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Some Congruences for Andrews' Partition Function $\overline{\mathcal{EO}}(n)$

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Abstract. Recently, Andrews introduced partition functions $\mathcal{EO}(n)$ and $\overline{\mathcal{EO}}(n)$ where the function $\mathcal{EO}(n)$ denotes the number of partitions of n in which every even part is less than each odd part and the function $\overline{\mathcal{EO}}(n)$ denotes the number of partitions enumerated by $\mathcal{EO}(n)$ in which only the largest even part appears an odd number of times. In this paper we obtain some congruences modulo 2, 4, 10 and 20 for the partition function $\overline{\mathcal{EO}}(n)$. We give a simple proof of the first Ramanujan-type congruences $\overline{\mathcal{EO}}\,(10n+8) \equiv 0 \pmod 5$ given by Andrews.

1. Introduction

A partition of a positive integer n is a nonincreasing sequence of positive integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ such that $\lambda_1 + \lambda_2 \cdots + \lambda_k = n$. Let $p(n)$ be the number of partitions of n. For example $p(5) = 7$. The seven partitions of 5 are 5, $4+1$, $3+2$, $3+1+1$, $2+2+1$, $2+1+1+1$, $1+1+1+1+1$. The generating function for $p(n)$ is given by

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty},$$

where throughout this paper, for any complex numbers a and $|\, q \,| < 1$ we define

$$(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}), \quad (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k).$$

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Almost a century back Ramanujan established the following identity [7],

$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty^6}, \tag{1.1}$$

which in fact implies Ramanujan's congruences for $p(n)$ modulo 5,

$$p(5n+4) \equiv 0 \pmod 5. \tag{1.2}$$

Recently, Andrews [2] introduced the partition function $\mathcal{EO}(n)$ which counts the number of partitions of n in which every even part is less than each odd part. For example, $\mathcal{EO}(6) = 7$. The seven partitions of 6 it enumerates are 6, $5+1$, $4+2$, $3+3$, $3+1+1+1$, $2+2+2$, $1+1+1+1+1+1$. In [2], Andrews shows that the generating function for $\mathcal{EO}(n)$ is

$$\sum_{n=0}^{\infty} \mathcal{EO}(n)q^n := \frac{1}{(1-q)(q^2;q^2)_\infty}. \tag{1.3}$$

Andrews [2], also defined the partition function $\overline{\mathcal{EO}}(n)$ which counts the number of partitions enumerated by $\mathcal{EO}(n)$ in which only the largest even part appears an odd number of times. For example, $\overline{\mathcal{EO}}(6) = 4$. The four partitions of 6 it enumerates are 6, $3+3$, $2+2+2$, $1+1+1+1+1+1$. In [2], Andrews shows that the generating function for $\overline{\mathcal{EO}}(n)$ is

$$\sum_{n=0}^{\infty} \overline{\mathcal{EO}}(n)q^n = \frac{(q^4;q^4)_\infty^3}{(q^2;q^2)_\infty^2}. \tag{1.4}$$

In Section 3 of this paper, we prove some congruences modulo 2 and 4 for the partition function $\overline{\mathcal{EO}}(n)$. In Section 4, we give a simple proof of Andrews' congruences

$$\overline{\mathcal{EO}}\,(10n+8) \equiv 0 \pmod 5,$$

and we prove some interesting congruences modulo 10 and 20. In the Section 5, we consider

$$\sum_{n=0}^{\infty} \mathcal{EO}_e(n)q^n := \frac{(q^4;q^4)_\infty^2}{(q^2;q^2)_\infty^2}, \tag{1.5}$$

where the function $\mathcal{EO}_e(n)$ counts the elements in the set of partitions which are enumerated by $\overline{\mathcal{EO}}(n)$ together with the partitions enumerated by $\mathcal{EO}(n)$ where all parts are odd and number of parts is even, i.e, $\mathcal{EO}_e(n)$ denotes the number of partitions enumerated by $\mathcal{EO}(n)$ in which only the largest even part appears an odd number of times except when parts are odd and number of parts is even. For example, $\mathcal{EO}_e(6) = 6$. The six partitions of 6 it enumerates are 6, $3+3$, $2+2+2$, $1+1+1+1+1+1$ (which are counted by $\overline{\mathcal{EO}}(n)$) and $5+1$ and $3+1+1+1$

