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On Weakly Prime and Weakly 2-absorbing Modules over Non-commutative Rings

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Abstract. Most of the research on weakly prime and weakly 2-absorbing modules is for modules over commutative rings. Only scatterd results about these notions with regard to non-commutative rings are available. The motivation of this paper is to show that many results for the commutative case also hold in the non-commutative case. Let R be a non-commutative ring with identity. We define the notions of a weakly prime and a weakly 2-absorbing submodules of R and show that in the case that R commutative, the definition of a weakly 2-absorbing submodule coincides with the original definition of A. Darani and F. Soheilnia. We give an example to show that in general these two notions are different. The notion of a weakly m-system is introduced and the weakly prime radical is characterized interms of weakly m-systems. Many properties of weakly prime submodules and weakly 2-absorbing submodules are proved which are similar to the results for commutative rings. Amongst these results we show that for a proper submodule N_i of an R_i-module M_i, for $i = 1, 2$, if $N_1 \times N_2$ is a weakly 2-absorbing submodule of $M_1 \times M_2$, then N_i is a weakly 2-absorbing submodule of M_i for $i = 1, 2$

1. Introduction

In 2007 Badawi [3] introduced the concept of 2-absorbing ideals of commutative rings with identity, which is a generalization of prime ideals, and investigated some properties. He defined a 2-absorbing ideal P of a commutative ring R with identity to be a proper ideal of R such that if $a, b, c \in R$ and $abc \in P$, then $ab \in P$ or $bc \in P$ or $ac \in P$. In 2011, Darani and Soheilnia [7] introduced the concepts of 2-absorbing and weakly 2-absorbing submodules of modules over commutative rings with identities. A proper submodule P of a module M over a commutative ring R with identity is said to be a 2-absorbing submodule (weakly 2-absorbing

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submodule) of M if whenever $a, b \in R$ and $m \in M$ with $abm \in P(0 \neq abm \in P)$, then $abM \subseteq P$ or $am \in P$ or $bm \in P$. One can see that 2-absorbing and weakly 2-absorbing submodules are generalizations of prime submodules. Moreover, it is obvious that 2-absorbing ideals are special cases of 2-absorbing submodules.

Throughout this paper, all rings are associative with identity elements (not necessarily commutative) and modules are unitary left modules. Let R be a ring and M be an R-module. We write $N \leq M$, if N is a submodule of M. In recent years the study of the absorbing properties of rings and modules, and related notions, have been topics of interest in ring and module theory. In [11] the notion of 2-absorbing modules over non-commutative rings was introduced. In this paper we study the notion of weakly prime and weakly 2-absorbing modules over non-commutative rings. We prove basic properties of weakly 2-absorbing submodules. In particular, we show that If R is a commutative ring then the notion of a weakly 2-absorbing submodule coincides with that of the original definition introduced by Darani and Soheilnia in [7]. For an R-module M and a submodule N of M we have $(N :_R M) = \{r \in R : rM \subseteq N\}$.

Following [9] a proper ideal P of the ring R is 2-absorbing if $aRbRc \subseteq P$ implies $ab \in P$ or $ac \in P$ or $bc \in P$ for a, b and c elements of R. Following [11] a proper submodule N of the R-module M is a 2-absorbing submodule of M if $aRbRx \subseteq N$ implies $ab \in (N :_R M)$ or $ax \in N$ or $bx \in N$ for $a, b \in R$ and $x \in M$. From [8] a proper submodule P of M is called a prime submodule of M if, for every ideal A of R and every submodule N of M, $AN \subseteq P$ implies either $N \subseteq P$ or $AM \subseteq P$. This is equivalent to $aRx \subseteq P$ implies $a \in (P : M)$ or $x \in P$ for $a \in R$ and $x \in M$. It is clear that a submodule N of an R-module M is prime if and only $P = (N :_R M)$ is a prime ideal of R.

From [10] A proper ideal P of the ring R is weakly prime if $0 \neq aRb \subseteq P$ implies $a \in P$ or $b \in P$ for a and b elements of R.

Definition 1.1.([1, Definition 3.3]) Let M be a left R-module. A proper submodule N of M is called a weakly prime submodule of M if whenever $r \in R$ and $m \in M$ with $0 \neq rRm \subseteq N$ then either $m \in N$ or $r \in (N :_R M)$.

Remark 1.2. Let p and q be two prime numbers. In the $\mathbb{Z}$-module $\mathbb{Z}pq$, the submodule (0) is weakly prime, but not prime.

Compare the next Theorem with [2, Corollary 2.3].

Theorem 1.3. *Let R be a ring, and M an R-module and N a weakly prime submodule of M. If N is not a prime submodule of M, then for any subset P of R such that $P \subseteq (N :_R M)$ we have $PN = 0$. In particular $(N :_R M)N = 0$.*

Proof. Suppose P is a subset of R such that $P \subseteq (N :_R M)$. Suppose $PN \neq 0$. We show that N is prime. Let $r \in R$ and $m \in M$ be such that $rRm \subseteq N$. If $rRm \neq 0$, then $r \in (N :_R M)$ or $m \in N$ since N is weakly prime. So assume $rRm = 0$. First assume $rN \neq 0$, say $rn \neq 0$ for some $n \in N$. Now $0 \neq rn \in rR(n + m) \subseteq N$ and N weakly prime, gives $r \in (N :_R M)$ or $(n + m) \in N$. Hence $r \in (N :_R M)$ or $m \in N$ since $n \in N$. So we can assume that $rN = 0$. Now suppose that $Pm \neq 0$, say $sm \neq 0$ for $s \in P \subseteq (N :_R M)$. We have $0 \neq sm \in (r+s)Rm \subseteq N$. Hence $(r+s) \in (N :_R M)$ or $m \in N$. So $r \in (N :_R M)$ or $m \in N$. Hence we can assume that $Pm = 0$. Since $PN \neq 0$, there exists $t \in P$ and $n \in N$ such that $tn \neq 0$. Now we have $0 \neq tn \in (r+t)R(n+m) \subseteq N$. Again, since N is weakly prime, we get $(r+t) \in (N :_R M)$ or $(m+n) \in N$. Hence $r \in (N :_R M)$ or $m \in N$. Thus N is a prime submodule. □

Compare (1) ⇔ (2) of the next Theorem with [2, Theorem 2.4]

Theorem 1.4. *Let N be a proper submodule of a left R-module M. Then the following are equivalent:*

(1) *N is a weakly prime submodule of M.*

(2) *For a left ideal P of R and submodule D of M with $0 \neq PD \subseteq N$, either $P \subseteq (N :_R M)$ or $D \subseteq N$.*

(3) *For any element $a \in R$ and $L \leq M$, if $0 \neq aRL \subseteq N$, then $L \subseteq N$ or $a \in (N :_R M)$.*

(4) *For any right ideal I of R and $L \leq M$, if $0 \neq IL \subseteq N$, then $L \subseteq N$ or $I \subseteq (N :_R M)$.*

(5) *For any element $a \in R$ and $L \leq M$, if $0 \neq RaRL \subseteq N$, then $L \subseteq N$ or $a \in (N :_R M)$.*

(6) *For any element $a \in R$ and $L \leq M$, if $0 \neq RaL \subseteq N$ then $L \subseteq N$ or $a \in (N :_R M)$.*

Proof.

