

Generalized Quasi-Einstein Metrics and Contact Geometry

GOUR GOPAL BISWAS

Department of Mathematics, University of Kalyani, Kalyani-741235, West Bengal, India

e-mail: ggbiswas6@gmail.com

UDAY CHAND DE*

Department of Pure Mathematics, University of Calcutta, 35 Ballygaunge Circular Road, Kolkata -700019, West Bengal, India

e-mail: uc_de@yahoo.com

AHMET YILDIZ

Education Faculty, Department of Mathematics, Inonu University, 44280, Malatya, Turkey

e-mail: a.yildiz@inonu.edu.tr

ABSTRACT. The aim of this paper is to characterize K -contact and Sasakian manifolds whose metrics are generalized quasi-Einstein metric. It is proven that if the metric of a K -contact manifold is generalized quasi-Einstein metric, then the manifold is of constant scalar curvature and in the case of a Sasakian manifold the metric becomes Einstein under certain restriction on the potential function. Several corollaries have been provided. Finally, we consider Sasakian 3-manifold whose metric is generalized quasi-Einstein metric.

1. Introduction

If the Ricci tensor S of a Riemannian manifold (M^n, g) , $n > 2$, satisfies the condition $Ric = \lambda g$, λ being a constant, then the manifold is named an Einstein manifold. According to Besse [4] this condition is called Einstein metric condition. The study of Einstein manifolds and their generalizations are very interesting in Riemannian and semi-Riemannian geometry. There are several generalizations of Einstein metric such as quasi-Einstein metric [8], m -quasi-Einstein metric [9],

* Corresponding Author.

Received February 15, 2021; revised January 30, 2022; accepted February 10, 2022.

2010 Mathematics Subject Classification: 53C25, 53C15.

Key words and phrases: GQE metrics, Almost contact manifolds, Contact manifolds, K -contact manifolds, Sasakian manifolds.

(m, ρ) -quasi-Einstein metric [18], generalized quasi-Einstein metric [10] and many others.

The idea of generalized quasi-Einstein metric in a Riemannian manifold of dimension n is introduced by Catino [10]. A metric g of an $n(> 2)$ dimensional Riemannian manifold M^n is called a *generalized quasi-Einstein metric* (shortly, GQE metric) if there exist three smooth functions ψ, α, β such that

$$(1.1) \quad S + H^\psi - \alpha d\psi \otimes d\psi = \beta g,$$

where H^ψ is the Hessian of the function ψ defined by

$$H^\psi(E, F) = g(\nabla_E \text{grad } \psi, F)$$

for all vector fields E, F in M^n . Here ∇ is the Riemannian connection and grad denotes the gradient operator. Obviously, when ψ is a constant, the metric becomes an Einstein metric.

For individual values of α and β , we get different type of metrics. They are

- i) *Gradient Ricci soliton* [7] for $\alpha = 0$ and $\beta \in \mathbb{R}$,
- ii) *Gradient almost Ricci soliton* [1] for $\alpha = 0$ and $\beta \in C^\infty(M^n)$,
- iii) *Gradient ρ -Einstein soliton* [11] for $\alpha = 0$, $\beta = \rho r + \lambda$ and $\lambda \in \mathbb{R}$, r being the scalar curvature,
- iv) *m -quasi-Einstein metric* for $\alpha = \frac{1}{m}$, $m \in \mathbb{N}$ and $\beta \in \mathbb{R}$,
- v) *gradient generalized m -quasi metric* [2] for $\alpha = \frac{1}{m}$, $m \in \mathbb{N}$ and $\beta \in C^\infty(M^n)$,
- vi) *(m, ρ) -quasi-Einstein metric* for $\alpha = \frac{1}{m}$, $m > 0$, $\beta = \rho r + \lambda$ and $\lambda \in \mathbb{R}$.