(only counted by $\mathcal{EO}(n)$ in which all parts are odd and the number of parts is even). We prove some arithmetic properties modulo 2 satisfied by $\mathcal{EO}_e(n)$. All of the proofs will follow from elementary generating function considerations and q–series manipulations. The paper concludes with a conjecture on $\overline{\mathcal{EO}}(n)$.

2. Preliminaries

We require the following definitions and lemmas to prove the main results in the next three sections. For $|ab| < 1$, Ramanujan's general theta function $f(a,b)$ is defined as

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}. \tag{2.1}$$

Using Jacobi's triple product identity [1, Theorem 2.8], (2.1) takes the shape

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty. \tag{2.2}$$

The special cases of $f(a,b)$ are

$$\phi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)^2_\infty(q^2;q^2)_\infty = \frac{(q^2;q^2)^5_\infty}{(q;q)^2_\infty(q^4;q^4)^2_\infty}, \tag{2.3}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} = \frac{(q^2;q^2)^2_\infty}{(q;q)_\infty}, \tag{2.4}$$

$$\phi(-q) := \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} = \frac{(q;q)^2_\infty}{(q^2;q^2)_\infty}. \tag{2.5}$$

Lemma 2.1.(Hirschhorn [6, p. 14, Eqn. 1.9.4]) *We have the following* 2*-dissection of* $\phi(q)$*,*

$$\phi(q) = \phi(q^4) + 2q\psi(q^8). \tag{2.6}$$

Lemma 2.2.(Hirschhorn [5] or Hirschhorn [6, p. 36, Eqn. 3.6.4]) *We have,*

$$\begin{aligned}(q;q)^3_\infty &= \sum_{n=0}^{\infty}(-1)^n(2n+1)q^{(n^2+n)/2} \\ &\equiv f(-q^{10},-q^{15}) - 3qf(-q^5,-q^{20}) \pmod 5.\end{aligned} \tag{2.7}$$

Lemma 2.3.(Hirschhorn [6, p. 105, Eqn. 10.7.6]) *We have the following beautiful identity due to Ramanujan,*

$$\frac{(q;q)_\infty^2(q^4;q^4)_\infty^2}{(q^2;q^2)_\infty} = \sum_{n=-\infty}^{\infty}(3n+1)q^{3n^2+2n}. \tag{2.8}$$

From the Binomial Theorem, for any positive integer, k,

$$(q^k;q^k)_\infty^5 \equiv (q^{5k};q^{5k})_\infty \pmod 5. \tag{2.9}$$

3. Congruences Modulo 2 and 4 for $\overline{\mathcal{EO}}(n)$

In this section we prove some congruences modulo 2 and 4 satisfied by $\overline{\mathcal{EO}}(n)$. We require the following generating functions to prove congruences for $\overline{\mathcal{EO}}(n)$.

Theorem 3.1. *We have,*

$$\sum_{n=0}^{\infty}\overline{\mathcal{EO}}(4n)q^n = \frac{(q^4;q^4)_\infty^5}{(q;q)_\infty^2(q^8;q^8)_\infty^2}, \tag{3.1}$$

$$\sum_{n=0}^{\infty}\overline{\mathcal{EO}}(4n+2)q^n = 2\frac{(q^2;q^2)_\infty^2(q^8;q^8)_\infty^2}{(q;q)_\infty^2(q^4;q^4)_\infty}, \tag{3.2}$$