(1) ⇒ (2) Suppose that N is a weakly prime submodule of M. If N is prime, then the result is clear from [5, Proposition 1.1]. So we can assume that N is weakly prime that is not prime. Let $0 \neq PD \subseteq N$ with $x \in D - N$. We show that $P \subseteq (N :_R M)$. Let $r \in P$. Now $rRx \subseteq rD \subseteq N$. If $0 \neq rRx$, then N weakly prime gives $r \in (N :_R M)$. So assume that $rRx = 0$. First suppose that $rD \neq 0$, say $rd \neq 0$ where $d \in D$. If $d \notin N$, then since $0 \neq rRd \subseteq N$ and N weakly prime $r \in (N :_R M)$. If $d \in N$, then $rR(d+x) = rRd \subseteq N$, so $r \in (N :_R M)$ or $(d+x) \in N$. Thus, $r \in (N :_R M)$; hence $P \subseteq (N :_R M)$. So we can assume that $rD = 0$. Suppose that $Px \neq 0$, say $ax \neq 0$ where $a \in P$. Now $0 \neq aRx \subseteq N$ and N weakly prime gives $a \in (N :_R M)$. As $(r+a)Rx = aRx \subseteq N$, we get $r \in (N :_R M)$, so $P \subseteq (N :_R M)$. Therefore, we can assume that $Px = 0$. Since $PD \neq 0$, there exist $b \in P$ and $d_1 \in D$ such that $bd_1 \neq 0$. As $(N :_R M)N = 0$ (by Theorem 1.3) and $0 \neq b(d_1+x) = bd_1 \in N$ we can divide the proof into the following two cases:

Case 1. $b \in (N :_R M)$ and $(d_1+x) \notin N$. Since $0 \neq (r+b)R(d_1+x) = bRd_1 \subseteq N$, we obtain $(r+b) \in (N :_R M)$, so $r \in (N :_R M)$. Hence $P \subseteq (N :_R M)$.

Case 2. $b \notin (N :_R M)$ and $(d_1+x) \in N$. As $0 \neq bRd_1 \subseteq N$ we have $d_1 \in N$, so $x \in N$ which is a contradiction. Thus $P \subseteq (N :_R M)$.

(2) $\Rightarrow$ (1) Suppose that $0 \neq sRm \subseteq N$ where $s \in R$ and $m \in M$. Take $I = Rs$ and $D = Rm$. Then $0 \neq ID \subseteq N$, so either $I \subseteq (N :_R M)$ or $D \subseteq N$; hence either $r \in (N :_R M)$ or $m \in N$. Thus N is weakly prime.

(2) $\Rightarrow$ (3) Let $a \in R$ and $L \leq M$ such that $0 \neq aRL \subseteq N$. Now $0 \neq RaL \subseteq N$ and from 2. $L \subseteq N$ or $a \in Ra \subseteq (N :_R M)$.

(3) $\Rightarrow$ (2) Let P be a left ideal of R and D a submodule of M with $0 \neq PD \subseteq N$. If $D \subseteq N$, then we are done. So suppose $D \nsubseteq N$. We will show that $P \subseteq (N :_R M)$. Let $a \in P$. Hence $aRD \subseteq N$. If $aRD \neq 0$ then it follows from (3) that $a \in (N :_R M)$ and we have $P \subseteq (N :_R M)$. So suppose $aRD = 0$. Because $PD \neq 0$, there exists $p \in P$ such that $pRD \neq 0$. We now have $0 \neq pRD = (a+p)RD \subseteq N$. It follows from (3) that $(a+p) \in (N :_R M)$ and we have $P \subseteq (N :_R M)$.

(3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) $\Leftrightarrow$ (6) is now easy to see. □

Remark 1.5. From [5] we know that if N is a prime submodule of an R-module M, then $(N :_R M)$ is a prime ideal of R. Suppose that N is weakly prime which is not prime. Contrary to what happens for a prime submodules, the ideal $(N :_R M)$ is not, in general, a weakly prime ideal of R. For example, let M denote the cyclic $\mathbb{Z}$-module $\mathbb{Z}/8\mathbb{Z}$. Take $N = \{0\}$. Certainly N is a weakly prime submodule of M, but $(N :_R M) = 8Z$ is not a weakly prime ideal of R, but we have the following result:

Proposition 1.6. *Let R be a ring with identity M a faithful R-module, and N a weakly prime submodule of M. Then $(N :_R M)$ is a weakly prime ideal of R.*

Proof. Assume that M is a faithful R module and let $0 \neq aRb \subseteq (N :_R M)$. Since M is a faithful R module we have $0 \neq aRbM \subseteq N$. It follows that $0 \neq (RaR)(RbR)M \subseteq N$. From Theorem 1.4 we have $(RaR)M \subseteq N$ or$(RbR)M \subseteq N$. Hence $a \in (N :_R M)$ or $b \in (N :_R M)$ and it follows that $(N :_R M)$ is a weakly prime ideal. □

From [12] we have that M is a multiplication module over a non-commutative ring if and only if $(N : M)M = N$ for each submodule N of M.

Proposition 1.7. *Let M be a multiplication R-module. If $(N : M)$ is a weakly prime ideal of R, then N is a weakly prime submodule of M.*

Proof. Let $0 \neq aRm \subseteq N$ with $m \in M$ and $a \notin (N : M)$. Since M is a multiplication module there is an ideal I of R such that $Rm = IM$, then $0 \neq RaIM \subseteq N$. Hence $0 \neq RaI \subseteq (N : M)$. Since $(N : M)$ is a weakly prime ideal of R, we have $Ra \subseteq (N : M)$ or $I \subseteq (N : M)$. Since $a \notin (N : M)$, we have $I \subseteq (N : M)$. Hence $Rm = IM \subseteq N$. Thus $m \in N$ and N is a weakly prime submodule of M. □

Remark 1.8. The converse of Proposition 1.7 is not true in general. Suppose that $M = \mathbb{Z} \times \mathbb{Z}$ is an $R = \mathbb{Z} \times \mathbb{Z}-$module and $N = 2\mathbb{Z} \times \{0\}$ is a submodule of M .$(N : M) = 0$ is a weakly prime ideal. We have $(0,0) \neq (2,0)(1,1) \in 2\mathbb{Z} \times \{0\}$. Now, neither $(2,0) \in (N:M)$ nor $(1,1) \in N$. Hence N is not weakly prime. Notice that M is not a multiplication module.

Lemma 1.9. *Let R be a ring, M an R-module and N a weakly prime submodule of M. If $0 \neq aRbRm \subseteq N$ and $am \notin N$, then $bM \subseteq N$ for all $a; b \in R$ and $m \in M$.*

Proof. Let $a, b \in R$ and $m \in M$. Assume that $0 \neq aRbRm \subseteq N$ and $am \notin N$. Now we have $0 \neq (RaR)(RbR)m \subseteq N$. From Theorem 1.4 we have $RaRM \subseteq N$ or $RbRm \subseteq N$. Since $am \notin N$ we have $RbRm \subseteq N$. Because $0 \neq aRbRm$ we must have $0 \neq bRm \subseteq N$. Now, since N is weakly prime we get $bM \subseteq N$ or $m \in N$. Since $am \notin N$,we must have $bM \subseteq N$ and we are done. □

The following result gives characterizations of weakly prime submodules.

