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Second Order Parallel Tensor on Almost Kenmotsu Manifolds

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Abstract. Let M be an almost Kenmotsu manifold of dimension $2n+1$ having non-vanishing ξ-sectional curvature such that $tr\ell > -2n-2$. We prove that any second order parallel tensor on M is a constant multiple of the associated metric tensor and obtained some consequences of this. Vector fields keeping curvature tensor invariant are characterized on M.

1. Introduction

In 1923, Eisenhart [13] proved that if a positive definite Riemannian manifold (M, g) admits a second order parallel symmetric covariant tensor other than the constant multiple of metric tensor, then it is reducible. In 1926, Levy [18] proved that a second order parallel symmetric non-singular tensor in a space of constant curvature

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is a constant multiple of the metric tensor. Using Ricci identities, Sharma in [24] gives a global approach to the Eisenhart problem, and generalized the Levy's theorem. This problem was studied by the same author in contact geometry [25, 26, 27] on different manifolds, for example for K-contact manifolds in [26]. Since then many geometers have investigated the Eisenhart problem on various contact manifolds: nearly Sasakian[30], P-Sasakian [7, 29], f-Kenmotsu manifold [4], $N(\kappa)$-quasi Einstein manifold [6], 3-dimensional normal paracontact geometry [1], contact manifolds having non-vanishing ξ-sectional curvature [14], (κ, μ)-contact metric manifold [19], almost Kenmotsu manifolds [34] and 3-dimensional non-cosymplectic normal almost contact pseudo-metric manifold of non-vanishing ξ-sectional curvature [32].

In contact geometry, Kenmotsu manifolds, introduced by Kenmotsu in [17], are one of the important classes of manifolds. Such manifolds were observed to be normal. Let $(M, \varphi, \xi, \eta, g)$ be an almost Kenmotsu structure (see Section 2) on a $(2n+1)$-dimensional differentiable manifold. The purpose of this paper is to study second order parallel tensors on M under certain conditions. Throughout the paper, we suppose that the almost Kenmotsu manifold is of dimension $2n+1$. Denoting the Ricci tensor by S, the tensor $\frac{1}{2}\pounds_\xi \varphi$ by h, where $\pounds$ denotes the Lie differentiation, and the operator $R(\cdot, \xi)\xi$ by ℓ, we prove the following.

Theorem 1.1. *Let α be a second order symmetric parallel tensor, and A be the $(1,1)$-tensor metrically equivalent to α on an almost Kenmotsu manifold M. The following hold*

(i) $A\xi = \alpha(\xi, \xi)\xi$, *if M has non-vanishing ξ-sectional curvature;*

(ii) $trA = \alpha(\xi, \xi) - tr(A(h^2 - 2\varphi h - \varphi(\nabla_\xi h))) + \alpha(\xi, \xi)S(\xi, \xi)$;

(iii) $tr(A\ell) = \alpha(\xi, \xi)S(\xi, \xi)$,

where tr denotes the trace.

Since the Ricci tensor S is a second order tensor, we have:

Corollary 1.1. *If an almost Kenmotsu manifold M is Ricci symmetric, and if M has non-vanishing ξ-sectional curvature, then*

(i) $Q\xi = -(2n + trh^2)\xi$;

(ii) *the scalar curvature* $r = \|Q\xi\|^2 - 2n - tr(h^2) - tr(Q(h^2 - 2\varphi h - \varphi(\nabla_\xi h)))$;

(iii) $tr(Q\ell) = \|Q\xi\|^2$,

where Q is the Ricci operator determined by $S(X,Y) = g(QX, Y)$.

Theorem 1.2. *Let M be an almost Kenmotsu manifold having non-vanishing ξ-sectional curvature such that $tr\ell > -2n-2$. The second order parallel tensor on M is a constant multiple of the associated metric tensor.*

Corollary 1.2. *If M is an almost Kenmotsu manifold having non-vanishing ξ-sectional curvature such that $tr\ell > -2n-2$ is Ricci symmetric, then it is Einsteinian.*

Since $tr\ell = S(\xi,\xi) = -2n$ and the ξ-sectional curvature $K(\xi, X) = -1$ for Kenmotsu manifold, we have the following.