The idea of a gradient ρ -Einstein soliton is introduced by Catino and Mazzieri [11] and studied in the papers ([12], [19]). In the paper [27], Venkatesha and Kumara studied gradient ρ -Einstein solitons on almost Kenmotsu manifolds. In [13], Chen studied m -quasi-Einstein structure in almost cosymplectic manifolds.

In the paper [10], Catino gave a local characterization of GQE metric with harmonic Weyl tensor and $C(\text{grad } \psi, \cdot, \cdot) = 0$, where C is the Weyl tensor. He proved that if the metric of a manifold (M^n, g) , $n \geq 3$ is a GQE metric with harmonic Weyl tensor and $C(\text{grad } \psi, \cdot, \cdot) = 0$, then M is locally warped product with $(m - 1)$ -dimensional Einstein fibers around any regular point of ψ . Recently, GQE manifolds have been studied by Mirshafezadeh and Bidabad ([22], [23]). So far our knowledge goes, contact or paracontact manifolds whose metrics are GQE metric have not been investigated. In the present paper we attempt to characterize K -contact and Sasakian manifolds whose metrics are GQE metric.

At first we obtain the expression of *Riemannian curvature tensor* and *Ricci tensor*

in a Riemannian manifold whose metric is GQE metric. Then we provide our main theorems. In the proof of the theorems we assume that the potential function ψ remains invariant under the characteristic vector field ξ , that is, $\mathcal{L}_\xi\psi = 0$, which implies that $\xi\psi = 0$, \mathcal{L}_ξ being the Lie-derivative in the direction of ξ . Precisely we prove the following theorems:

Theorem 1.1. *The scalar curvature of a K-contact manifold with GQE metric is constant, provided the potential function ψ remains invariant under the characteristic vector field ξ .*

Theorem 1.2. *A Sasakian manifold with GQE metric is an Einstein manifold, provided the potential function ψ remains invariant under the Reeb vector field ξ .*

2. Preliminaries

Let M^{2n+1} be a smooth manifold. Let η be a 1-form, ξ be a vector field and φ be a $(1, 1)$ -tensor field. The triple (η, ξ, φ) is called an *almost contact structure (acs)* if

$$(2.1) \quad I = -\varphi^2 + \eta \otimes \xi, \quad \eta(\xi) = 1,$$

where I is the identity map. Obviously $\varphi\xi = 0$ and $\eta \circ \varphi = 0$. The *acs* is called *normal* if the almost complex structure J on the product manifold $M^{2n+1} \times \mathbb{R}$ defined by

$$J \left(E, \gamma \frac{d}{dt} \right) = \left(\varphi E - \gamma \xi, \eta(E) \frac{d}{dt} \right)$$

for all $E \in \chi(M^{2n+1})$ and $\gamma \in C^\infty(M^{2n+1} \times \mathbb{R})$, is integrable. Here $\chi(M^{2n+1})$ denotes the tangent space of M^{2n+1} . Blair [5] proved that the *acs* is normal if and only if $[\varphi, \varphi] + 2\eta \otimes \xi = 0$, where $[\varphi, \varphi]$ denotes the *Nijenhuis tensor* of φ defined by

$$[\varphi, \varphi](E, F) = \varphi^2[E, F] + [\varphi E, \varphi F] - \varphi[\varphi E, F] - \varphi[E, \varphi F], \quad \forall E, F \in \chi(M^{2n+1}).$$

If there exists a Riemannian metric g on M^{2n+1} such that

$$(2.2) \quad g = g(\varphi \cdot, \varphi \cdot) + \eta \otimes \eta,$$

then the manifold M^{2n+1} together with (η, ξ, φ, g) is said to be an *almost contact metric manifold* (shortly, *acm manifold*). On *acm manifold* we can define the fundamental 2-form Φ defined by $\Phi = g(\cdot, \varphi \cdot)$. When $d\eta = \Phi$, the *acm manifold* is called a *contact metric (cm) manifold*. On a *cm manifold* $\eta \wedge (d\eta)^n$ is a non-vanishing $(2n + 1)$ -form. Contact metric manifolds have been studied by several authors such as ([14], [15], [20], [24]-[26], [28]) and many others.