Theorem 1.10. *Let M be an R-module. The following asserations are equivalent:*

(1) *P is a weakly prime submodule of M.*

(2) *$(P:Rx) = (P:M) \cup (0:Rx)$ for any $x \in M - P$.*

(3) *$(P:Rx) = (P:M)$ or $(P:Rx) = (0:Rx)$ for any $x \in M - P$.*

Proof. (1) $\Rightarrow$ (2) Let $r \in (P:Rx)$ and $x \notin P$. Then $rRx \subseteq P$. Suppose $rRx \neq 0$. Hence $r \in (P:M)$ because P is weakly prime and $x \notin P$. If $rRx = 0$, then $r \in (0:Rx)$. Thus $(P:Rx) \subseteq (P:M) \cup (0:Rx)$. Now if $r \in (P:M) \cup (0:Rx)$ then either $r \in (P:M)$ or $r \in (0:Rx)$. Hence, when $r \in (0:Rx)$, $rRx = 0 \subseteq P$ and so $r \in (P:Rx)$. If $r \in (P:M)$ then $rM \subseteq P$, and this implies $rRx \subseteq rM \subseteq P$. Hence $r \in (P:Rx)$ and therefore $(P:Rx) = (P:M) \cup (0:Rx)$. (2) $\Rightarrow$ (3) Is obvious. (3) $\Rightarrow$ (1) Suppose that $0 \neq rRx \subseteq P$ with $r \in R$ and $x \in M - P$. Then $r \in (P:Rx)$ and $r \notin (0:Rx)$. It follows from (3) that $r \in (P:Rx) = (P:M)$, as required. □

Proposition 1.11. *Let M_1 and M_2 be unitary R-modules over a ring R. Let $M = M_1 \oplus M_2$ and $N \subseteq M_1 \oplus M_2$. Then the following are satisfied:*

(1) $N = Q \oplus M_2$ is a weakly prime submodule of M if and only if Q is a weakly prime submodule of M_1 and $r \in R$, $x \in M_1$ with $rRx = 0$, but $x \notin Q$, $r \notin (Q:M_1)$ implies $rM_2 = 0$.

(2) $N = M_1 \oplus Q$ is a weakly prime submodule of M if and only if Q is a weakly prime submodule of M_2 and $r \in R$, $x \in M_2$ with $rRx = 0$, but $x \notin Q$, $r \notin (Q:M_2)$ implies $rM_1 = 0$.

Proof. We will prove (1) and the proof of (2) will be similar. ($\Rightarrow$) Let $N = Q \oplus M_2$ be a weakly prime submodule of M. Let $0 \neq rRq \subseteq Q$, $q \notin Q$. Then $(q,0) \notin Q \oplus M_2$, while $0 \neq rR(q,0) \subseteq Q \oplus M_2$. Since $N = Q \oplus M_2$ is a weakly prime submodule of M we have $r \in (M_1 \oplus M_2 : Q \oplus M_2)$. Hence $rM_1 \subseteq Q$ and Q is a weakly prime submodule of M_1. Now, suppose $r \in R$, $x \in M_1$ such that $rRx = 0$, but $x \notin Q$, $r \notin (Q:M_1)$. Assume that $rM_2 \neq 0$, so there esists $m \in M_2$ such that $rm \neq 0$. Thus $(0,0) \neq rR(x,m) = (rRx, rRm) = (0, rRm) \subseteq Q \oplus M_2 = N$. N is a weakly prime submodule of M, so either $(x,m) \in Q \oplus M_2$ or $r \in (Q \oplus M_2 : M_1 \oplus M_2)$. Thus either $x \in Q$ or $r \in (Q : M_1)$ which is a contradiction with hypothesis, hence $rM_2 = 0$. ($\Leftarrow$) Let $r \in R$ and $(x,y) \in M$. Assume $(0,0) \neq rR(x,y) \subseteq Q \oplus M_2$, so if $rRx \neq 0$, then $x \in Q$ or $r \in (Q : M_1)$, since Q is a weakly prime submodule of M_1. Thus either $(x,y) \in Q \oplus M_2 = N$ or $r \in (N : M)$. If $rRx = 0$, suppose $x \notin Q$, $r \notin (Q, M_1)$. Then by hypothesis $rM_2 = 0$ and so $rRy \subseteq rM_2 = 0$. Hence $rR(x,y) = (0,0)$ which is a contradiction. Thus either $x \in Q$ or $r \in (Q : M_1)$ and hence either $(x,y) \in Q \oplus M_2 = N$ or $r \in (Q \oplus M_2 : M_1 \oplus M_2)$. □

Remark 1.12. Let M_1 and M_2 be R-modules. If (0) is a prime submodule of M_1, then $(0) \oplus M_2$ is a weakly prime submodule of $M_1 \oplus M_2$.

Proof. Let $r \in R$ and $(x,y) \in M$. If $(0,0) \neq rR(x,y) \subseteq (0) \oplus M_2$, then $rRx = 0$ and $rRy \subseteq M_2$. Since (0) is a prime submodule of M_1, either $x = 0$ or $r \in ((0) : M_1)$. Hence either $(x,y) = (0,y) \in (0) \oplus M_2$ or $r \in ((0) \oplus M_2 : M_1 \oplus M_2)$, that is $(0) \oplus M_2$ is a weakly prime submodule of $M_1 \oplus M_2$. □

Proposition 1.13. *Let M_1 and M_2 be R-modules. If $U \oplus W$ is a weakly prime submodule of $M_1 \oplus M_2$, then U and W are weakly prime submodules of M_1 and M_2 respectively.*

Proof. The proof is straight forward so it is omitted. □

Remark 1.14. The converse of Proposition 1.13 is not true in general as the following example shows.

Example 1.15. Suppose $M = \mathbb{Z} \oplus \mathbb{Z}$ is a $\mathbb{Z}$-module and consider the submodule $N = p\mathbb{Z} \oplus \{0\}$ of M. $p\mathbb{Z}$ is a prime submodule of the $\mathbb{Z}$-module $\mathbb{Z}$ and hence also a weakly prime submodule and $\{0\}$ is a weakly prime submodule of the $\mathbb{Z}$ module $\mathbb{Z}$. $N = p\mathbb{Z} \oplus \{0\}$ is not weakly prime since $(0,0) \neq p(1,0) \in p\mathbb{Z} \oplus \{0\}$ but $p \notin (p\mathbb{Z} \oplus \{0\} :_{\mathbb{Z}} \mathbb{Z} \oplus \mathbb{Z})$ and $(1,0) \notin p\mathbb{Z} \oplus \{0\}$.

2. The Weakly Prime Radical

We begin this section with the definition of weakly m-systems.

Definition 2.1 Let R be a ring and M be an R-module. A nonempty set $S \subseteq M \backslash \{0\}$ is called a weakly m-system if, for each ideal A of R, and for all submodules $K, L \subseteq M$, if $(K+L) \cap S \neq \emptyset$, $(K+AM) \cap S \neq \emptyset$, and $AL \neq 0$ then $(K+AL) \cap S \neq \emptyset$.

Proposition 2.2. *Let M be an R-module. Then a submodule P of M is weakly prime if and only if $M \backslash P$ is a weakly m-system.*

Proof. Suppose $S = M \backslash P$. Let A be an ideal in R and K and L be submodules of M such that $(K+L) \cap S \neq \emptyset$, $(K+AM) \cap S \neq \emptyset$ and $AL \neq 0$. If $(K+AL) \cap S = \emptyset$ then $K + AL \subseteq P$. Hence $AL \subseteq P$ and since P is weakly prime, and $AL \neq 0$, $L \subseteq P$ or $AM \subseteq P$. It follows that $(K+L) \cap S = \emptyset$ or $(K+AM) \cap S = \emptyset$, a contradiction. Therefore, S is a weakly m-system in M. Conversely, let $S = M \backslash P$ be a weakly m-system in M. Suppose $AL \subseteq P$ and $AL \neq 0$, where A is an ideal of R and L is a submodule M. If $L \nsubseteq P$ and $AM \nsubseteq P$, then $L \cap S \neq \emptyset$ and $AM \cap S \neq \emptyset$. Thus, $AL \cap S \neq \emptyset$, a contradiction. Therefore, P is a weakly prime submodule of M. □

The following proposition offers several characterizations of a weakly m-system S when it is the complement of a submodule.

