} I_S & \xrightarrow{u} & A_S \\ {\scriptstyle i}\downarrow & & \\ S_S & & \end{array}$$

where I is a poideal of S. Define S-poset map $\bar{u} : I_S \to A_S \times S_S$ by $\bar{u}(s) = (u(s), s)$ for each $s \in I_S$. By the assumption, there exists an S-poset map $v : S_S \to A_S \times S_S$ such that $vi = \bar{u}$.

$$\begin{array}{ccc} I_S & \xrightarrow{\bar{u}} & A_S \times S_S \\ {\scriptstyle i}\downarrow & \nearrow_{v} & \\ S_S & & \end{array}$$

Now by composition v with the projection $\pi_A : A_S \times S_S \to A_S$, we get A_S is a weakly regularly d-injective. □

For a pomonoid S recall that an element $s \in S$ is called *regular* if there exists $t \in S$ such that $sts = s$. One calls S a *regular pomonoid* if all its elements are regular.

Theorem 3.4. *Let S be a pomonoid whose identity element is the top element. Then the following statements are equivalent:*
(i) *All S-posets are principally weakly regularly d-injective.*
(ii) *All principal right poideals of S are principally weakly regularly d-injective.*
(iii) *All generators in* **Pos**-S *are principally weakly regularly d-injective.*
(iv) *S is a regular pomonoid.*

Proof. The equivalence of (i) and (iii) comes from (the proof of) Proposition 3.3. The implication (iv) $\Rightarrow$ (i) is in [18, Theorem 3.6] and the implication (i) $\Rightarrow$ (ii) is trivial, so it is enough for us to show the implication (ii) $\Rightarrow$ (iv).

So assume (ii). For every $s \in S$, consider the down-closed embedding $i : \downarrow sS \to S_S, x \mapsto x$. It has a left inverse f, as $\downarrow sS$ is principally weakly regularly d-injective. Taking $f(1) = z$, we have $z \leq st$ for some $t \in S$ and

$$s = f(s) = f(1)s = zs \leq sts.$$

On the other hand, $sts \leq s$, as 1 is the top element of S. Therefore $sts = s$, showing that s is a regular element. As this was for any s, S is a regular pomonoid. □

Recall from [4] that a pomonoid S which has no proper non-empty left (right) poideal is said to be left (right) *simple*.

Corollary 3.5. *If all generators in* **Pos**-S *are Emb-injective then S is left simple.*

Proof. From the hypothesis and Theorem 3.1, we conclude that all complete S-posets are Emb-injective. It follows then from [4, Theorem 3.9] that S is left simple. □

Proposition 3.6. *For any pomonoid S the following statements are equivalent:*
(i) *All generators in* **Pos**-S *are complete S-posets.*
(ii) *All S-posets are complete.*

Proof. First assume (i). Let A_S be an S-poset. Consider the generator $A_S \times S_S$, which is a complete S-poset by assumption. Since the order on the product $A_S \times S_S$ is the componentwise order, joins are computed componentwise in the product as well. That is, for a subset $T \subseteq A_S \times B_S$ we have $\bigvee T = (\bigvee \pi_A(T), \bigvee \pi_B(T))$ where π_A and π_B are canonical projections on A_S and B_S, respectively. Therefore, for any subset $B \subseteq A$, $\bigvee B$ exists and so A_S is complete, giving (ii).

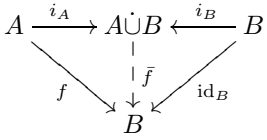
The converse implication is trivial. □

We state the following result from [7, Proposition 3.17] that will be used later on. We give a direct proof of it here, for the convenience of the reader .

Proposition 3.7. *Let S be a pomonoid and $B_S \in$* **Pos**-S. *Suppose $f : A_S \to B_S$ is an Emb-injective object in* **Pos**-S/B_S. *Then f is a split epimorphism in* **Pos**-S.

Proof. By the universal property of the coproduct S-poset $A \dot{\cup} B$ (the disjoint union of A and B) there exists a unique S-poset map $\bar{f} : A \dot{\cup} B \to B$ such that the following

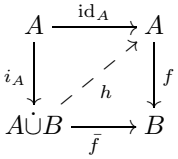
diagram commutes where i_A and i_B are injection S-poset maps.