Given a *cm manifold* M^{2n+1} we can define a symmetric $(1, 1)$ -tensor field $h = \frac{1}{2}\mathcal{L}_\xi\varphi$, where \mathcal{L}_ξ denotes the Lie derivative along the vector field ξ , which satisfy

$$(2.3) \quad h\xi = 0, \quad h\varphi + \varphi h = 0$$

$$(2.4) \quad \nabla_E \xi = -\varphi E - \varphi hE$$

$$(2.5) \quad (\nabla_E \varphi)F + (\nabla_{\varphi E} \varphi)\varphi F = 2g(E, F)\xi - \eta(F)(E + hE + \eta(E)\xi)$$

for all $E, F \in \chi(M^{2n+1})$. We denote R for Riemannian curvature tensor and Q for Ricci operator defined by

$$(2.6) \quad R(E, F) = [\nabla_E, \nabla_F] - \nabla_{[E, F]},$$

$$S(E, F) = g(QE, F).$$

According to Blair [5] $h = 0$ if and only if the Reeb vector field ξ is Killing. If ξ is a Killing vector field, then the cm manifold M^{2n+1} is called K -contact manifold [5]. On a K -contact manifold M^{2n+1} the following relations hold:

$$(2.7) \quad \nabla_E \xi = -\varphi E$$

$$(2.8) \quad Q\xi = 2n\xi$$

$$(2.9) \quad R(\xi, E)F = (\nabla_E \varphi)F$$

$$(2.10) \quad (\nabla_E \varphi)F + (\nabla_{\varphi E} \varphi)\varphi F = 2g(E, F)\xi - \eta(F)(E + \eta(E)\xi)$$

for all $E, F \in \chi(M^{2n+1})$. Taking covariant derivative of (2.8) along $E \in \chi(M^{2n+1})$, we obtain

$$(2.11) \quad (\nabla_E Q)\xi = Q\varphi E - 2n\varphi E.$$

Since ξ is Killing, $\mathcal{L}_\xi Q = 0$. By direct computation

$$(2.12) \quad (\nabla_\xi Q)E = Q\varphi E - \varphi QE.$$

A normal cm manifold is said to be a *Sasakian manifold*. A necessary and sufficient condition for an acm manifold M^{2n+1} to be Sasakian is that

$$(2.13) \quad (\nabla_E \varphi)F = g(E, F)\xi - \eta(F)E$$

for all $E, F \in \chi(M^{2n+1})$. A cm manifold is Sasakian if and only if

$$(2.14) \quad R(E, F)\xi = \eta(F)E - \eta(E)F.$$

Every Sasakian manifold is K -contact manifold, but the converse is not true, in general. However in 3-dimensional manifold K -contact and Sasakian manifolds are

equivalent [21]. The relations (2.7)-(2.10) also hold in Sasakian manifolds. The Ricci tensor in a Sasakian 3-manifold is given by [6]

$$(2.15) \quad S = \frac{r-2}{2}g + \frac{6-r}{2}\eta \otimes \eta.$$

From the above we see that if $r = 6$ then the manifold is an Einstein manifold and conversely. Since in a three dimensional manifold, Einstein and space of constant sectional curvature are equivalent, a Sasakian 3-manifold is of *constant sectional curvature* 1 if and only if $r = 6$.

3. Generalized Guasi-Einstein Metric in a Riemannian Manifold

In this section we deduce the expression of R and S on a Riemannian manifold with GQE metric.