Proposition 2.3. *Let R be a ring and M be an R-module. Let P be a proper submodule of M, and let $S := M \backslash P$. Then the following statements are equivalent:*

(1) *P is weakly prime;*

(2) *S is a weakly m-system;*

(3) *for each left ideal $A \subseteq R$, and for every submodule $L \leqq M$, if $L \cap S \neq \emptyset$, $AM \cap S \neq \emptyset$ and $AL \neq 0$ then $AL \cap S \neq \emptyset$;*

(4) *for each ideal $A \subseteq R$, and for every $m \in M$, if $Rm \cap S \neq \emptyset$, $AM \cap S = \emptyset$ and $AL \neq 0$, then $ARm \cap S \neq \emptyset$;*

(5) *for each $a \in R$, and for each $m \in M$, if $Rm \cap S \neq \emptyset$, $aM \cap S \neq \emptyset$ and $aRm \neq 0$, then $aRm \cap S \neq \emptyset$.*

Proof. (1) $\Leftrightarrow$ (2) follows from Proposition 2.2. (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) is clear (5) $\Rightarrow$ (1). Suppose $a \in R$ and $m \in M$ with $0 \neq aRm \subseteq P$. If $Rm \nsubseteq P$ and $aM \nsubseteq P$, then $Rm \cap S \neq \emptyset$ and $aM \cap S \neq \emptyset$ and $aRm \neq 0$. From (5) $aRm \cap S \neq \emptyset$. Hence $aRm \nsubseteq P$ a contradiction. Hence $Rm \subseteq P$ or $aM \subseteq P$ and P is weakly prime. □

Proposition 2.4. *Let M be an R-module, $S \subseteq M$ be a weakly m-system, and let P be a submodule of M maximal with respect to the property that P is disjoint from S. Then P is a weakly prime submodule.*

Proof. Suppose $0 \neq AL \subseteq P$, where A is an ideal of R and $L \leqq M$. If $L \nsubseteq P$ and $AM \nsubseteq P$, then by the maximal property of P, we have, $(P + L) \cap S \neq \emptyset$ and $(P + AM) \cap S \neq \emptyset$. Thus, since S is a weakly m-system $(P + AL) \cap S \neq \emptyset$ and it follows that $P \cap S \neq \emptyset$, a contradiction. Thus, P must be a weakly prime submodule. □

Next we need a generalization of the notion of $\sqrt{N}$ for any submodule N of M. We adopt the following:

Definition 2.5. Let R be a ring and M be an R-module. For a submodule N of M, if there is a weakly prime submodule containing N, then we define $\sqrt{N} := \{m \in M$: every weakly m-system containing m meets $N\}$. If there is no weakly prime submodule containing N, then we put $\sqrt{N} = M$.

Theorem 2.6. *Let M be an R-module and $N \leq M$. Then either $\sqrt{N} = M$ or $\sqrt{N}$ equals the intersection of all the weakly prime submodules of M containing N.*

Proof. Suppose that $\sqrt{N} \neq M$. This means that $\{P | P$ is a weakly prime submodule of M and $N \subseteq P\} \neq \emptyset$. We first prove that $\sqrt{N} \subseteq \{P | P$ is a weakly prime submodule of M and $N \subseteq P\}$. Let $m \in \sqrt{N}$ and P be any weakly prime submodule of M containing N. Consider the m-system $M\backslash P$. This m-system cannot contain m, for otherwise it meets N and hence also P. Therefore, we have $m \in P$. Conversely, assume $m \notin \sqrt{N}$. Then, by Definition 2.5, there exists an m-system S containing m which is disjoint from N. By Zorn's Lemma, there exists a submodule $P \supseteq N$ which is maximal with respect to being disjoint from S. By Proposition 2.4, P is a weakly prime submodule of M, and we have $m \notin P$, as desired. □

3. Weakly 2-absorbing Submodules

From [11] we have the following:

Definition 3.1. Let P be a proper ideal of a ring R. Then P is a 2-absorbing ideal of R if $aRbR \subseteq P$ implies $ab \in P$ or $bc \in P$ or $ac \in P$ for all $a; b; c \in R$.

Definition 3.2. Let R be a ring and N be a proper submodule of an R-module M. Then N is 2-absorbing submodule of M if $aRbRm \subseteq N$ implies $abM \subseteq N$ i.e. $ab \in (N :_R M)$ or $am \in N$ or $bm \in N$ for all $a; b \in R$ and $m \in M$.

Remark 3.3. If R is a commutative ring then this notion of a 2-absorbing submodule coincides with that of Darani and Soheilnia [7].

We now have the following:

Definition 3.4. Let R be a ring and N be a proper submodule of an R-module M. Then N is a *weakly 2-absorbing submodule* of M if $0 \neq aRbRm \subseteq N$ implies $abM \subseteq N$ i.e. $ab \in (N :_R M)$ or $am \in N$ or $bm \in N$ for all $a; b \in R$ and $m \in M$.

Remark 3.5. Every 2-absorbing submodule is weakly 2-absorbing but the converse does not necessarily hold. For example consider the case where $R = \mathbb{Z}$, $M = \mathbb{Z}/30\mathbb{Z}$ and $N = 0$. Then $2.3.(5 + 30\mathbb{Z}) = 0 \in N$ while $2.3 \notin (N :_R M)$, $2.(5 + 30Z) \notin N$ and $3.(5+30Z) \notin N$. Therefore N is not 2-absorbing while it is weakly 2-absorbing.

Proposition 3.6. *Let $x \in M$ and $a \in R$. Then if $ann_l(x) \subseteq (Rx : M)$, the submodule Rx is 2-absorbing if and only if Rx is weakly 2- absorbing.*

Proof. Let Rx be a weakly 2-absorbing submodule of M and suppose $r, s \in R$ and $m \in M$ with $rRsRm \subseteq Rx$. Since Rx is a weakly 2-absorbing submodule, we may assume $rRsRm = 0$, otherwise Rx is 2-absorbing. Now $rRsR(x + m) \subseteq Rx$. If $rRsR(x+m) \neq 0$ then we have $rs \in (Rx : M)$ or $r(x+m) \in Rx$ or $s(x+m) \in Rx$, as Rx is a weakly 2-absorbing submodule. Hence $rs \in (Rx : M)$ or $rm \in Rx$ or $sm \in Rx$. Now let $rRsR(x+m) = 0$. Then $rRsRm = 0$ implies $rRsRx = 0$. Hence $rs \in ann_l(x) \subseteq (Rx : M)$. Thus Rx is 2-absorbing. □

Proposition 3.7. *Let R be a ring and N be a proper submodule of an R-module M. If N is weakly prime, then it is weakly 2-absorbing.*

Proof. Assume N is a weakly prime submodule of the R-module M and $0 \neq aRbRm \subseteq N$ for all $a; b \in R$ and $m \in M$. Suppose $am \notin N$. It now follows from Proposition 1.9 that $bM \subseteq N$ and consequently $abM \subseteq N$. Hence N is weakly 2-absorbing. □

Compare the following theorem with that of [7, Theorem 2.3(ii)].

