KYUNGPOOK Math. J. 61(2021), 455-459
https://doi.org/10.5666/KMJ.2021.61.3.455
pISSN 1225-6951 eISSN 0454-8124

Oh's 8-Universality Criterion is Unique

Scott Duke Kominers
Harvard Business School, Department of Economics, and Center of Mathematical Sciences and Applications, Harvard University, Rock Center 219, Soldiers Field, Boston, MA 02163, USA
e-mail : kominers@fas.harvard.edu

Abstract. We partially characterize criteria for the n-universality of positive-definite integer-matrix quadratic forms. We then obtain the uniqueness of Oh's 8-universality criterion [11] as a corollary.

1. Introduction

A degree-two homogeneous polynomial in n independent variables is called a *quadratic form* (or just *form*) *of rank* n. For a rank-n quadratic form $Q(x_1, \ldots, x_n) = \sum_{i,j} a_{ij}x_ix_j$ (where $a_{ij} = a_{ji}$), the matrix given by $L = (a_{ij})$ is the *Gram Matrix* of a $\mathbb{Z}$-lattice L equipped with a symmetric bilinear form $B(\cdot, \cdot)$ such that $B(L, L) \subseteq \mathbb{Z}$. Then, $Q(\mathbf{x}) = \mathbf{x}^T L\mathbf{x} = B(L\mathbf{x}, \mathbf{x})$ for $\mathbf{x} \in \mathbb{R}^n$.

A rank-n quadratic form Q is said to *represent* an integer k if there exists an $\mathbf{x} \in \mathbb{Z}^n$ such that $Q(\mathbf{x}) = k$. More generally, a $\mathbb{Z}$-lattice L *represents* another $\mathbb{Z}$-lattice ℓ if there exists a $\mathbb{Z}$-linear, bilinear form-preserving injection $\ell \to L$. A quadratic form is called *universal* if it represents all positive integers. Analogously, a lattice is called n*-universal* if it represents all rank-n positive-definite integer-matrix $\mathbb{Z}$-lattices. Connecting the two notions of universality, we observe that a rank-n quadratic form Q is universal if and only if it is 1-universal, as for an integer k,

$$k = Q(x_1, \ldots, x_n) \iff Q(x_1x, \ldots, x_nx) = kx^2.$$

In 1993, Conway and Schneeberger announced their celebrated *Fifteen Theorem*, giving a criterion characterizing the universal positive-definite integer-matrix

Received June 8, 2020; revised August 30, 2020; accepted November 23, 2020.
2020 Mathematics Subject Classification: 11E20, 11E25.
Key words and phrases: n-universal lattice, 8-universal lattice, universality criteria, quadratic form, additively indecomposable.
This work was supported by a Harvard Mathematics Department Highbridge Fellowship, the National Science Foundation (grant SES-1459912), and the Ng Fund and the Mathematics in Economics Research Fund of the Harvard Center of Mathematical Sciences and Applications.

quadratic forms. Specifically, they showed that any positive-definite integer-matrix form that represents the set of nine critical numbers

$$\{1, 2, 3, 5, 6, 7, 10, 14, 15\}$$

is universal (see [1, 3]). Kim, Kim, and Oh [6] presented an analogous criterion for 2-universality, showing that a positive-definite integer-matrix lattice is 2-universal if and only if it represents the set of forms

$$\mathcal{S}_2 = \left\{ \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 2 & 0 \\ 0 & 3 \end{smallmatrix}\right), \left(\begin{smallmatrix} 3 & 0 \\ 0 & 3 \end{smallmatrix}\right), \left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right), \left(\begin{smallmatrix} 2 & 1 \\ 1 & 3 \end{smallmatrix}\right), \left(\begin{smallmatrix} 2 & 1 \\ 1 & 4 \end{smallmatrix}\right) \right\}.$$

Oh [11] gave a similar criterion for 8-universality, which we state in Theorem 4.1 of Section 4.

A set $\mathcal{S}$ of rank-n lattices having the property that a lattice L is n-universal if and only if L represents every lattice in $\mathcal{S}$ is called an *n-criterion set*. Thus, for example, the set $\mathcal{S}_2$ obtained by Kim, Kim, and Oh [6] is a 2-criterion set and the set of integers found by Conway [3] naturally gives the 1-criterion set

$$\mathcal{S}_1 = \left\{ x^2, 2x^2, 3x^2, 5x^2, 6x^2, 7x^2, 10x^2, 14x^2, 15x^2 \right\}.$$

The set $\mathcal{S}_1$ is known to be the unique minimal 1-criterion set (see [7]), in the sense that if $\mathcal{S}'_1$ is a 1-criterion set, then $\mathcal{S}_1 \subseteq \mathcal{S}'_1$. The author [9] obtained an analogous uniqueness result for the 2-criterion set $\mathcal{S}_2$.

Kim, Kim, and Oh [7] have proven that n-criterion sets exist for all positive integers n. However, the problems of finding and determining the uniqueness of these sets have proven to be difficult (see the discussion in [7]). Here, we advance both problems: We obtain two simple (partial) characterization results for arbitrary n-criterion sets, from which we obtain the uniqueness of Oh's 8-universality criterion as a corollary.