Proposition 3.1. *In a Riemannian manifold (M^{2n+1}, g) with GQE metric, the tensors R and S satisfy*

$$(3.1) \quad \begin{aligned} R(E, F)grad \psi &= (\nabla_F Q)E - (\nabla_E Q)F + (E\beta)F - (F\beta)E \\ &+ \{(E\alpha)(F\psi) - (F\alpha)(E\psi)\}grad \psi \\ &- \alpha\{(F\psi)QE - (E\psi)QF\} + \alpha\beta\{(F\psi)E - (E\psi)F\} \end{aligned}$$

and

$$(3.2) \quad \begin{aligned} (1 - \alpha)S(F, grad \psi) &= \frac{1}{2}(Fr) - 2n(F\beta) - g(grad \psi, grad \psi)(F\alpha) \\ &+ \{g(grad \alpha, grad \psi) - \alpha r + 2n\alpha\beta\}(F\psi) \end{aligned}$$

for all $E, F \in \chi(M^{2n+1})$.

Proof. From (1.1) it follows that

$$(3.3) \quad \nabla_F grad \psi = -QF + \beta F + \alpha g(grad \psi, F)grad \psi.$$

where $H^\psi =$ Hessian of the function ψ is defined by

$$H^\psi(E, F) = g(\nabla_E grad \psi, F)$$

for all vector fields E, F in M^{2n+1} . Taking covariant derivative of (3.3) in the direction $E \in \chi(M^{2n+1})$, we obtain

$$(3.4) \quad \begin{aligned} \nabla_E \nabla_F grad \psi &= -\nabla_E(QF) + (E\beta)F + \beta \nabla_E F + (E\alpha)g(grad \psi, F)grad \psi \\ &+ \alpha(Eg(grad \psi, F))grad \psi + \alpha g(grad \psi, F)\nabla_E grad \psi. \end{aligned}$$

Interchanging E and F in the foregoing equation, we have

$$(3.5) \quad \begin{aligned} \nabla_F \nabla_E grad \psi &= -\nabla_F(QE) + (F\beta)E + \beta \nabla_F E + (F\alpha)g(grad \psi, E)grad \psi \\ &+ \alpha(Fg(grad \psi, E))grad \psi + \alpha g(grad \psi, E)\nabla_F grad \psi. \end{aligned}$$

Using (3.3)-(3.5) in (2.6), we get (3.1). Contracting the equation (3.1) and applying the well known formulas $Er = tr\{F \rightarrow (\nabla_E Q)F\}$ and $\frac{1}{2}Er = div Q$, we get the second result. \square

4. Proof of the Main Results

Proof of the Theorem 1.1. Replacing E by ξ in (3.1) and using (2.11) and (2.12), we have

$$\begin{aligned} R(\xi, F)grad \psi &= \varphi QF - 2n\varphi F + (\xi\beta)F - (F\beta)\xi \\ &+ \{(\xi\alpha)(F\psi) - (F\alpha)(\xi\psi)\}grad \psi \\ (4.1) \quad &- \alpha\{2n(F\psi)\xi - (\xi\psi)QF\} + \alpha\beta\{(F\psi)\xi - (\xi\psi)F\}. \end{aligned}$$

Taking inner product of the foregoing equation with E and using (2.9), we infer

$$\begin{aligned} -g((\nabla_F \varphi)E, grad \psi) &= g(\varphi QF, E) - 2ng(E, \varphi F) \\ &+ (\xi\beta)g(E, F) - (F\beta)\eta(E) \\ &+ (\xi\alpha)(E\psi)(F\psi) - (\xi\psi)(E\psi)(F\alpha) \\ &- \alpha\{2n(F\psi)\eta(E) - (\xi\psi)g(QF, E)\} \\ (4.2) \quad &+ \alpha\beta\{(F\psi)\eta(E) - (\xi\psi)g(E, F)\}. \end{aligned}$$

Replacing E by φE and F by φF in (4.2), entail that

$$\begin{aligned} -g((\nabla_{\varphi F} \varphi)\varphi E, grad \psi) &= g(Q\varphi F, E) - 2ng(E, \varphi F) \\ &+ (\xi\beta)g(\varphi E, \varphi F) + (\xi\alpha)g(\varphi E, grad \psi)g(\varphi F, grad \psi) \\ &- (\xi\psi)g(\varphi E, grad \psi)g(\varphi F, grad \alpha) \\ (4.3) \quad &+ \alpha(\xi\psi)g(Q\varphi F, \varphi E) - \alpha\beta(\xi\psi)g(\varphi E, \varphi F). \end{aligned}$$