Theorem 3.8. *The intersection of each pair of weakly prime submodules of an R-module M is a weakly 2-absorbing submodule of M.*

Proof. Let N and K be two weakly prime submodules of M. If $N = K$, then $N \cap K$ is a weakly prime submodule of M so that $N \cap K$ is a weakly 2-absorbing submodule of M. Assume that N and K are distinct. Since N and K are proper submodules of M, it follows that $N \cap K$ is a proper submodule of M. Next, let $a, b \in R$ and $m \in M$ be such that $0 \neq aRbRm \subseteq N \cap K$ but $am \notin N \cap K$ and $ab \notin (N \cap K : M)$. Then, we can conclude that (a) $am \notin N$ or $am \notin K$, and (b) $ab \notin (N :_R M)$ or $ab \notin (K :_R M)$. These two conditions give 4 cases:

(1) $am \notin N$ and $ab \notin (N :_R M)$;

(2) $am \notin N$ and $ab \notin (K :_R M)$;

(3) $am \notin K$ and $ab \notin (N :_R M)$;

(4) $am \notin K$ and $ab \notin (K :_R M)$.

We first consider Case(1). Since $0 \neq aRbRm \subseteq N \cap K \subseteq N$ and $am \notin N$, it follows from Proposition 1.9 that $bM \subseteq N$. This is a contradiction because $ab \notin (N :_R M)$. Hence Case(1) does not occur. Similarly, Case(4) is not possible. Next, Case(2) is considered. Again, we obtain that $bM \subseteq N$ and then $bm \in N$. Since $0 \neq aRbRm \subseteq K$ it follows that $0 \neq (RaR)(RbR)m) \subseteq K$. Hence, from the fact that K is weakly prime and from Theorem 1.4 it follows that $aM \subseteq RaRM \subseteq K$ or $bm \in RbRm \subseteq K$ If $aM \subseteq K$, then $abM \subseteq aM \subseteq K$ which contradicts $ab \notin (K :_R M)$. Thus $bm \in K$. Hence $bm \in N \cap K$. The proof of Case(3) is similar to that of Case(2). Hence $N \cap K$ is a weakly 2-absorbing submodule of M. □

Definition 3.9. Let N be a weakly 2-absorbing submodule of M. (a, b, m) is called a triple-zero of N if $aRbRm = 0$, $ab \notin (N :_R M)$, $am \notin N$ and $bm \notin N$.

The following result is an analogue of [6, Theorem 1].

Theorem 3.10. *Let N be weakly 2-absorbing submodule of M and (a, b, m) be a triple-zero of N for some $a, b \in R$ and $m \in M$. Then the followings hold.*

(1) $aRbN = a(N :_R M)m = b(N :_R M)m = 0$.

(2) $a(N :_R M)N = b(N :_R M)N = (N :_R M)bN = (N :_R M)bm = (N :_R M)^2m = 0$.

Proof. Suppose that (a, b, m) is a triple-zero of N for some a, $b \in R$ and $m \in M$.

(1) Assume that $aRbN \neq 0$. Then there is an element $n \in N$ such that $aRbRn \neq 0$. Now $aRbR(m + n) = aRbRm + aRbRn = aRbRn \neq 0$ since $aRbRm = 0$ because (a, b, m) is a triple-zero of N. Since $0 \neq aRbR(m + n) \subseteq N$ and N weakly 2-absorbing we have $ab \in (N :_R M)$ or $a(m+n) \in N$ or $b(m+n) \in N$. Since (a, b, m) is a triple-zero of N, $ab \notin (N :_R M)$. Hence $a(m + n) \in N$

or $b(m+n) \in N$ and consequently $am \in N$ or $bm \in N$ a contradiction. Hence $aRbN = 0$. Now, we suppose that $a(N :_R M)m \neq 0$. Thus there exists an element $r \in (N :_R M)$ such that $arm \neq 0$. Hence $aR(r+b)rm = aRrRm + aRbRm = aRrRm$. Since $0 \neq arm \in aRrRm \subseteq N$ and N weakly 2-absorbing we have $a(r+b) \in (N :_R M)$ or $am \in N$ or $(r+b)m \in N$. Hence $ab \in (N :_R M)$ or $am \in N$ or $bm \in N$ a contradiction since (a, b, m) is a triple-zero of N. Similarly, it can be easily seen that $b(N :_R M)m = 0$.

(2) Assume that $a(N :_R M)N \neq 0$. Then there are $r \in (N :_R M)$, $n \in N$ such that $arn \neq 0$. By (1), we get $a(b+r)(m+n) = abm+abn+arm+arn = arn \neq 0$. Now $0 \neq aR(b+r)R(m+n) \subseteq N$. Therefore, we have $a(b+r) \in (N :_R M)$ or $a(m+n) \in N$ or $(b+r)(m+n) \in N$ and we obtain $ab \in (N :_R M)$ or $am \in N$ or $bm \in N$, a contradiction. Hence $a(N :_R M)N = 0$. In a similar way, we get $b(N :_R M)N = 0$. Now, we suppose that $(N :_R M)bN \neq 0$. Then there are $r \in (N :_R M)$, $n \in N$ such that $rbn \neq 0$. Now, from above $(a+r)b(n+m) = abn + abm + rbn + rbm = rbn \neq 0$. Hence $0 \neq (a+r)RbR(n+m) \subseteq N$ and since N is weakly 2-absorbing $(a+r)b \in (N :_R M)$ or $(a+r)(n+m) \in N$ or $b(n+m) \in N$. Hence $ab \in (N :_R M)$ or $am \in N$ or $bm \in N$ a contradiction since (a, b, m) is a triple-zero of N. Now, we suppose that $(N :_R M)bm \neq 0$. Then there is $r \in (N :_R M)$ such that $rbm \neq 0$. Hence $0 \neq rbm = (a+r)bm \in (a+r)RbRm = rRbRm \subseteq N$. Since N is weakly 2-absorbing, we have $(a+r)b \in (N :_R M)$ or $(a+r)m \in N$ or $bm \in N$. Therefore $ab \in (N :_R M)$ or $am \in N$ or $bm \in N$ a contradiction since (a, b, m) is a triple-zero of N. Hence $(N :_R M)bm = 0$. Lastly, we show that $(N :_R M)^2 m = 0$. Let $(N :_R M)^2 m \neq 0$. Thus there exist $r, s \in (N :_R M)$ where $rsm \neq 0$. By (1), we get $(a+r)(b+s)m = rsm \neq 0$. Thus we have $0 \neq (a+r)R(b+s)Rm \subseteq N$. Hence $(a+r)(b+s) \in (N :_R M)$ or $(a+r)m \in N$ or $(b+s)m \in N$.

Consequently, $ab \in (N :_R M)$ or $am \in N$ or $bm \in N$ a contradiction, since (a, b, m) is a triple-zero of N. Therefore $(N : M)^2 m = 0$. □

The following result is an analogue of [6, Lemma 1].