$$\sum_{n=0}^{\infty}\overline{\mathcal{EO}}(8n)q^n = \frac{(q^2;q^2)_\infty^5(q^4;q^4)_\infty^3}{(q;q)_\infty^5(q^8;q^8)_\infty^2}, \tag{3.3}$$

$$\sum_{n=0}^{\infty}\overline{\mathcal{EO}}(8n+2)q^n = 2\frac{(q^4;q^4)_\infty^7}{(q;q)_\infty^3(q^2;q^2)_\infty(q^8;q^8)_\infty^2}, \tag{3.4}$$

$$\sum_{n=0}^{\infty}\overline{\mathcal{EO}}(8n+4)q^n = 2\frac{(q^2;q^2)_\infty^7(q^8;q^8)_\infty^2}{(q;q)_\infty^5(q^4;q^4)_\infty^3}, \tag{3.5}$$

$$\sum_{n=0}^{\infty}\overline{\mathcal{EO}}(8n+6)q^n = 4\frac{(q^2;q^2)_\infty(q^4;q^4)_\infty(q^8;q^8)_\infty^2}{(q;q)_\infty^3}. \tag{3.6}$$

Proof. From (1.4), we have

$$\sum_{n=0}^{\infty}\overline{\mathcal{EO}}(n)q^n = \frac{(q^4;q^4)_\infty^3}{(q^2;q^2)_\infty^2},$$

since there are no terms on the right in which the power of q is odd, we have

$$\overline{\mathcal{EO}}(2n+1) = 0,$$

thus by using (2.6), we obtain

$$\sum_{n=0}^{\infty}\overline{\mathcal{EO}}(2n)q^n = \frac{(q^2;q^2)_\infty^3}{(q;q)_\infty^2} = (q^2;q^2)_\infty^3\frac{(q^4;q^4)_\infty^2}{(q^2;q^2)_\infty^5}\phi(q)$$

$$= \frac{(q^4;q^4)_\infty^2}{(q^2;q^2)_\infty^2}\left(\phi(q^4)+2q\psi(q^8)\right). \tag{3.7}$$

It follows that

$$\sum_{n=0}^{\infty}\overline{\mathcal{EO}}(4n)q^n = \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty^2}\phi(q^2) = \frac{(q^4;q^4)_\infty^5}{(q;q)_\infty^2(q^8;q^8)_\infty^2}$$

and

$$\sum_{n=0}^{\infty}\overline{\mathcal{EO}}(4n+2)q^n = 2\frac{(q^2;q^2)_\infty^2}{(q;q)_\infty^2}\psi(q^4) = 2\frac{(q^2;q^2)_\infty^2(q^8;q^8)_\infty^2}{(q;q)_\infty^2(q^4;q^4)_\infty},$$

which is our (3.1) and (3.2). We have

$$\sum_{n=0}^{\infty}\overline{\mathcal{EO}}(4n)q^n = \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty^2}\phi(q^2)$$

$$= (q^2;q^2)_\infty^2\phi(q^2)\frac{(q^4;q^4)_\infty^2}{(q^2;q^2)_\infty^5}\phi(q)$$

$$= \frac{(q^4;q^4)_\infty^2}{(q^2;q^2)_\infty^3}\phi(q^2)\phi(q)$$

$$= \frac{(q^4;q^4)_\infty^2}{(q^2;q^2)_\infty^3}\phi(q^2)\left(\phi(q^4)+2q\psi(q^8)\right). \tag{3.8}$$