Corollary 1.3. *Any Ricci symmetric Kenmotsu manifold is Einsteinian.*

Corollary 1.4. *An affine Killing vector field on M which has non-vanishing ξ-sectional curvature such that $tr\ell > -2n-2$ is homothetic.*

Blair, Koufogiorgos and Papantoniou [3] introduced (κ,μ)-nullity distributions on a contact metric manifold generalizing the notion of κ-nullity distributions by defining

$$N_p(\kappa,\mu) = \{Z \in T_pM : R(U,V)Z = \kappa[g(V,Z)U - g(U,Z)V] \\ + \mu[g(V,Z)hU - g(U,Z)hV]\},$$

for any $p \in M$, where $\kappa, \mu \in \mathbb{R}$.

Corollary 1.5. *A second order parallel tensor on an almost Kenmotsu manifold with $\xi \in N(\kappa,\mu)$ is a constant multiple of the associated metric tensor.*

The above corollary has been proved by Wang and Liu in [34]. Recently, Dileo-Pastore [12] introduced $(\kappa,\mu)'$-nullity distribution on an almost Kenmotsu manifold which is defined by

$$N_p(\kappa,\mu)' = \{Z \in T_pM : R(U,V)Z = \kappa[g(V,Z)U - g(U,Z)V] \\ + \mu[g(V,Z)h'U - g(U,Z)h'V]\},$$

for any $p \in M$, where $h' = h \circ \varphi$ and $\kappa, \mu \in \mathbb{R}$. Here we recall the following results due to Dileo-Pastore [11, 12].

Lemma 1.1.([12, Proposition 4.1]) *Let M be an almost Kenmotsu manifold such that $h' \neq 0$ with $\xi \in N(\kappa,\mu)'$. Then $\kappa < -1$, $\mu = -2$ and $spect(h') = \{0, \lambda, -\lambda\}$ with 0 as simple eigenvalue and $\lambda = \sqrt{-1-k}$.*

Lemma 1.2.([11, Theorem 6]) *Let M be a locally symmetric almost Kenmotsu manifold such that $h' \neq 0$ and $R(X,Y)\xi = 0$ for any $X, Y \in \mathcal{D}$, where $\mathcal{D} = ker(\eta)$. Then, M is locally isometric to $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$.*

Lemma 1.3.([12, Corollary 4.2]) *Let M be an almost Kenmotsu manifold such that $h' \neq 0$ with $\xi \in N(\kappa,\mu)'$. Then M is locally symmetric if and only if $spect(h') = \{0, 1, -1\}$ that is if and only if $k = -2$.*

Lemma 1.4.([12, Proposition 4.3]) *Let M be an almost Kenmotsu manifold such that $h' \neq 0$ with $\xi \in N(\kappa,\mu)'$. Then the ξ-sectional curvature satisfies*

$$K(\xi, X) = \begin{cases} \kappa - 2\lambda, & if X \in [\lambda]' \\ \kappa + 2\lambda, & if X \in [-\lambda]', \end{cases}$$

where $[\lambda]'$ denotes the eigenspace of h' related to the eigenvalue λ.

We use these to prove:

Theorem 1.3. *Let M be an almost Kenmotsu manifold with $\xi \in N(\kappa,\mu)'$ and $h' \neq 0$. If M admits a second order parallel tensor, then either the second order parallel tensor is a constant multiple of the associated metric tensor or M is locally isometric to $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$.*

In [34], Wang and Liu proved the above theorem in another way. Now waving the hypothesis non-vanishing ξ-sectional curvature in Theorem 1.2 by $\nabla_\xi h = 0$, we prove:

Theorem 1.4. *Let M be an almost Kenmotsu manifold with $\nabla_\xi h = 0$ such that $tr\ell > -2n-2$. Then the second order parallel tensor on M is a constant multiple of the associated metric tensor.*

In [23] Naik et al. proved that every vector field which leaves the curvature tensor invariant are Killing in a $(\kappa,\mu)'$-almost Kenmotsu manifold with $h' \neq 0$ and $\kappa \neq -2$. Here, as an application of Theorem 1.4, we prove the following.