Proposition 3.11. *Assume that N is a weakly 2-absorbing submodule of an R-module M that is not 2-absorbing. Then $(N :_R M)^2 N = 0$. In particular, $(N :_R M)^3 \subseteq Ann(M)$.*

Proof. Suppose that N is a weakly 2-absorbing submodule of an R-module M that is not 2-absorbing. Then there is a triple-zero (a, b, m) of N for some $a, b \in R$ and $m \in M$. Assume that $(N :_R M)^2 N \neq 0$. Thus there exist $r, s \in (N :_R M)$ and $n \in N$ with $rsn \neq 0$. By Theorem 3.10, we get $(a+r)(b+s)(n+m) = rsn \neq 0$. Then we have $0 \neq (a+r)R(b+s)R(n+m) \subseteq N$. Since N is weakly 2-absorbing we have $(a+r)(b+s) \in (N :_R M)$ or $(a+r)(n+m) \in N$ or $(b+s)(n+m) \in N$ and so $ab \in (N :_R M)$ or $am \in N$ or $bm \in N$, which is a contradiction.

Thus $(N :_R M)^2 N = 0$. We get $(N :_R M)^3 \subseteq ((N :_R M)^2 N : M) = (0 : M) = \mathrm{Ann}(M)$. □

4. On a Question from Badawi and Yousefian

In [4], the authors asked the following question:

Question. Suppose that L is a weakly 2-absorbing ideal of a ring R and $0 \neq IJK \subseteq L$ for some ideals I, J, K of R. Does it imply that $IJ \subseteq L$ or $IK \subseteq L$ or $JK \subseteq L$?

This section is devoted to studying the above question and its generalization in modules over non-commutative rings.

Definition 4.1. Let N be a weakly 2-absorbing submodule of an R-module M and let $0 \neq I_1 I_2 K \subseteq N$ for some ideals I_1, I_2 of R and some submodule K of M. N is called free triple-zero in regard to I_1, I_2, K if (a, b, m) is not a triple-zero of N for every $a \in I_1, b \in I_2$ and $m \in K$.

The following result and its proof are analogous of [6, Lemma 2].

Lemma 4.2. *Let N be a weakly 2-absorbing submodule of M. Assume that $aRbK \subseteq N$ for some $a, b \in R$ and some submodule K of M where (a, b, m) is not a triple-zero of N for every $m \in K$. If $ab \notin (N :_R M)$, then $aK \subseteq N$ or $bK \subseteq N$.*

Proof. Assume that $aK \nsubseteq N$ and $bK \nsubseteq N$. Then there are $x, y \in K$ such that $ax \notin N$ and $by \notin N$. We get $bx \in N$ since N is a weakly 2-absorbing submodule, (a, b, x) is not a triple-zero of N, $ab \notin (N :_R M)$ and $ax \notin N$. In a similar way, $ay \in N$. Now, $aRbR(x + y) \subseteq N$ and since $(a, b, x + y)$ is not a triple-zero of N and $ab \notin (N :_R M)$ we have $a(x + y) \in N$ or $b(x + y) \in N$. Assume that $a(x + y) = (ax + ay) \in N$. As $ay \in N$, we get $ax \in N$, a contradiction. Assume that $b(x + y) = (bx + by) \in N$. As $bx \in N$, we get $by \in N$, a contradiction again. Hence we obtain that $aK \subseteq N$ or $bK \subseteq N$. □

Let N be a weakly 2-absorbing submodule of an R-module M and $I_1 I_2 K \subseteq N$ for some for some ideals I_1, I_2 of R and some submodule K of M where N is free triple-zero in regard to I_1, I_2, K. Note that if $a \in I_1, b \in I_2$ and $m \in K$, then $ab \in (N :_R M)$ or $am \in N$ or $bm \in N$.

The following result and its proof are analogous of [6, Theorem 1] and its proof.

Theorem 4.3. *Assume that N is a weakly 2-absorbing submodule of an R-module M and $0 \neq IJK \subseteq N$ for some ideals I, J of R and some submodule K of M where N is free triple-zero in regard to I, J, K. Then $IJ \subseteq (N :_R M)$ or $IK \subseteq N$ or $JK \subseteq N$.*

Proof. Let N be a weakly 2-absorbing submodule of an R-module M and $0 \neq IJK \subseteq N$ for some ideals I, J of R and some submodule K of M where N is free triple-zero in regard to I, J, K. Suppose $IJ \nsubseteq (N :_R M)$. We show that $IK \subseteq N$ or $JK \subseteq N$. Assume $IK \nsubseteq N$ and $JK \nsubseteq N$. Then $a_1 K \nsubseteq N$ and $a_2 K \nsubseteq N$ where $a_1 \in I$ and $a_2 \in J$. From Lemma 4.2 $a_1 a_2 \in (N :_R M)$ since $a_1 R a_2 K \subseteq IJK \subseteq N$ and $a_1 K \nsubseteq N$ and $a_2 K \nsubseteq N$. By our assumption, there are $b_1 \in I$ and $b_2 \in J$ such that $b_1 b_2 \notin (N :_R M)$. By Lemma 4.2, we get $b_1 K \subseteq N$ or $b_2 K \subseteq N$ since $b_1 R b_2 K \subseteq IJK \subseteq N$ and $b_1 b_2 \notin (N :_R M)$. We have the following cases: Case (1)

$b_1K \subseteq N$ and $b_2K \nsubseteq N$: Since $a_1Rb_2K \subseteq IJK \subseteq N$ and $a_1K \nsubseteq N$ and $b_2K \nsubseteq N$ it follows from Lemma 4.2 that $a_1b_2 \in (N :_R M)$. Since $b_1K \subseteq N$ and $a_1K \nsubseteq N$, we conclude $(a_1+b_1)K \nsubseteq N$. On the other hand since $(a_1+b_1)Rb_2K \subseteq N$ and neither $(a_1+b_1)K \subseteq N$ nor $b_2K \subseteq N$, we get that $(a_1+b_1)b_2 \in (N :_R M)$ by Lemma 4.2. But, because $(a_1+b_1)b_2 = (a_1b_2+b_1b_2) \in (N :_R M)$ and $(a_1+b_1)b_2 \in (N :_R M)$, we get $b_1b_2 \in (N :_R M)$ which is a contradiction. Case (2) $b_2K \subseteq N$ and $b_1K \nsubseteq N$: By a similar argument to case (1) we get a contradiction. Case (3) $b_1K \subseteq N$ and $b_2K \subseteq N$: $b_2K \subseteq N$ and $a_2K \nsubseteq N$ gives $(a_2+b_2)K \nsubseteq N$. But $a_1R(a_2+b_2)K \subseteq N$ and neither $a_1K \subseteq N$ nor $(a_2+b_2)K \subseteq N$, hence $a_1(a_2+b_2) \in (N :_R M)$ by Lemma 4.2. Since $a_1a_2 \in (N :_R M)$ and $(a_1a_2+a_1b_2) \in (N :_R M)$, we have $a_1b_2 \in (N :_R M)$. Since $(a_1+b_1)Ra_2K \subseteq N$ and neither $a_2K \subseteq N$ nor $(a_1+b_1)K \subseteq N$, we conclude $(a_1+b_1)a_2 \in (N :_R M)$ by Lemma 4.2. But $(a_1+b_1)a_2 = a_1a_2+b_1a_2$, so $(a_1a_2+b_1a_2) \in (N :_R M)$ and since $a_1a_2 \in (N :_R M)$, we get $b_1a_2 \in (N :_R M)$. Now, since $(a_1+b_1)R(a_2+b_2)K \subseteq N$ and neither $(a_1+b_1)K \subseteq N$ nor $(a_2+b_2)K \subseteq N$, we have $(a_1+b_{1)}(a_2+b_2) = (a_1a_2+a_1b_2+b_1a_2+b_1b_2) \in (N :_R M)$ by Lemma 4.2. But $a_1a_2, a_1b_2, b_1a_2 \in (N :_R M)$, so $b_1b_2 \in (N :_R M)$ which is a contradiction. Consequently $IK \subseteq N$ or $JK \subseteq N$. □