Adding (4.2) and (4.3) and using (2.10), we get

$$\begin{aligned} &-2g(E, F)(\xi\psi) + \eta(E)((F\psi) + \eta(F)(\xi\psi)) \\ &= g(\varphi QF + Q\varphi F, E) - 4ng(E, \varphi F) \\ &+ (\xi\beta)(g(E, F) + g(\varphi E, \varphi F)) - (F\beta)\eta(E) \\ &+ (\xi\alpha)((E\psi)(F\psi) + g(\varphi E, grad \psi)g(\varphi F, grad \psi)) \\ &- (\xi\psi)((E\psi)(F\alpha) + g(\varphi E, grad \psi)g(\varphi F, grad \alpha)) \\ &- \alpha\{2n(F\psi)\eta(E) - (\xi\psi)g(QF, E)\} \\ &+ \alpha\beta\{(F\psi)\eta(E) - (\xi\psi)g(E, F)\} \\ (4.4) \quad &+ \alpha(\xi\psi)g(Q\varphi F, \varphi E) - \alpha\beta(\xi\psi)g(\varphi E, \varphi F). \end{aligned}$$

Anti-symmetrizing the above equation, it follows that

$$\begin{aligned} &(1 + 2n\alpha - \alpha\beta)((F\psi)\eta(E) - (E\psi)\eta(F)) \\ &= 2g(\varphi QF + Q\varphi F, E) - 8ng(E, \varphi F) \\ &+ (E\beta)\eta(F) - (F\beta)\eta(E) + (\xi\psi)((E\alpha)(F\psi) - (F\alpha)(E\psi)) \\ (4.5) \quad &+ (\xi\psi)(g(\varphi E, grad \alpha)g(\varphi F, grad \psi) - g(\varphi F, grad \alpha)g(\varphi E, grad \psi)). \end{aligned}$$

Now we assume that the potential function ψ remains invariant under the characteristic vector field ξ , that is, $\xi\psi = 0$. Then the equation (4.5) reduces to

$$(4.6) \quad \begin{aligned} & (1 + 2n\alpha - \alpha\beta)((F\psi)\eta(E) - (E\psi)\eta(F)) \\ & = 2g(\varphi QF + Q\varphi F, E) - 8ng(E, \varphi F) + (E\beta)\eta(F) - (F\beta)\eta(E). \end{aligned}$$

Replacing E by φE and F by φF in the equation (4.6), we infer

$$(4.7) \quad \varphi QF + Q\varphi F = 4n\varphi F$$

for all vector field F on M^{2n+1} . Suppose $\{e_1, e_2, \dots, e_n, \varphi e_1, \varphi e_2, \dots, \varphi e_n, \xi\}$ is a φ -basis of (M^{2n+1}, g) . Then $g(\varphi Qe_i, \varphi e_i) = g(Qe_i, e_i)$ for $i = 1, 2, \dots, n$. We compute

$$\begin{aligned} r &= \sum_{i=1}^n g(Qe_i, e_i) + \sum_{i=1}^n g(Q\varphi e_i, \varphi e_i) + g(Q\xi, \xi) \\ &= \sum_{i=1}^n g(\varphi Qe_i + Q\varphi e_i, \varphi e_i) + 2n \\ &= 2n(2n + 1). \end{aligned}$$

This finishes the proof.