Theorem 1.5. *Let M be an almost Kenmotsu manifold with $\nabla_\xi h = 0$ such that $tr\ell > -2n-2$. Then every vector field keeping curvature tensor invariant are homothetic.*

The *m-Bakry-Emery Ricci tensor* is a natural extension of the Ricci tensor to smooth metric measure spaces and is given by

$$S_f^m = S + \mathrm{Hess} f - \frac{1}{m} df \otimes df,$$

where f is a smooth function on M and m is an integer such that $0 < m \leq \infty$. If S_f^m is a constant multiple of the metric g, then the Riemannian manifold (M,g) is called *m-quasi-Einstein manifold* (see [5, 15] and the references therein). Now applying Theorem 1.2, Theorem 1.3 and Theorem 1.4, we deduce the following statement.

Theorem 1.6. *Let M be an almost Kenmotsu manifold either having non-vanishing ξ-sectional curvature such that $tr\ell > -2n-2$ or $\xi \in N(\kappa,\mu)'$ and $h' \neq 0$ or $\nabla_\xi h = 0$ such that $tr\ell > -2n-2$. Then the m-Bakry-Emery Ricci tensor $S_f^m = S + Hess f - \frac{1}{m} df \otimes df$ is parallel if and only if M is m-quasi-Einstein manifold.*

A *Ricci soliton* on a Riemannian manifold (M,g) is defined by

$$\pounds_V g + 2S + 2\rho g = 0,$$

where V is a smooth vector field and ρ is a constant. In the context of contact geometry and paracontact geometry, Ricci solitons are studied in [8, 9, 10, 20, 21, 22, 28, 31, 33]. In the similar vein as Theorem 1.6. we state the following.

Theorem 1.7. *Let M be an almost Kenmotsu manifold either having non-vanishing ξ-sectional curvature such that $tr\ell > -2n-2$ or $\xi \in N(\kappa,\mu)'$ and $h' \neq 0$ or $\nabla_\xi h = 0$ such that $tr\ell > -2n-2$. Then the the second order tensor $\pounds_V g + 2S$ is parallel if* *and only if M admits a Ricci soliton.*

2. Preliminaries

An *almost contact metric structure* on a $(2n+1)$-Riemannian manifold M is a quadruple (φ, ξ, η, g), where φ is an endomorphism, ξ a global vector field, η a 1-form and g a Riemannian metric, such that

$$\varphi^2 = -I + \eta \otimes \xi, \qquad \eta(\xi) = 1, \tag{2.1}$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y). \tag{2.2}$$

We easily obtain from (2.1) that $\varphi\xi = 0$ and $\eta \circ \varphi = 0$ (see [2]).

A manifold M with (φ, ξ, η, g) structure is said to be an *almost contact metric manifold*. We define the *fundamental 2-form* on $(M, \varphi, \xi, \eta, g)$ by $\Phi(X, Y) = g(X, \varphi Y)$. If the 1-form η is closed and $d\Phi = 2\eta \wedge \Phi$, then M is said to be an *almost Kenmotsu manifold* [16]. It is well known that a normal almost Kenmotsu manifold is a Kenmotsu manifold.

The two tensor fields $\ell := R(\cdot, \xi)\xi$ and $h : \frac{1}{2}\pounds_\xi \varphi$ are known to be symmetric and satisfy

$$\ell\xi = 0 \quad h\xi = 0, \quad trh = 0, \quad h\varphi + \varphi h = 0. \tag{2.3}$$

Further, one has the following formulas:

$$\nabla_\xi \varphi = 0, \tag{2.4}$$

$$\nabla_X \xi = -\varphi^2 X - \varphi h X, \tag{2.5}$$

$$(\nabla_\xi h)X = -\varphi X - 2hX - \varphi h^2 X - \varphi R(X, \xi)\xi, \tag{2.6}$$

$$R(X, \xi)\xi - \varphi R(\varphi X, \xi)\xi = 2(\varphi^2 X - h^2 X), \tag{2.7}$$

$$tr\ell = S(\xi, \xi) = -2n - trh^2. \tag{2.8}$$

3. Proof of Theorems

Now we prove the results stated in Section 1.