It follows that

$$\sum_{n=0}^{\infty}\overline{\mathcal{EO}}(8n)q^n = \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty^3}\phi(q)\phi(q^2)$$

$$= \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty^3}\frac{(q^2;q^2)_\infty^5}{(q;q)_\infty^2(q^4;q^4)_\infty^2}\frac{(q^4;q^4)_\infty^5}{(q^2;q^2)_\infty^2(q^8;q^8)_\infty^2}$$

$$= \frac{(q^2;q^2)_\infty^5(q^4;q^4)_\infty^3}{(q;q)_\infty^5(q^8;q^8)_\infty^2}$$

and

$$\sum_{n=0}^{\infty}\overline{\mathcal{EO}}(8n+4)q^n = 2\frac{(q^2;q^2)_\infty^2}{(q;q)_\infty^3}\phi(q)\psi(q^4) = 2\frac{(q^2;q^2)_\infty^7(q^8;q^8)_\infty^2}{(q;q)_\infty^5(q^4;q^4)_\infty^3},$$

which is our (3.3) and (3.5). We have

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{\mathcal{EO}}(4n+2)q^n &= 2\frac{(q^2;q^2)_\infty^2}{(q;q)_\infty^2}\psi(q^4)\\ &= 2(q^2;q^2)_\infty^2\psi(q^4)\frac{(q^4;q^4)_\infty^2}{(q^2;q^2)_\infty^5}\phi(q)\\ &= 2\frac{(q^4;q^4)_\infty^2}{(q^2;q^2)_\infty^3}\psi(q^4)\phi(q)\\ &= 2\frac{(q^4;q^4)_\infty^2}{(q^2;q^2)_\infty^3}\psi(q^4)\left(\phi(q^4)+2q\psi(q^8)\right). \qquad (3.9)\end{aligned}$$

It follows that

$$\sum_{n=0}^{\infty}\overline{\mathcal{EO}}(8n+2)q^n = 2\frac{(q^2;q^2)_\infty^2}{(q;q)_\infty^3}\psi(q^2)\phi(q^2) = 2\frac{(q^4;q^4)_\infty^7}{(q;q)_\infty^3(q^2;q^2)_\infty(q^8;q^8)_\infty^2}$$

and

$$\sum_{n=0}^{\infty}\overline{\mathcal{EO}}(8n+6)q^n = 4\frac{(q^2;q^2)_\infty^2}{(q;q)_\infty^3}\psi(q^2)\psi(q^4) = 4\frac{(q^2;q^2)_\infty(q^4;q^4)_\infty(q^8;q^8)_\infty^2}{(q;q)_\infty^3},$$

which is our (3.4) and (3.6). □

We have the following congruences.

Corollary 3.2. *For all* $n \geq 0$,

$$\overline{\mathcal{EO}}(2n+1) = 0, \qquad (3.10)$$

$$\overline{\mathcal{EO}}(4n+2) \equiv 0 \pmod{2}, \qquad (3.11)$$

$$\overline{\mathcal{EO}}(8n+4) \equiv 0 \pmod{2}, \qquad (3.12)$$

$$\overline{\mathcal{EO}}(8n+6) \equiv 0 \pmod{4}. \qquad (3.13)$$

Remark 3.3. The congruences (3.11)–(3.13) were obtained earlier by Andrews et al. [4]. Andrews et al. [3] introduced a partition function $p_\nu(n)$ which counts the number of partitions of n in which the parts are distinct and all odd parts are less than twice the smallest part.

$$\sum_{n=0}^{\infty} p_\nu(n)q^n = \nu(-q), \qquad (3.14)$$

where $\nu(q)$ is a mock theta function. Andrews [2, Corollary 5.2] noted that

$$p_\nu(2n) = \overline{\mathcal{EO}}(2n). \qquad (3.15)$$

He proved the congruences using the properties of mock theta function, whereas we use the q-series identities.

4. Congruences Modulo 5, 10 and 20 for $\overline{\mathcal{EO}}(n)$

In this section we prove some congruences modulo 5, 10 and 20 for $\overline{\mathcal{EO}}(n)$. In the next theorem, we give a simple proof of the Andrews' result [2, Eqn. 1.6], which can be tracked back to [3, Thrm. 6.7]. He used the properties of mock theta functions to prove the congruence, whereas we manipulate the q-series identities to get the result.