5. Weakly 2-absorbing Submodules of Product Modules

Proposition 5.1. *Let $R = R_1 \times R_2$ and $M = M_1 \times M_2$ where M_1 is an R_1 module and $0 \neq M_2$ is an R_2 module. If N_1 is a proper submodule of M_1 then the following statements are equivalent:*

(1) *N_1 is a 2-absorbing submodule of M_1;*

(2) *$N_1 \times M_2$ is a 2-absorbing submodule of $M_1 \times M_2$;*

(3) *$N_1 \times M_2$ is a weakly 2-absorbing submodule of $M_1 \times M_2$.*

Proof. (1) $\Leftrightarrow$ (2) follows from [11, Theorem 2.5]. (2) $\Rightarrow$ (3) is clear. We show (3) $\Rightarrow$ (1) Let $a, b \in R_1$ and $x \in M_1$ such that $aR_1bR_1x \subseteq N_1$. For every $0 \neq y \in M_2$ we have $(a,1)(b,1)(x,y) = (abx,y) \neq (0,0)$. Now $(0,0) \neq (abx,y) \in aR_1bR_1x \times 1R_11R_1y \subseteq N_1 \times M_2$. Since $N_1 \times M_2$ is a weakly 2-absorbing submodule of $M_1 \times M_2$, we get $(a,1)(b,1) \in (M_1 \times M_2 : N_1 \times M_2)$ or $(a,1)(x,y) \in N_1 \times M_2$ or $(b,1)(x,y) \in N_1 \times M_2$. Hence $ab \in (N_1 : M_1)$ or $ax \in N_1$ or $bx \in N_1$. □

Proposition 5.2. *Let $R = R_1 \times R_2$ and $M = M_1 \times M_2$ where $0 \neq M_1$ is an R_1 module and M_2 is an R_2module. If N_2 is a proper submodule of M_2 then the following statements are equivalent:*

(1) *N_2 is a 2-absorbing submodule of M_1;*

(2) *$M_1 \times N_2$ is a 2-absorbing submodule of $M_1 \times M_2$;*

(3) *$M_1 \times N_2$ is a weakly 2-absorbing submodule of $M_1 \times M_2$.*

Proof. Similar to Proposition 5.1. □

Proposition 5.3. *Let $R = R_1 \times R_2$ and $M = M_1 \times M_2$ where M_1 is an R_1 module and $0 \neq M_2$ is an R_2module. Let $N_1 \neq M_1$. If N_1 is a weakly prime submodule of M_1 and 0 a prime submodule of M_2 then $N_1 \times \{0\}$ is a weakly 2-absorbing submodule of $M_1 \times M_2$.*

Proof. Assume $(0, 0) \neq (a, b)R(c, d)R(x, y) \subseteq N_1 \times \{0\}$ where $(a, b) \in R, (c, d) \in R$ and $(x, y) \in M$. Hence $0 \neq aR_1cR_1x \subseteq N_1$ and $bR_2dR_2y = 0$. Since N_1 is a weakly prime submodule of M_1 we get $a \in (N_1 : M_1)$ or $c \in (N_1 : M_1)$ or $x \in N_1$. Also, since 0 is a prime submodule of M_2 and $bR_2dR_2y = 0$ we have $b \in (0 : M_2)$ or $d \in (0 : M_2)$ or $y = 0$. In any of the above cases we have $(a, b)(c, d) \in (N_1 \times \{0\} : M)$ or $(a, b)(x, y) \in N_1 \times \{0\}$ or $(c, d)(x, y) \in N_1 \times \{0\}$. □

Proposition 5.4. *Let $R = R_1 \times R_2$ and $M = M_1 \times M_2$ where $0 \neq M_1$ is an R_1 module and $0 \neq M_2$ is an R_2 module. If $N = N_1 \times N_2$ is a weakly 2-absorbing submodule of M, $N_1 \neq M_1$, and $N_2 \neq M_2$, then N_1 and N_2 are weakly prime submodules of M_1 and M_2 respectively.*

Proof. Let $0 \neq rRx \subseteq N_1$, where $r \in R_1$ and $x \in M_1$. Consider $z \in M_2 \backslash N_2$. Then $(0, 0) \neq (1, 0)R(r, 1)R(x, z) \subseteq N$ and as N is weakly 2-absorbing, $(1, 0)(r, 1) \in (N : M)$ or $(r, 1)(x, z) \in N$ or $(1, 0)(x, z) \in N$. Note that since $z \in M_2 \backslash N_2$, $(r, 1)(x, z) \notin N$. Thus $(1, 0)(r, 1) \in (N : M) = (N_1 : M_1) \times (N_2 : M_2)$ or $(1, 0)(x, z) \in N$. Therefore, $r \in (N_1 : M_1)$ or $x \in N_1$. This shows that N_1 is a weakly prime submodule of M_1. Similarly we can show that N_2 is a weakly prime submodule of M_2. □

Proposition 5.5. *Let N_i be a proper submodule of an R_i-module M_i, for $i = 1, 2$. If $N_1 \times N_2$ is a weakly 2-absorbing submodule of $M_1 \times M_2$, then*

(1) *N_1 is a weakly 2-absorbing submodule of M_1,*

(2) *N_2 is a weakly 2-absorbing submodule of M_2.*

Proof.

(1) Suppose that $N_1 \times N_2$ is a weakly 2-absorbing submodule of $M_1 \times M_2$. Let $a_1, a_2 \in R_1$ and $m \in M_1$ such that $0 \neq a_1R_1a_2R_1m \subseteq N_1$. Clearly, $(0, 0) \neq (a_1, 1)(R_1 \times R_2)(a_2, 1)(R_1 \times R_2)(m, m_2)$ for any $m_2 \in N_2$. Hence $(0, 0) \neq (a_1, 1)(R_1 \times R_2)(a_2, 1)(R_1 \times R_2)(m, m_2) \subseteq a_1R_1a_2R_1m \times 1R_21R_2m_2 \subseteq N_1 \times N_2$. Since $N_1 \times N_2$ is a weakly 2-absorbing submodule of $M_1 \times M_2$, $(a_1, 1)(a_2, 1) \in (N_1 \times N_2 : M_1 \times M_2)$ or $(a_1, 1)(m, m_2) \in N_1 \times N_2$ or $(a_2, 1)(m, m_2) \in N_1 \times N_2$. Consequently $a_1a_2 \in (N_1 : M_1)$ or $a_1m \in N_1$ or $a_2m \in N_1$. Hence N_1 is a weakly 2-absorbing submodule of M_1.

(2) This follows as in part (1). □

The converse of the above proposition is no true in general:

Example 5.6. Suppose that $M = \mathbb{Z}\times \mathbb{Z}$ is an $R = \mathbb{Z}\times \mathbb{Z}$-module and $N = p\mathbb{Z}\times\{0\}$ is a submodule of M where $p\mathbb{Z}$ is a prime submodule and hence a weakly 2-absorbimg submodule of the $\mathbb{Z}$ module $\mathbb{Z}$ and $\{0\}$ is weakly 2-absorbing.submodule of the $\mathbb{Z}$ module $\mathbb{Z}$. Then $(N : M) = 0$. Assume that $(0, 0) \neq (p, 1)(1, 0)(1, 1) \in p\mathbb{Z} \times \{0\}$. Then neither $(p, 1)(1, 0) \in (N : M)$ nor $(p, 1)(1, 1) \in N$ nor $(1, 0)(1, 1) \in N$. Hence N is not weakly 2-absorbing.