Suppose $d\alpha \wedge d\psi = 0$. Then

$$(4.8) \quad (E\alpha)(F\psi) - (F\alpha)(E\psi) = 0$$

for all $E, F \in \chi(M^{2n+1})$, which implies $(E\alpha)grad\psi - (E\psi)grad\alpha = 0$, that is, $grad\alpha$ and $grad\psi$ are collinear. Conversely, if $grad\alpha$ and $grad\psi$ are collinear then $d\alpha \wedge d\psi = 0$. Using (4.8) in (4.5), we get

$$\begin{aligned} & (1 + 2n\alpha - \alpha\beta)((F\psi)\eta(E) - (E\psi)\eta(F)) \\ & = 2g(\varphi QF + Q\varphi F, E) - 8ng(E, \varphi F) \\ & + (E\beta)\eta(F) - (F\beta)\eta(E). \end{aligned}$$

Proceeding in the similar way as in the proof of Theorem 1.1, it follows that the manifold is of constant scalar curvature. Hence, we can state the following:

Corollary 4.1. *The scalar curvature of a K-contact manifold with GQE metric is constant, provided $grad\alpha$ and $grad\psi$ are collinear.*

Proof of the Theorem 1.2. Let (M^{2n+1}, g) be a Sasakian manifold with GQE metric. In a Sasakian manifold the relation $\varphi Q = Q\varphi$ holds. Therefore $\nabla_\xi Q = 0$.

Using (2.13) in (4.2), we get

$$\begin{aligned}
 -g(E, F)(\xi\psi) + (F\psi)\eta(E) &= g(\varphi QF, E) - 2ng(E, \varphi F) \\
 &+ (\xi\beta)g(E, F) - (F\beta)\eta(E) \\
 &+ (\xi\alpha)(E\psi)(F\psi) - (\xi\psi)(E\psi)(F\alpha) \\
 &- \alpha\{2n(F\psi)\eta(E) - (\xi\psi)g(QF, E)\} \\
 (4.9) \qquad \qquad \qquad &+ \alpha\beta\{(F\psi)\eta(E) - (\xi\psi)g(E, F)\}.
 \end{aligned}$$

Anti-symmetrizing the equation (4.9), we infer

$$\begin{aligned}
 &(1 + 2n\alpha - \alpha\beta)((F\psi)\eta(E) - (E\psi)\eta(F)) \\
 &= 2g(\varphi QF, E) - 4ng(E, \varphi F) \\
 &+ (E\beta)\eta(F) - (F\beta)\eta(E) \\
 (4.10) \qquad \qquad \qquad &+ (\xi\psi)\{(E\alpha)(F\psi) - (F\alpha)(E\psi)\}.
 \end{aligned}$$

Replacing E by φE and F by φF in (4.10), we have

$$\begin{aligned}
 0 &= 2g(\varphi QF, E) - 4ng(E, \varphi F) \\
 (4.11) \qquad &+ (\xi\psi)\{g(\varphi E, \text{grad } \alpha)g(\varphi F, \text{grad } \psi) - g(\varphi F, \text{grad } \alpha)g(\varphi E, \text{grad } \psi)\}.
 \end{aligned}$$

Again replacing E by φE in (4.11) and applying (2.8), we obtain

$$\begin{aligned}
 S(E, F) &= 2ng(E, F) - \frac{1}{2}(\xi\psi)\{g(\varphi^2 E, \text{grad } \alpha)g(\varphi F, \text{grad } \psi) \\
 (4.12) \qquad &- g(\varphi F, \text{grad } \alpha)g(\varphi^2 E, \text{grad } \psi)\}
 \end{aligned}$$

for all vector fields E, F on M^{2n+1} . Contracting the equation, we get

$$r = 2n(2n + 1) + (\xi\psi)g(\varphi(\text{grad } \alpha), \text{grad } \psi).$$

Suppose that, the potential function ψ remains invariant under the characteristic vector field ξ , that is, $\xi\psi = 0$. Then from (4.12), we see that $S = 2ng$.

This finishes the proof.