Theorem 4.1. *For all* $n \geq 0$,

$$\overline{\mathcal{EO}}\,(10n+8) \equiv 0 \pmod 5. \tag{4.1}$$

Proof. Applying (2.9) in (1.4), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} \overline{\mathcal{EO}}(2n)q^n &= \frac{(q^2;q^2)_\infty^3}{(q;q)_\infty^2} = \frac{(q^2;q^2)_\infty^3 (q;q)_\infty^3}{(q;q)_\infty^5} \\ &\equiv \frac{(q^2;q^2)_\infty^3 (q;q)_\infty^3}{(q^5;q^5)_\infty} \pmod 5.\end{aligned} \tag{4.2}$$

From (2.7), we have

$$(q;q)_\infty^3 \equiv J_0 + J_1 \pmod 5, \tag{4.3}$$

where J_i contains terms in which the power of q is congruent to i modulo 5, then

$$(q^2;q^2)_\infty^3 \equiv J_0^* + J_2^* \pmod 5, \tag{4.4}$$

where J_i^* contains terms in which the power of q is congruent to i modulo 5. Substituting (4.3) and (4.4) in (4.2), we have

$$\sum_{n=0}^{\infty} \overline{\mathcal{EO}}(2n)q^n \equiv \frac{1}{(q^5;q^5)_\infty} (J_0 + J_1)\,(J_0^* + J_2^*) \pmod 5. \tag{4.5}$$

There are no terms on the right in which the power of q is 4 modulo 5, so

$$\sum_{n=0}^{\infty} \overline{\mathcal{EO}}(2(5n+4))q^{5n+4} \equiv 0 \pmod 5,$$

from which we deduce (4.1). □

In the next theorem, we derive two congruences modulo 10 from the generating functions (3.2) and (3.5).

Theorem 4.2. *For all* $n \geq 0$,

$$\overline{\mathcal{EO}}(20n+18) \equiv 0 \pmod{10}, \tag{4.6}$$

$$\overline{\mathcal{EO}}(40n+28) \equiv 0 \pmod{10}. \tag{4.7}$$

Proof. Using (2.9) in (3.2), we have

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{\mathcal{EO}}(4n+2)q^n &= 2\frac{1}{(q;q)_\infty^2}\frac{(q^2;q^2)_\infty^2(q^8;q^8)_\infty^2}{(q^4;q^4)_\infty}\\ &= 2\frac{(q;q)_\infty^3}{(q;q)_\infty^5}\frac{(q^2;q^2)_\infty^2(q^8;q^8)_\infty^2}{(q^4;q^4)_\infty}\\ &\equiv 2\frac{(q;q)_\infty^3}{(q^5;q^5)_\infty}\frac{(q^2;q^2)_\infty^2(q^8;q^8)_\infty^2}{(q^4;q^4)_\infty} \pmod{10}. \quad (4.8)\end{aligned}$$

Replacing q by q^2 in (2.8), we have

$$\frac{(q^2;q^2)_\infty^2(q^8;q^8)_\infty^2}{(q^4;q^4)_\infty} = \sum_{n=-\infty}^{\infty}(3n+1)q^{6n^2+4n} \equiv R_0^* + R_1^* + R_2^* \pmod{5}, \quad (4.9)$$

where R_i^* contains terms in which the power of q is congruent to i modulo 5. Substituting (4.3) and (4.9) in (4.8), we obtain

$$\sum_{n=0}^{\infty}\overline{\mathcal{EO}}(4n+2)q^n \equiv 2\frac{1}{(q^5;q^5)_\infty}(J_0+J_1)(R_0^*+R_1^*+R_2^*) \pmod{10}. \quad (4.10)$$