Since we first circulated this paper, there has been renewed attention in characterizing criterion sets: Elkies, Kane, and the author [5] identified several families of lattices for which there exist multiple universality criteria of different sizes, including one based on the $\mathbb{Z}^n$ and E_8 lattices that builds on our work here. More recently, Lee [10] and Kim, Lee, and Oh [8] showed that the minimal n-criterion sets are *not* unique for $n \geq 9$, and introduced an elegant theory of recoverable lattices that substantially generalizes [5]. (See also recent work of Chan and Oh [2] characterizing classes of exceptional sets for rank-n quadratic forms, which in some sense can be thought of as building blocks for criterion sets.)

2. Notation and Terminology

We use the lattice-theoretic language of quadratic form theory. A complete introduction to this approach may be found in [12]. In addition, we use the lattice notation of [4], under which I_n is the rank-n lattice of the form $\langle 1, \ldots, 1 \rangle$ and E_8 is the unique even unimodular lattice of rank 8.

For a $\mathbb{Z}$-lattice (or hereafter, just *lattice*) L with basis $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$, we write $L \cong \mathbb{Z}\mathbf{x}_1 + \cdots + \mathbb{Z}\mathbf{x}_n$. If L is of the form $L = L_1 \oplus L_2$ for sublattices L_1 and L_2 of L with $B(L_1, L_2) = 0$, then we write $L \cong L_1 \perp L_2$ and say that L_1 and L_2 are *orthogonal*.

For a sublattice ℓ of $L_1 \perp L_2$ that can be expressed in the form

$$\ell \cong \mathbb{Z}(\mathbf{x}_{1,1} + \mathbf{x}_{2,1}) + \cdots + \mathbb{Z}(\mathbf{x}_{1,n} + \mathbf{x}_{2,n})$$

with $\mathbf{x}_{i,j} \in L_i$, we denote $\ell(L_i) := \mathbb{Z}\mathbf{x}_{i,1} + \cdots + \mathbb{Z}\mathbf{x}_{i,n}$. We naturally extend this notation to lattices ℓ represented by $L_1 \perp L_2$. We then say that a lattice is *additively indecomposable* if either $\ell(L_1) \cong 0$ or $\ell(L_2) \cong 0$ whenever $L_1 \perp L_2$ represents ℓ. Otherwise, we say that ℓ is *additively decomposable*.

3. Partial Characterization of n-Criterion Sets

In this section, we prove two results that partially characterize the contents of arbitrary n-criterion sets.

Proposition 3.1. *Any n-criterion set must include the lattice I_n.*

Proof. If $\mathcal{T}$ is a finite, nonempty set of rank-n lattices not containing I_n, then every lattice $T \in \mathcal{T}$ may be written in the form $T \cong I_k \perp T'$, where $0 \leq k < n$, the sublattice T' is of rank $n-k$, and the first minimum of T' is larger than 1. Indeed, any I_k-sublattice of T is unimodular and therefore splits T; the condition on T' follows from Minkowski reduction.

We may therefore write $\mathcal{T}$ in the form

$$\mathcal{T} = \bigcup_{k=0}^{n-1} \{I_k \perp T_{k,i}\}_{i=1}^{i_k},$$

where $0 < |\mathcal{T}| = \sum_{k=0}^{n-1} i_k$ and each $T_{k,i}$ is a rank-$(n-k)$ lattice with first minimum greater than 1. Then, the lattice

$$I_{n-1} \perp \left(\left(\perp_{i=1}^{i_0} T_{0,i} \right) \perp \cdots \perp \left(\perp_{i=1}^{i_{n-1}} T_{n-1,i} \right) \right)$$

represents all of $\mathcal{T}$ but does not represent I_n. It follows that $\mathcal{T}$ is not an n-criterion set; hence, any n-criterion set must contain I_n. □

Proposition 3.2. *Let $\mathcal{E}$ be the set of additively indecomposable unimodular lattices of rank n. If $\mathcal{E} \neq \emptyset$, then any n-criterion set must include at least one lattice $E \in \mathcal{E}$.*

Proof. Suppose that $\mathcal{E} \neq \emptyset$. If $\mathcal{T} = \{T_i\}_{i=1}^{k}$ is a finite, nonempty set of rank-n lattices with $\mathcal{T} \cap \mathcal{E} = \emptyset$, then every lattice $T_i \in \mathcal{T}$ is either additively decomposable or not unimodular (or both). Now, we consider the lattice

$$T_1 \perp \cdots \perp T_k,$$

which of course represents all of $\mathcal{T}$ by construction.