In fact,

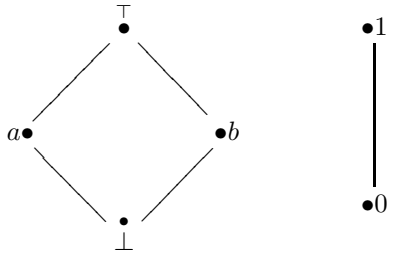
$$\bar f(x) = \begin{cases} f(x) & \text{if } x \in A \\ x & \text{if } x \in B. \end{cases}$$

Now, let us consider the following commutative square



Since f is an Emb-injective object in **Pos**-S/B_S, there exists a unique S-poset map $h : A\dot\cup B \to A$ such that $fh = \bar f$ and $hi_A = \mathrm{id}_A$. So $fhi_B = \bar f i_B = \mathrm{id}_B$, which shows that f is a split epimorphism in **Pos**-S. □

Remark 3.8. There exist split epimorphisms in **Pos**-S which are not Emb-injective in **Pos**-S/B_S. To present an example, take an arbitrary pomonoid S and let X and B be, respectively, the first and second lattices shown in the following diagram:



Evidently, X is an S-poset with the action defined by $\top s = \top$ and $as = bs = \bot s = a$ for all $s \in S$, also we consider B with the trivial action as an S-poset. Define the S-poset map $f : X_S \to B_S$, by $f(a) = f(b) = f(\bot) = 0$ and $f(\top) = 1$. Then f is a convex map. We show that it is not a regular injective object in **Pos**-S/B_S. Since $f^{-1}(0) = \{\bot, a, b\}$ is not a complete lattice, the authors in [6] showed that it is not Emb-injective in **Pos**-S/B_S.

On the other hands, define the S-poset map $g : B_S \to X_S$ by $g(0) = \bot, g(1) = \top$. Then we have $fg = \mathrm{id}_B$, so f is a split epimorphism. Therefore, the converse of the above proposition is not true generally. □

Next recall that for a given poset P and a pomonoid S, the cofree S-poset on P is the set $P^{(S)}$ of all monotone maps from S to P, with pointwise order and action given by $(fs)(t) = f(st)$ for $s, t \in S$ and $f \in P^{(S)}$ (see also [3, Theorem 13]).

Corollary 3.9. *Suppose $f : A_S \to B_S$ is an Emb-injective object in* **Pos**-S/B_S. *If A is a complete lattice which is also a retract of the cofree S-poset $A^{(S)}$, then A_S and B_S are Emb-injective object in* **Pos**-S.

Proof. By hypothesis we conclude that $A^{(S)}$ is an Emb-injective S-poset (see [4, Theorem 3.3]). Also it is straightforward to see that a retract of a Emb-injective S-poset is Emb-injective and so we get A_S is an Emb-injective S-poset. Also, by Proposition 3.7 the S-poset map f is a split epimorphism. Consequently B_S being a retract of an Emb-injective S-poset is an Emb-injective S-poset. □

At the rest of this section, we investigate some connections between Emb-injectivity in **Pos**-S/B_S and generators and cyclic projectives in **Pos**-S.

Theorem 3.10. *If $f : A_S \to B_S$ is an Emb-injective object in* **Pos**-S/B_S *and B_S is a generator in* **Pos**-S *then A_S is a generator. Further, ${}_{\mathrm{End}(A_S)}A$ is a cyclic projective in* $\mathrm{End}(A_S)$-**Pos**.

Proof. Since $f : A_S \to B_S$ is Emb-injective object in **Pos**-S/B_S, by Proposition 3.7, there exists $g : B_S \to A_S$ in **Pos**-S such that $fg = \mathrm{id}_B$. As B_S is a generator in **Pos**-S and f is an epimorphism, A_S is also a generator (see [14]). Now, applying this fact and [14, Theorem 2.2], we get that ${}_{\mathrm{End}(A_S)}A$ is a cyclic projective. □

Theorem 3.11. *Suppose $f : A_S \to B_S$ is an Emb-injective object in* **Pos**-S/B_S *where A_S is a cyclic projective S-poset. Then B_S is a cyclic projective S-poset. Moreover, ${}_{\mathrm{End}(B_S)}B$ is a generator in* $\mathrm{End}(B_S)$-**Pos**.

Proof. Since $f : A_S \to B_S$ is Emb-injective object in **Pos**-S/B_S, by Proposition 3.7, there exists $g : B_S \to A_S$ in **Pos**-S such that $fg = \mathrm{id}_B$. Also, A_S is a cyclic projective in **Pos**-S hence by Proposition 2.1, there exist two S-poset maps $S_S \underset{\gamma}{\overset{\pi}{\rightleftarrows}} A_S$ such that $\pi\gamma = \mathrm{id}_A$. This yields $f\pi\gamma g = \mathrm{id}_B$ which shows that B_S is a retract of S_S. We get B_S is a cyclic projective S-poset by Proposition 2.1, so by [14, Proposition 3.1], we conclude that ${}_{\mathrm{End}(B_S)}B$ is a generator in $\mathrm{End}(B_S)$-**Pos**. □

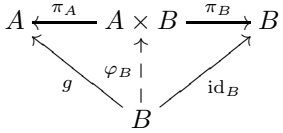
Theorem 3.12. *Suppose $f : A_S \to B_S$ is an Emb-injective object in* **Pos**-S/B_S. *Then all of the following hold.*
(i) $\mathbf{Pos}_S(B_S, A_S)$ *is a generator in* **Pos**-$\mathrm{End}(B_S)$.
(ii) $\mathbf{Pos}_S(A_S, B_S)$ *is a generator in* $\mathrm{End}(B_S)$-**Pos**.
(iii) $\mathbf{Pos}_S(B_S, A_S)$ *is a cyclic projective in* $\mathrm{End}(A_S)$-**Pos**.
(iv) $\mathbf{Pos}_S(A_S, B_S)$ *is a cyclic projective in* **Pos**-$\mathrm{End}(A_S)$.