Proof of Theorem 1.1. Suppose that α is a symmetric $(0,2)$-tensor and A be the $(1,1)$-tensor metrically equivalent to α, that is, $g(AX, Y) = \alpha(X, Y)$. Note that $\nabla\alpha = 0$ implies $\nabla A = 0$, and so

$$R(X, Y)AZ = AR(X, Y)Z.$$

Therefore

$$g(R(X, Y)Z, AW) = -g(R(X, Y)AW, Z) = -g(R(X, Y)W, AZ).$$

Thus we get

$$\begin{aligned} g(R(Z,AW)X,Y) &= g(R(X,Y)Z,AW) \\ &= -g(R(X,Y)W,AZ) = -g(R(W,AZ)X,Y). \end{aligned}$$

Setting $X = Z = \xi$ in the above equation, we get

$$g(R(\xi,AW)\xi,Y) + g(R(W,A\xi)\xi,Y) = 0. \tag{3.1}$$

For $W = \xi$ and $Y = A\xi$, equation (3.1) becomes

$$g(R(\xi,A\xi)\xi,A\xi) = 0. \tag{3.2}$$

Since the ξ-sectional curvature $K(\xi,X)$ is non-vanishing, equation (3.2) implies that $A\xi = f\xi$, for some scalar function f on M. As $g(\xi,\xi) = 1$, we have

$$f = g(A\xi,\xi) = \alpha(\xi,\xi).$$

This proves (i). Applying φ to (2.6) shows that

$$R(\xi,X)\xi = h^2X - \varphi^2X - 2\varphi hX - \varphi(\nabla_\xi h)X. \tag{3.3}$$

Using (3.3) in (3.1), we get

$$g(h^2AW + AW - \eta(AW)\xi - 2\varphi hAW - \varphi(\nabla_\xi h)AW, Y) + g(R(W,A\xi)\xi,Y) = 0.$$

If $\{e_i\}$ is a local orthonormal basis, then putting $W = Y = e_i$ in the above equation and summing it over i, leads to

$$tr(h^2A) + trA - \alpha(\xi,\xi) - 2tr(\varphi hA) - tr(\varphi(\nabla_\xi h)A) + \alpha(\xi,\xi)S(\xi,\xi) = 0.$$

This gives (ii). Now plugging X by AX in (3.3) and then contracting with respect to X gives

$$tr(A\ell) = tr(\varphi(\nabla_\xi h)A) - trA + \alpha(\xi,\xi) + 2tr(\varphi hA) - tr(h^2A).$$

Using this in (ii) yields (iii). This finishes the proof. □

Proof of Corollary 1.1. As the Ricci tensor S is parallel, it follows from Theorem 1.1 that

$$Q\xi = S(\xi,\xi)\xi. \tag{3.4}$$

Now (i) follows from (2.8). Note that from (3.4), we have

$$S(\xi,\xi)S(\xi,\xi) = g(Q\xi,Q\xi) = \|Q\xi\|^2.$$

Hence (ii) and (iii) follows directly from (ii) and (iii) of Theorem 1.1. □

Proof of Theorem 1.2. Let $\{e_i, \varphi e_i, \xi\}_{i=1}^n$ be a local orthonormal basis such that $he_i = \lambda_i e_i$. Then $h\varphi e_i = -\lambda_i \varphi e_i$. Hence as $h\xi = 0$, we have

$$trh^2 = 2\sum_{i=1}^{n} \lambda_i^2.$$

But by hypothesis, $tr\ell = -2n - trh^2 > -2n - 2$, and so $trh^2 < 2$. Therefore $\sum_{i=1}^n \lambda_i^2 < 1$ which means $\lambda_i^2 < 1$ for each i. This fact will be used in the rest of analysis for this theorem.

Let α be a $(0,2)$-tensor such that $\nabla\alpha = 0$, and A be the dual $(1,1)$-type tensor which is metrically equivalent to α, that is, $\alpha(X,Y) = g(AX,Y)$. We will analyse the symmetric and anti-symmetric cases of second order tensor separately.

First, suppose that α is symmetric. Then from item (i) of Theorem 1.1, we get

$$A\xi = \alpha(\xi,\xi)\xi. \tag{3.5}$$

To show $\alpha(\xi,\xi)$ is constant, we differentiate it along X to obtain

$$\begin{aligned} X(\alpha(\xi,\xi)) &= 2\alpha(\nabla_X\xi,\xi) = 2g(-\varphi^2X - \varphi hX, A\xi) \\ &= 2\alpha(\xi,\xi)g(-\varphi^2X - \varphi hX, \xi) = 0. \end{aligned}$$