If $T_1 \perp \cdots \perp T_k$ were to represent some $E \in \mathcal{E}$, then under any such representation we would have $E(T_i) \cong 0$ for all but one i (with $1 \le i \le k$) because E is additively indecomposable. Then, for some i (again, with $1 \le i \le k$), the lattice T_i would represent E. In that case, as E is unimodular, the associated sublattice of T_i would split T_i as $T_i \cong E \perp T'$—and since both E and T_i are of rank n, we would have $T' \cong 0$; hence, $T_i \cong E$. But this is impossible because T_i is either additively decomposable or not unimodular, whereas $E \in \mathcal{E}$ is both additively indecomposable and unimodular.

Thus, we have found a lattice that represents all of $\mathcal{T}$ but cannot represent any $E \in \mathcal{E}$. As $\mathcal{E} \neq \emptyset$ by hypothesis, we see that $\mathcal{T}$ must not be an n-criterion set; the result follows. □

Remark 3.3. It is clear that direct analogues of Propositions 3.1 and 3.2 hold in the more general setting of S-universal lattices discussed in [7]. In particular, suppose that S is an infinite set of lattices. Then, if $n = \max\{k : I_k \in \mathsf{S}\} > 0$, any finite set $\mathcal{S}_{\mathsf{S}} \subset \mathsf{S}$ with the property that a lattice L represents every $\ell \in \mathsf{S}$ if and only if L represents every $\ell \in \mathcal{S}_{\mathsf{S}}$ must contain I_n. Similarly, such a set $\mathcal{S}_{\mathsf{S}}$ must contain an additively indecomposable unimodular lattice if S does.

4. Uniqueness of The 8-Criterion Set

Oh [11] obtained the following 8-criterion set.

Theorem 4.1.([11, remark on Theorem 3.1]) *The set $\mathcal{S}_8 = \{I_8, E_8\}$ is an 8-criterion set.*

The set $\mathcal{S}_8$ is clearly a *minimal* 8-criterion set, as for each $\ell \in \mathcal{S}_8$ there is a lattice that represents $\mathcal{S}_8 \setminus \ell$ but does not represent ℓ. (The single lattice in $\mathcal{S}_8 \setminus \ell$ suffices.) Meanwhile, our characterization results imply the following corollary, which strengthens Theorem 4.1.

Corollary 4.2. *Every 8-criterion set must contain $\mathcal{S}_8$ as a subset.*

Proof. As E_8 is the unique additively indecomposable unimodular lattice of rank 8, the result follows directly from Propositions 3.1 and 3.2. □

Corollary 4.2, when combined with Theorem 4.1, shows that $\mathcal{S}_8$ is the unique minimal 8-criterion set.

Acknowledgements. The author is grateful to Pablo Azar, Noam D. Elkies, Andrea J. Hawksley, Sonia Jaffe, Paul M. Kominers, and especially Ravi Jagadeesan for helpful comments and suggestions, and particularly thanks an anonymous referee for pointing out a problem with the original form of Proposition 3.2.

References

[1] M. Bhargava, *On the Conway-Schneeberger fifteen theorem*, Quadratic forms and their applications (Dublin, 1999), 27–37, Contemp. Math. **272**, Amer. Math. Soc., Providence, RI, 2000.

[2] W. K. Chan and B.-K. Oh, *On the exceptional sets of integral quadratic forms*, Int. Math. Res. Notices, https://doi.org/10.1093/imrn/rnaa382.

[3] J. H. Conway, *Universal quadratic forms and the fifteen theorem*, Quadratic forms and their applications (Dublin, 1999), 2326, Contemp. Math. **272**, Amer. Math. Soc., Providence, RI, 2000.

[4] J. H. Conway and N. J. A. Sloane, *Low-dimensional lattices I. Quadratic forms of small determinant*, Proc. Roy. Soc. London Ser. A, **418(1854)**(1988), 17–41.

[5] N. D. Elkies, D. M. Kane and S. D. Kominers, *Minimal* $\mathcal{S}$*-universality criteria may vary in size*, J. Théor. Nombres Bordeaux, **25(3)**(2013), 557–563.

[6] B. M. Kim and M.-H. Kim and B.-K. Oh, 2*-universal positive definite integral quinary quadratic forms*, Integral quadratic forms and lattices (Seoul, 1998), 5162, Contemp. Math. **249**, Amer. Math. Soc., Providence, RI, 1999.

[7] B. M. Kim and M.-H. Kim and B.-K. Oh, *A finiteness theorem for representability of quadratic forms by forms*, J. Reine Angew. Math., **581**(2005), 23–30.

[8] K. Kim and J. Lee and B.-K. Oh, *Minimal universality criterion sets on the representations of quadratic forms*, (2020), preprint, arXiv:2009.04050.

[9] S. D. Kominers, *Uniqueness of the* 2*-universality criterion*, Note Mat., **28(2)**(2008), 203–206.

[10] J. Lee, *Minimal* S*-universality criterion sets*, Seoul National University, Thesis (Ph.D.), 2020.

[11] B.-K. Oh, *Universal* $\mathbb{Z}$*-lattices of minimal rank*, Proc. Amer. Math. Soc., **128(3)**(2000), 683–689.

[12] O. T. O'Meara, *Introduction to quadratic forms*, Springer-Verlag, New York, 2000.