There are no terms on the right in which the power of q is 4 modulo 5, so

$$\sum_{n=0}^{\infty}\overline{\mathcal{EO}}(4(5n+4)+2)q^{5n+4} \equiv 0 \pmod{10},$$

from which we deduce (4.6). Using (2.9) in (3.5), we have

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{\mathcal{EO}}(8n+4)q^n &= 2\frac{(q^2;q^2)_\infty^5}{(q;q)_\infty^5(q^4;q^4)_\infty^2}\frac{(q^2;q^2)_\infty^2(q^8;q^8)_\infty^2}{(q^4;q^4)_\infty}\\ &= 2(q^4;q^4)_\infty^3\frac{(q^2;q^2)_\infty^5}{(q;q)_\infty^5(q^4;q^4)_\infty^5}\frac{(q^2;q^2)_\infty^2(q^8;q^8)_\infty^2}{(q^4;q^4)_\infty}\\ &\equiv 2(q^4;q^4)_\infty^3\frac{(q^{10};q^{10})_\infty}{(q^5;q^5)_\infty(q^{20};q^{20})_\infty}\frac{(q^2;q^2)_\infty^2(q^8;q^8)_\infty^2}{(q^4;q^4)_\infty} \pmod{10}. \quad (4.11)\end{aligned}$$

From (2.7), we have

$$(q^4;q^4)_\infty^3 \equiv J_0^{**} + J_4^{**} \pmod{5}, \quad (4.12)$$

where J_i^{**} contains terms in which the power of q is congruent to i modulo 5. Substituting (4.9) and (4.12) in (4.11), we obtain

$$\sum_{n=0}^{\infty}\overline{\mathcal{EO}}(8n+4)q^n \equiv 2\frac{(q^{10};q^{10})_\infty}{(q^5;q^5)_\infty(q^{20};q^{20})_\infty}(J_0^{**}+J_4^{**})(R_0^*+R_1^*+R_2^*) \pmod{10}. \quad (4.13)$$

There are no terms on the right in which the power of q is 3 modulo 5, so

$$\sum_{n=0}^{\infty} \overline{\mathcal{EO}}(8(5n+3)+4)q^{5n+3} \equiv 0 \pmod{10},$$

from which we deduce (4.7). □

In the next theorem, we derive a congruences modulo 20 from the generating function (3.6).

Theorem 4.3. *For all $n \geq 0$,*

$$\overline{\mathcal{EO}}(40n+38) \equiv 0 \pmod{20}. \tag{4.14}$$

Proof. Using (2.9) in (3.6), we have

$$\begin{aligned}
\sum_{n=0}^{\infty} \overline{\mathcal{EO}}(8n+6)q^n &= 4\frac{1}{(q;q)_\infty^3}\frac{(q^4;q^4)_\infty^2}{(q^2;q^2)_\infty}\frac{(q^2;q^2)_\infty^2(q^8;q^8)_\infty^2}{(q^4;q^4)_\infty} \\
&= 4\frac{1}{(q;q)_\infty^5}\frac{(q;q)_\infty^2(q^2;q^2)_\infty^4}{(q^2;q^2)_\infty}\frac{(q^2;q^2)_\infty^2(q^8;q^8)_\infty^2}{(q^4;q^4)_\infty} \\
&\equiv 4\frac{1}{(q^5;q^5)_\infty}\frac{(q;q)_\infty^2(q^2;q^2)_\infty^4}{(q^2;q^2)_\infty}\frac{(q^2;q^2)_\infty^2(q^8;q^8)_\infty^2}{(q^4;q^4)_\infty} \pmod{20}. \qquad (4.15)
\end{aligned}$$

From (2.8), we have

$$\frac{(q;q)_\infty^2(q^4;q^4)_\infty^2}{(q^2;q^2)_\infty} = \sum_{n=-\infty}^{\infty}(3n+1)q^{3n^2+2n} \equiv R_0+R_2+R_3 \pmod{5}, \tag{4.16}$$

where R_i contains terms in which the power of q is congruent to i modulo 5. Substituting (4.9) and (4.16) in (4.15), we obtain

$$\sum_{n=0}^{\infty} \overline{\mathcal{EO}}(8n+6)q^n \equiv 4\frac{1}{(q^5;q^5)_\infty}(R_0+R_2+R_3)(R_0^*+R_1^*+R_2^*) \pmod{20}. \tag{4.17}$$