Now differentiating (3.5) along X yields

$$A(-\varphi^2X - \varphi hX) = \alpha(\xi,\xi)(-\varphi^2X - \varphi hX).$$

Replacing X by φX, it follows that

$$A(\varphi X - hX) = \alpha(\xi,\xi)(\varphi X - hX). \tag{3.6}$$

Now putting $X = e_i$ and $X = \varphi e_i$ in (3.6) respectively gives

$$A(\varphi e_i) - \lambda_i A(e_i) = \alpha(\xi,\xi)\{\varphi e_i - \lambda_i e_i\}, \tag{3.7}$$

and

$$\lambda_i A(\varphi e_i) - A(e_i) = \alpha(\xi,\xi)\{\lambda_i \varphi e_i - e_i\}. \tag{3.8}$$

Multiplying λ_i to (3.7) and then subtracting it with (3.8) shows

$$(\lambda_i^2 - 1)A(e_i) = \alpha(\xi,\xi)(\lambda_i^2 - 1)e_i.$$

Similarly, we can find

$$(\lambda_i^2 - 1)A(\varphi e_i) = \alpha(\xi,\xi)(\lambda_i^2 - 1)\varphi e_i.$$

Since $\lambda_i^2 < 1$, we have $A(e_i) = \alpha(\xi,\xi)e_i$ and $A(\varphi e_i) = \alpha(\xi,\xi)\varphi e_i$ for every $i = 1,\dots,n$. But $A\xi = \alpha(\xi,\xi)\xi$, and so

$$AX = \alpha(\xi,\xi)X,$$

for any $X \in TM$, that is, α is a constant multiple of g. Now suppose that α is skew-symmetric. Note that $\nabla\alpha = 0$ implies

$$g(R(X,Y)AZ,W) = g(AR(X,Y)Z,W).$$

The skew-symmetry of α then gives

$$g(R(X,Y)AZ,W) + g(R(X,Y)Z,AW) = 0. \tag{3.9}$$

For $X = A^2\xi$, $Y = Z = \xi$ and $W = A\xi$, we find

$$g(R(A^2\xi,\xi)\xi,A^2\xi) = 0. \tag{3.10}$$

From (3.10), the hypothesis $K(\xi,X)$ is non-vanishing imply that $A^2\xi = f\xi$. Clearly,

$$f = -g(A\xi,A\xi) = -\|A\xi\|^2. \tag{3.11}$$

One can easily verify the constancy of f by differentiating (3.11) along any vector field and getting the derivative 0. Now differentiating $A^2\xi = -\|A\xi\|^2\xi$ along φX yields

$$A^2(\varphi X - hX) = -\|A\xi\|^2(\varphi X - hX).$$

Next, we take $X = e_i$ and φe_i successively in the above equation and argue as before in order to obtain $A^2 e_i = -\|A\xi\|^2 e_i$ and $A^2(\varphi e_i) = -\|A\xi\|^2\varphi e_i$, for every $i = 1,2,\ldots,n$. So, $A^2X = -\|A\xi\|^2X$ for any X orthogonal to ξ. Hence, we obtain

$$A^2 = -\|A\xi\|^2 I.$$

If $\|A\xi\| \neq 0$, then $J = \|A\xi\|^{-1}A$ defines a Kaehlerian structure on M leading to a contradiction that M is odd dimensional. Thus $\|A\xi\| = 0$, and so $A\xi = 0$. Differentiating it along φX gives

$$A(\varphi X - hX) = 0.$$

Now, as before we put $X = e_i$ and φe_i successively in the above equation to conclude that $AX = 0$, for any $X \perp \xi$. Since $A\xi = 0$, we obtain $AX = 0$ for any $X \in TM$. This completes the proof. □

Proof of Corollary 1.4. A vector field V such that $\pounds_V\nabla = 0$ is called an affine Killing vector field. Note that $\pounds_V\nabla = 0$ is equivalent to $\nabla(\pounds_V g) = 0$. Now the result follows from Theorem 1.2. □

Proof of Corollary 1.5. Let M be an almost Kenmotsu manifold with $\xi \in N(\kappa,\mu)'$, that is

$$R(U,V)\xi = \kappa\{\eta(V)U - \eta(U)V\} + \mu\{\eta(V)hU - \eta(U)hV\}, \tag{3.12}$$

for all $U,V \in TM$. It then follows from Dileo-Pastore [12] that $k = -1$ and $h = 0$. Hence (3.12) gives $tr\ell = -2n$, $K(\xi,X) = -1$ and so the conclusion follows from Theorem 1.2. □

Proof of Theorem 1.3. Observe that, if X is an eigenvector of h with eigenvalue λ, and thus $h\varphi X = -\lambda\varphi X$, then $X + \varphi X$ is an eigenvector of h' with eigenvalue $-\lambda$, while $X - \varphi X$ is eigenvector with eigenvalue λ. Thus, it follows that h and h' admit the same eigenvalues.