If $\alpha = 0$ and $\beta \in \mathbb{R}$, from (4.12) we see that $S = 2ng$ and the manifold is an Einstein manifold. Also the equation (4.10) reduces to $(F\psi)\eta(E) - (E\psi)\eta(F) = 0$, that is $\text{grad } \psi = (\xi\psi)\xi$. Now applying $g(\nabla_{\varphi E}\text{grad } \psi, \varphi F) = g(\nabla_{\varphi F}\text{grad } \psi, \varphi E)$, we obtain $(\xi\psi)g(E, \varphi F) = 0$. This implies $\xi\psi = 0$. Therefore $\text{grad } \psi = 0$, that is, ψ is constant. Thus, we can state that:

Corollary 4.2. *A Sasakian manifold whose metric satisfies gradient Ricci soliton equation is an Einstein manifold and the potential function is constant.*

The corollary 4.2 has been proved by He and Zhu [17].

If $\alpha = 0$ and $\beta \in C^\infty(M)$, from the equation (4.12) we have $S = 2ng$. Then the equation becomes $(F\psi)\eta(E) - (E\psi)\eta(F) = (E\beta)\eta(F) - (F\beta)\eta(E)$. Thus for any $E \perp \xi$, we have $g(\text{grad } \psi, E) = -(E\beta)$. Since β is a non-zero function, ψ is non-constant. Also $\text{grad } \psi$ is not perpendicular to $E \perp \xi$. Thus we get the following:

Corollary 4.3. *A Sasakian manifold whose metric satisfies gradient almost Ricci soliton equation is an Einstein manifold. Moreover, neither ψ is a constant function nor $\text{grad } \psi$ is perpendicular to the vector field $E \perp \xi$.*

The second part of the Corollary 4.3 is also proved in the paper [3]. If $\alpha = \frac{1}{m}$, $m \in \mathbb{N}$ and $\beta \in C^\infty(M)$, from (4.12), we see that $S = 2ng$. Thus, we can state that:

Corollary 4.4. *A Sasakian manifold with m -quasi-Einstein metric is an Einstein manifold.*

The above result has also been obtained in [16]. Now we consider GQE metric on Sasakian 3-manifold. Using (2.15) in (4.11), it follows that

$$(4.13) \quad \begin{aligned} 0 &= (r - 6)g(E, \varphi F) + (\xi\psi)\{g(\varphi E, \text{grad } \alpha)g(\varphi F, \text{grad } \psi) \\ &\quad - g(\varphi F, \text{grad } \alpha)g(\varphi E, \text{grad } \psi)\}. \end{aligned}$$

If the potential function ψ remains invariant under the characteristic vector field ξ , from the above equation we have $r = 6$. Thus, we can state that

Corollary 4.5. *A Sasakian 3-manifold with GQE metric is a manifold of constant sectional curvature 1, provided the potential function ψ remains invariant under the Reeb vector field ξ .*

Remark 1. It can be easily shown that in a 3-dimensional Sasakian manifold the φ -sectional curvature is equal to $\frac{r-4}{2}$. Under the hypothesis of Corollary 4.5, we can prove that the scalar curvature of a 3-dimensional Sasakian manifold is constant. Therefore the φ -sectional curvature is constant and the manifold becomes a 3-dimensional Sasakian space form [5], provided the potential function remains invariant under the Reeb vector field ξ .

Acknowledgements. The authors would like to thank the referees and editor for reviewing the paper carefully and their valuable comments to improve the quality of the paper.

References

- [1] A. Barros, R. Batista and E. Ribeiro, Jr, *Rigidity of gradient almost Ricci solitons*, Illinois J. Math., **56**(2012), 1267–1279.
- [2] A. Barros and J. N. Gomes, *A compact gradient generalized quasi-Einstein metric with constant scalar curvature*, J. Math. Anal. Appl., **401**(2013), 702–705.
- [3] N. Basu and B. Bhattacharyya, *Gradient Ricci almost soliton in Sasakian manifolds*, Acta Mathematica Academiae Paedagogicae Nyíregyháziensis, **32**(2016), 161–164.
- [4] A. L. Besse, *Einstein manifolds*, Springer-Verlag, Berlin(1987).