Proof. Since $f : A_S \to B_S$ is Emb-injective object in **Pos**-S/B_S, in view of Proposition 3.7, there exists $g : B_S \to A_S$ such that $fg = \mathrm{id}_B$. Applying the functors $\mathbf{Pos}_S(B_S, -)$ and $\mathbf{Pos}_S(-, B_S)$ to the identity map id_{B_S} we can easily get the assertions (i) and (ii), respectively. Again by applying the functors $\mathbf{Pos}_S(-, A_S)$ and

$\mathbf{Pos}_S(A_S, -)$ to the above identity, in light of Proposition 2.1, we can deduce that the statements (iii) and (iv) are true. □

Proposition 3.13. *Let A_S be an S-poset. Then in any of the following cases $\mathbf{Pos}_S(A_S \times B_S, B_S)$ is a generator in* $\mathrm{End}(B_S)$-**Pos**, *for every $B_S \in$***Pos**-S*:*
(i) *A_S is an Emb-injective S-poset.*
(ii) *$f : A_S \to B_S$ is an Emb-injective object in* **Pos**-S/B_S.

Proof. (i) Consider the second projection S-poset map $\pi_B : A_S \times B_S \to B_S$. The authors in [6] have showed that it is an Emb-injective object in **Pos**-S/B_S. Consequently, by Theorem 3.12(ii), we get the result.
(ii) By Proposition 3.7, there exists an S-poset map $g : B_S \to A_S$ such that $fg = \mathrm{id}_B$. By the universal property of the product S-poset $A \times B$ there exists a unique S-poset map $\varphi_B : B_S \to A \times B$ (indeed $b \mapsto (g(b), b)$) such that the following diagram commutes:



i.e., $\pi_B\varphi_B = \mathrm{id}_B$ and $\pi_A\varphi_B = g$. Applying the functor $\mathbf{Pos}_S(-, B_S)$ to the first identity above we obtain

$$\mathrm{End}(B_S) = \mathbf{Pos}_S(B, B) \underset{\bar{\varphi}_B}{\overset{\bar{\pi}_B}{\rightleftarrows}} \mathbf{Pos}_S(A \times B, B)$$

such that $\bar{\varphi}_B\bar{\pi}_B = \mathrm{id}_{\mathrm{End}(B_S)}$. This means that $\mathrm{End}(B_S)$ is a retract of $\mathbf{Pos}_S(A \times B, B)$ as we needed (see Theorem 2.6 again). □

Proposition 3.13. *Suppose that B_S is in* **Pos**-S, ${}_TA_S$ *is a T-S-biposet, and $A \times B$ is a cyclic projective S-poset. If $f : A_S \to B_S$ is an Emb-injective object in* **Pos**-S/B_S *and $\lambda : T \to \mathrm{End}(A_S)$, defined as in (1.1), is an isomorphism then ${}_TA$ is a generator in T-***Pos**.

Proof. Consider the second projection S-poset map $\pi_A : A \times B \to A_S$ and the unique S-poset map $\varphi_A : A_S \to A \times B$ for which $\pi_A\varphi_A = \mathrm{id}_A$. That is, let $\varphi_A(a) = (a, f(a))$. Since $A \times B$ is a cyclic projective S-poset by assumption, there exist S-poset maps $A \times B \underset{\pi}{\overset{\gamma}{\rightleftarrows}} S_S$ such that $\pi\gamma = \mathrm{id}_{A\times B}$. Applying the functor $\mathbf{Pos}_S(-, A_S)$ to the former identity and knowing that the composition $\pi_A\pi\gamma\varphi_A = \mathrm{id}_A$, we obtain

$$T \cong \mathbf{Pos}_S(A, A) \underset{\bar{\varphi}_A}{\overset{\bar{\pi}_A}{\rightleftarrows}} \mathbf{Pos}_S(A \times B, A) \underset{\bar{\gamma}}{\overset{\bar{\pi}}{\rightleftarrows}} \mathbf{Pos}_S(S, A) \cong_T A$$

in which $\bar{\varphi}_A\bar{\pi}_A = \mathrm{id}_{\mathbf{Pos}_S(A,A)}$ and $\bar{\gamma}\bar{\pi} = \mathrm{id}_{\mathbf{Pos}_S(S,A)}$. Thus, T is a retract of ${}_TA$ and hence ${}_TA$ is a generator in **Pos**-S. □

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