First suppose that $\kappa \neq -2$. Note that, if X is such that $X \perp \xi$ and $hX = \lambda X$, then from Lemma , we see λ is different from $+1$ and -1. Thus Lemma implies that $K(\xi, X)$ is non-vanishing. If the second order parallel tensor α is symmetric, then part (i) of Theorem 1.1 shows that

$$A\xi = \alpha(\xi,\xi)\xi. \tag{3.13}$$

That $\alpha(\xi,\xi)$ is constant, can be verified by differentiating it and getting the derivative equal to 0. Now differentiating (3.13) along X and φX, where $X \perp \xi$ and $hX = \lambda X$, gives

$$(\lambda^2 - 1)A(X) = \alpha(\xi,\xi)(\lambda^2 - 1)X,$$

and

$$(\lambda^2 - 1)A(\varphi X) = \alpha(\xi,\xi)(\lambda^2 - 1)\varphi X.$$

Since λ is different from $+1$ and -1, we obtain

$$AX = \alpha(\xi,\xi)X,$$

for any $X \in TM$, that is, α is a constant multiple of g. Now suppose that α is skew-symmetric. Then as in the proof of Theorem 1.2, we obtain $A^2\xi = -\|A\xi\|^2$, where $\|A\xi\|^2$ is constant. Differentiating it along X and φX successively, where $X \perp \xi$ and $hX = \lambda X$, gives

$$(\lambda^2 - 1)A^2(X) = -\|A\xi\|^2(\lambda^2 - 1)X,$$

and

$$(\lambda^2 - 1)A^2(\varphi X) = -\|A\xi\|^2(\lambda^2 - 1)\varphi X.$$

Thus, we obtain

$$A^2 = -\|A\xi\|^2 I.$$

Proceeding the similar manner as in the Theorem 1.2 one gets $AX = 0$ for any $X \in TM$.

Now suppose that $\kappa = -2$ and $\xi \in N(\kappa,\mu)'$, that is,

$$R(U,V)\xi = \kappa\{\eta(V)U - \eta(U)V\} + \mu\{\eta(V)h'U - \eta(U)h'V\}.$$

Then Lemma shows M is locally symmetric. Now it follows from Lemma that M is locally isometric to $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$, since $R(X,Y)\xi = 0$ for any $X, Y \in \mathcal{D}$. This completes the proof. □

Proof of Theorem 1.4. Let $\{e_i, \varphi e_i, \xi\}_{i=1}^n$ be a local orthonormal basis as considered in Theorem 1.2. If the second order parallel tensor α is symmetric, then

$$g(R(X,Y)Z, AW) + g(R(X,Y)W, AZ) = 0.$$

Putting $Y = Z = W = \xi$ in above, we get

$$g(R(X,\xi)\xi, A\xi) = 0. \tag{3.14}$$

Note that (2.6) takes the form

$$R(X,\xi)\xi = \varphi^2 X + 2\varphi h X - h^2 X. \tag{3.15}$$

Using (3.15) in (3.14), and putting $X = e_i$ and φe_i successively in the resulting equation gives

$$(1+\lambda_i^2)g(e_i, A\xi) - 2\lambda_i g(\varphi e_i, A\xi) = 0,$$

and

$$(1+\lambda_i^2)g(\varphi e_i, A\xi) - 2\lambda_i g(e_i, A\xi) = 0,$$

from which we obtain

$$\{(1+\lambda_i^2)^2 - 4\lambda_i^2\}g(e_i, A\xi) = 0,$$

and

$$\{(1+\lambda_i^2)^2 - 4\lambda_i^2\}g(\varphi e_i, A\xi) = 0.$$

Since λ_i is different from $+1$ and -1, we get $g(X, A\xi) = 0$ for any $X \perp \xi$. Hence $A\xi = \alpha(\xi,\xi)\xi$. Now arguing in the similar manner as in Theorem 1.2, one can conclude that