- [5] D. E. Blair, *Riemannian geometry of contact and symplectic manifolds*, Progress in math., **203**(2010), Birkhauser.
- [6] D. E. Blair, T. Koufogiorgos and R. Sharma, *A classification of 3-dimensional contact metric manifold with $Q\varphi = \varphi Q$* , Kodai Math. J., **33**(2010), 361–368.
- [7] H. D. Cao, *Recent progress on Ricci soliton*, Adv. Lect. Math., **11**(2009), 1–38.
- [8] J. Case, *The non-existence of quasi-Einstein metrics*, Pacific J. Math., **248**(2010), 277–284.
- [9] J. Case, Y. -J. Shu and G. Wei, *Rigidity of quasi-Einstein metrics*, Diff. Geom. Appl., **29**(2011), 93–100.
- [10] G. Catino, *Generalized quasi-Einstein manifolds with harmonic Weyl tensor*, Math. Z., **271**(2012), 751–756.
- [11] G. Catino and L. Mazzieri, *Gradient Einstein solitons*, Non-linear anal., **132**(2016), 66–94.
- [12] G. Catino, L. Mazzieri and S. Mongodi, *Rigidity of gradient Einstein shrinkers*, Commun. Contemp. Math., **17**(2015), 1550046, 18pp.
- [13] X. M. Chen, *Quasi-Einstein structures and almost cosymplectic manifolds*, RACSAM, **72**(2020), 14pp.
- [14] U. C. De and P. Majhi, *On a type of contact metric manifolds*, Lobachevskii J. Math., **34**(2013), 89–98.
- [15] U. C. De and A. Sarkar, *On the quasi-conformal curvature tensor of a (κ, μ) -contact metric manifold*, Math. Reports, **14**(2012), 115–129.
- [16] A. Ghosh, *Generalized m -quasi-Einstein metric within the framework of Sasakian and K -contact manifolds*, Ann. Polon. Math., **115**(2015), 33–41.
- [17] C. He and M. Zhu, *Ricci solitons on Sasakian manifolds*, arXiv : 1109.4407V2 [Math.DG] 26 sep 2011.
- [18] G. Huang and Y. Wei, *The classification of (m, ρ) -quasi-Einstein manifolds*, Ann. Global Anal. Geom., **44**(2013), 269–282.
- [19] G. Huang, *Integral pinched gradient shrinking ρ -Einstein solitons*, J. Math. Anal. Appl., **451**(2017), 1025–1055.
- [20] J. B. Jun and A. De, *On $N(\kappa)$ -contact metric manifolds satisfying certain curvature conditions*, Kyungpook Math. J., **51**(2011), 457–468.
- [21] J. B. Jun, I. -B. Kim and U. K. Kim, *On 3-dimensional almost contact metric manifolds*, Kyungpook Math. J., **34**(1999), 293–301.
- [22] M. A. Mirshafezadeh and B. Bidabad, *On generalized quasi-Einstein manifolds*, Adv. Pure Appl. Math., **10**(2019), 193–202.
- [23] M. A. Mirshafezadeh and B. Bidabad, *On the Rigidity of generalized quasi-Einstein manifolds*, Bull. Malays. Math. Sci. Soc., **43**(2020), 2029–2042.
- [24] C. Özgür, *Contact metric manifolds with cyclic-parallel Ricci tensor*, Differ. Geom. Dyn. Syst., **4**(2002), 21–25.
- [25] C. Özgür and S. Sular, *On $N(\kappa)$ -contact metric manifolds satisfying certain conditions*, SUT J. Math., **44**(2008), 89–99.

- [26] R. Sharma, *Certain results on K -contact and (κ, μ) -contact manifolds*, J. Geom., **89**(2008), 138–147.
- [27] V. Venkatesha and H. A. Kumara, *Gradient ρ -Einstein soliton on almost Kenmotsu manifolds*, Ann. Univ. Ferrara Sez. VII Sci. Mat., **65**(2019), 375–388.
- [28] A. Yıldız, U. C. De, C. Murathan and K. Arslan, *On the Weyl projective curvature tensor of an $N(\kappa)$ -contact metric manifold*, Mathematica Pannonica, **21**(2010), 129–142.