There are no terms on the right in which the power of q is 4 modulo 5, so

$$\sum_{n=0}^{\infty} \overline{\mathcal{EO}}(8(5n+4)+6)q^{5n+4} \equiv 0 \pmod{20},$$

from which we deduce (4.14). □

5. Congruences for $\mathcal{EO}_e(n)$

In this section we prove some congruences modulo 2 for $\mathcal{EO}_e(n)$.

Theorem 5.1.

$$\sum_{n=0}^{\infty} \mathcal{EO}_e(4n)q^n = \frac{(q^4;q^4)_\infty^5}{(q;q)_\infty^3 (q^8;q^8)_\infty^2}, \tag{5.1}$$

$$\sum_{n=0}^{\infty} \mathcal{EO}_e(4n+2)q^n = 2\frac{(q^2;q^2)_\infty^2 (q^8;q^8)_\infty^2}{(q;q)_\infty^3 (q^4;q^4)_\infty}. \tag{5.2}$$

Proof. From (1.5), we have

$$\sum_{n=0}^{\infty} \mathcal{EO}_e(n)q^n = \frac{(q^4;q^4)_\infty^2}{(q^2;q^2)_\infty^2},$$

since there are no terms on the right in which the power of q is odd, we have

$$\mathcal{EO}_e(2n+1) = 0,$$

by using (2.6), we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} \mathcal{EO}_e(2n)q^n &= \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty^2} = (q^2;q^2)_\infty^2 \frac{(q^4;q^4)_\infty^2}{(q^2;q^2)_\infty^5}\phi(q) \\ &= \frac{(q^4;q^4)_\infty^2}{(q^2;q^2)_\infty^3}\left(\phi(q^4) + 2q\psi(q^8)\right). \end{aligned} \tag{5.3}$$

It follows that

$$\sum_{n=0}^{\infty} \mathcal{EO}_e(4n)q^n = \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty^3}\phi(q^2) = \frac{(q^4;q^4)_\infty^5}{(q;q)_\infty^3 (q^8;q^8)_\infty^2}$$

and

$$\sum_{n=0}^{\infty} \mathcal{EO}_e(4n+2)q^n = 2\frac{(q^2;q^2)_\infty^2}{(q;q)_\infty^3}\psi(q^4) = 2\frac{(q^2;q^2)_\infty^2 (q^8;q^8)_\infty^2}{(q;q)_\infty^3 (q^4;q^4)_\infty},$$

which is our (5.1) and (5.2). □

We have the following congruences.

Corollary 5.2. *For all $n \geq 0$,*

$$\mathcal{EO}_e(2n+1) = 0, \tag{5.4}$$

$$\mathcal{EO}_e(4n+2) \equiv 0 \pmod{2}. \tag{5.5}$$

6. Conclusion

Andrews [2, Problem 4], proposed to further investigate the properties of $\overline{\mathcal{EO}}\,(n)$. We conclude the paper with the following conjecture. Using maple, we found the following congruences hold up to n = 2000.

Conjecture 6.1. *For all* $n \geq 0$,

(6.1) $$\overline{\mathcal{EO}}\,(50n+18) \equiv 0 \pmod{20},$$

(6.2) $$\overline{\mathcal{EO}}\,(50n+28) \equiv 0 \pmod{20},$$

(6.3) $$\overline{\mathcal{EO}}\,(50n+38) \equiv 0 \pmod{20},$$

(6.4) $$\overline{\mathcal{EO}}\,(50n+48) \equiv 0 \pmod{20}.$$

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