$$AX = \alpha(\xi,\xi)X,$$

for any $X \in TM$, that is, α is a constant multiple of g. If α is skew-symmetric, then we have equation (3.9). Putting $Y = Z = \xi$ and $W = A\xi$ in (3.9), we find

$$g(R(X,\xi)\xi, A^2\xi) = 0.$$

Then using (3.15) in above, and putting $X = e_i$ and φe_i successively we get $g(X, A\xi) = 0$ for any $X \perp \xi$. Hence $A\xi = \alpha(\xi,\xi)\xi$, and similar to the proof of Theorem 1.2, we obtain $AX = 0$ for any $X \in TM$. This finishes the proof. □

Proof of Theorem 1.5. Let $\{e_i, \varphi e_i, \xi\}_{i=1}^n$ be a local orthonormal basis such that $he_i = \lambda_i e_i$. The hypothesis $tr\ell > -2n-2$ shows that $\sum_{i=1}^n \lambda_i^2 < 1$ which means $\lambda_i^2 < 1$, that is, $\lambda_i \neq (+1,-1)$ for each i.

Now the condition $\pounds_V R = 0$ implies

$$(\pounds_V g)(R(X,Y)Z, W) + (\pounds_V g)(R(X,Y)W, Z) = 0.$$

Let G be a $(1,1)$-tensor field defined by $g(GX, Y) = (\pounds_V g)(X,Y)$. Then

$$g(R(X,Y)Z, GW) + g(R(X,Y)W, GZ) = 0. \tag{3.16}$$

Taking $Y = Z = W = \xi$ in (3.16), we get

$$g(R(X,\xi)\xi, G\xi) = 0.$$

Using (3.15) in above, and putting $X = e_i$ and φe_i successively in the resulting equation gives

$$(1+\lambda_i^2)g(e_i, G\xi) - 2\lambda_i g(\varphi e_i, G\xi) = 0,$$

and

$$(1+\lambda_i^2)g(\varphi e_i, G\xi) - 2\lambda_i g(e_i, G\xi) = 0,$$

from which we obtain

$$\{(1+\lambda_i^2)^2 - 4\lambda_i^2\}g(e_i, G\xi) = 0,$$

and

$$\{(1+\lambda_i^2)^2 - 4\lambda_i^2\}g(\varphi e_i, G\xi) = 0.$$

Since λ_i is different from $+1$ and -1, we get $g(X, G\xi) = 0$ for any $X \perp \xi$. Hence $G\xi = g(G\xi, \xi)\xi$. Now putting $Y = Z = \xi$ in (3.16), using (3.15), and taking $X = e_i$ and φe_i successively, we obtain

$$\{(1+\lambda_i^2)^2 - 4\lambda_i^2\}(g(G\xi, \xi)g(e_i, W) - g(Ge_i, W) = 0,$$

and

$$\{(1+\lambda_i^2)^2 - 4\lambda_i^2\}(g(G\xi, \xi)g(\varphi e_i, W) - g(G\varphi e_i, W) = 0.$$

So that, we have $GX = g(G\xi, \xi)X$ for any $X \perp \xi$, and hence

$$G = g(G\xi, \xi)I,$$

that is,

$$\pounds_V g = 2fg$$

for some function f. Thus V is a conformal vector field, and we have

$$(\pounds_V S)(Y, Z) = -(2n-1)(\nabla_Y df)Z - (\text{div grad}f)g(X, Y).$$

Since $\pounds_V R = 0$ implies $\pounds_V S = 0$, the above equation gives $\nabla df = 0$, and hence

$$\nabla(df \otimes df) = 0.$$

Since $df \otimes df$ is a $(0,2)$-tensor, it follows from Theorem 1.2 that $df \otimes df = cg$, for some constant c. Thus

$$(Yf)\text{grad}f = cY, \tag{3.17}$$

which for $Y = \text{grad}f$ gives $\|\text{grad}f\|^2\text{grad}f = c\ \text{grad}f$. Now, if $\{e_i\}$ is an orthonormal basis then putting $Y = e_i$ in (3.17), taking inner product with e_i and summing over i yields

$$\|\text{grad}f\|^2 = c(2n+1).$$

Consequently, we obtain f is constant and hence V is homothetic. □

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