Proposition 5.7. *Let N_i be a proper submodule of an R-module M_i, for $i = 1, 2$ Then the following conditions are equivalent:*

(1) *$N_1 \times M_2$ is a weakly 2-absorbing submodule of $M_1 \times M_2$;*

(2) (a) *N_1 is a weakly 2-absorbing submodule of M_1;*
   (b) *For each $a_1, a_2 \in R$ and $m \in M_1$ such that $a_1Ra_2Rm = 0$ if $a_1a_2 \notin (N_1 : M_1)$ and $a_1m \notin N_1$ and $a_2m \notin N_1$ then $a_1Ra_2M_2 = 0$.*

Proof. (1) $\Rightarrow$ (2).

(a) Suppose $N_1\times M_2$ is a weakly 2-absorbing submodule of $M_1\times M_2$. Let $a_1, a_2 \in R$ and $m \in M_1$ such that $0 \neq a_1Ra_2Rm \subseteq N_1$. Now $(0, 0) \neq (a_1, 0)(R \times R)(a_2, 0)(R \times R)(m, 0) \subseteq N_1 \times M_2$. Hence $N_1 \times M_2$ a weakly 2-absorbing submodule of $M_1 \times M_2$ gives $(a_1a_2, 0) = (a_1, 0)(a_2, 0) \in (N_1 \times M_2 : M_1 \times M_2)$ or $(a_1, 0)(m, 0) \in N_1 \times M_2$ or $(a_2, 0)(m, 0) \in N_1 \times M_2$. Consequently $a_1a_2 \in (N_1 : M_1)$ or $a_1m \in N_1$ or $a_2m \in N_1$. Hence N_1 is a weakly 2-absorbing submodule of M_1

(b) Let $a_1Ra_2Rm = 0$ with $a_1a_2 \notin (N_1 : M_1)$ and $a_1m \notin N_1$ and $a_2m \notin N_1$ for $a_1, a_2 \in R$ and $m \in M_1$. Suppose $a_1Ra_2M_2 \neq 0$. Hence there exists $m_2 \in M_2$ such that $a_1Ra_2m_2 \neq 0$ and therefore $(0, 0) \neq a_1Ra_2(m, m_2) \subseteq a_1Ra_2Rm \times a_1Ra_2Rm_2 = (a_1, 1)(R\times R)(a_2, 1)(R\times R)(m, m_2) \subseteq N_1 \times M_2$. Since $N_1 \times M_2$ is a weakly 2-absorbing submodule of $M_1 \times M_2$ we have $(a_1, 1)(a_2, 1) \in (N_1 \times M_2 : M_1 \times M_2)$ or $(a_1, 1)(m, m_2) \in N_1 \times M_2$ or $(a_2, 1)(m, m_2) \in N_1 \times M_2$. Hence $a_1a_2 \in (N_1 : M_1)$ or $a_1m \in N_1$ or $a_2m \in N_1$ a contradiction. Hence $a_1Ra_2M_2 = 0$.

(2) $\Rightarrow$ (1).

Let $a_1, a_2 \in R$ and $(m_1, m_2) \in M_1 \times M_2$ such that $(0, 0) \neq (a_1, a_1)(R \times R)(a_2, a_2)(R \times R)(m_1, m_2) \subseteq N_1 \times M_2$. If $0 \neq a_1Ra_2Rm_1$ then $0 \neq a_1Ra_2Rm_1 \subseteq N_1$ and N_1 a weakly 2-absorbing submodule of M_1 gives $a_1a_2 \in (N_1 : M_1)$ or $a_1m_1 \in N_1$ or $a_2m_2 \in N_1$. Hence $(a_1, a_1)(a_2, a_2) \in (N_1 \times M_2 : M_1\times M_2)$ or $(a_1, a_1)(m_1, m_2) \in N_1\times M_2$ or $(a_2, a_2)(m_1, m_2) \in N_1\times M_2$. Thus $N_1 \times M_2$ is a weakly 2-absorbing submodule of $M_1 \times M_2$. If $a_1Ra_2Rm_1 = 0$, then $a_1Ra_2Rm_2 \neq 0$ and therefore $a_1Ra_2M_2 \neq 0$. By b. $a_1a_2 \in (N_1 : M_1)$ or $a_1m_1 \in N_1$ or $a_2m_2 \in N_1$. Thus $(a_1, a_1)(a_2, a_2) \in (N_1 \times M_2 : M_1 \times M_2)$ or $(a_1, a_1)(m_1, m_2) \in N_1 \times M_2$ or $(a_2, a_2)(m_1, m_2) \in N_1 \times M_2$. Hence $N_1 \times M_2$ is a weakly 2-absorbing submodule of $M_1 \times M_2$. □

Proposition 5.8. *Let N_i be a submodule of an R_i-module M_i, for $i = 1, 2, 3$. If N is a weakly 2-absorbing submodule of $M_1 \times M_2 \times M_3$, then $N = \{(0, 0, 0)\}$ or N is a 2-absorbing submodule of $M_1 \times M_2 \times M_3$.*

Proof. Suppose that N is a weakly 2-absorbing submodule of $M_1 \times M_2 \times M_3$ that is not 2-absorbing . We will show that $N = \{(0, 0, 0)\}$. Now suppose that $N_1 \times N_2 \times N_3 \neq \{0\} \times \{0\} \times \{0\}$. Thus $N_i \neq \{0\}$, for some $i = 1, 2, 3$. We claim that $N_1 \neq \{0\}$. There exists $m_1 \in N_1$ such that $m_1 \neq 0$. To show that $N_2 = M_2$ or $N_3 = M_3$. Assume that $N_2 \neq M_2$ and $N_3 \neq M_3$. Thus there exist $m_2 \in M_2$ and $m_3 \in M_3$ such that $m_2 \notin N_2$ and $m_3 \notin N_3$. Since $(1, 0, 1)(1, 1, 0)(m_1, m_2, m_3) = (m_1, 0, 0) \neq (0, 0, 0)$, we have $(0, 0, 0) \neq (1, 0, 1)(R_1 \times R_2 \times R_3)(1, 1, 0)(R_1 \times R_2 \times R_3)(m_1, m_2, m_3) \subseteq N_1 \times N_2 \times N_3$. Now, because $N_1 \times N_2 \times N_3$ is a weakly 2-absorbing submodule of $M_1 \times M_2 \times M_3$, we have $(1, 0, 1)(1, 1, 0) \in (N_1 \times N_2 \times N_3 : M_1 \times M_2 \times M_3)$ or $(1, 0, 1)(m_1, m_2, m_3) \in N_1 \times N_2 \times N_3$ or $(1, 1, 0)(m_1, m_2, m_3) \in N_1 \times N_2 \times N_3$. Hence $m_2 \in N_2$ or $m_3 \in N_3$ a contradiction. Therefore $N = N_1 \times M_2 \times N_3$ or $N = N_1 \times N_2 \times M_3$. If $N = N_1 \times M_2 \times N_3$, then $(0, 1, 0) \in (N_1 \times M_2 \times N_3 : M_1 \times M_2 \times M_3)$. By Proposition 3.11, $\{0\} \times M_2 \times \{0\} = (0, 1, 0)^2 N \subseteq (N : N_1 \times M_2 \times N_3) = \{(0, 0, 0)\}$, which is a contradiction. Hence $N = \{(0, 0, 0)\}$. □

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