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The Universal Property of Inverse Semigroup Equivariant KK-theory

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Abstract. Higson proved that every homotopy invariant, stable and split exact functor from the category of C^*-algebras to an additive category factors through Kasparov's KK-theory. By adapting a group equivariant generalization of this result by Thomsen, we generalize Higson's result to the inverse semigroup and locally compact, not necessarily Hausdorff groupoid equivariant setting.

1. Introduction

In [3], Cuntz noted that if F is a homotopy invariant, stable and split exact functor from the category of separable C^*-algebras to the category of abelian groups then Kasparov's KK-theory acts on F, that is, every element of $KK(A,B)$ induces a natural map $F(A) \to F(B)$. Higson [4], on the other hand, developed Cuntz' findings further and proved that every such functor F factorizes through the category $\mathbf{K}$ consisting of separable C^*-algebras as the object class and KK-theory together with the Kasparov product as the morphism class, that is, F is the composition $\hat{F} \circ \kappa$ of a universal functor κ from the class of C^*-algebras to $\mathbf{K}$ and a functor $\hat{F}$ from $\mathbf{K}$ to abelian groups.

In [12], Thomsen generalized Higson's findings to the group equivariant setting by replacing everywhere in the above statement algebras by equivariant algebras, $*$-homomorphisms by equivariant $*$-homomorphisms and KK-theory by equivariant KK-theory (the proof is however far from such a straightforward replacement). Meyer [9] used a different approach in generalizing Higson's result, and generalized it to the setting of action groupoids $G \ltimes X$.

In this note we extend Higson's universality result to the inverse semigroup equivariant setting.

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More specifically, the following will be shown. Let G be a countable unital inverse semigroup and denote by $\mathbf{C}^*$ the category consisting of (ungraded) separable G-equivariant C^*-algebras as objects and G-equivariant $*$-homomorphisms as morphisms. Denote by $\mathbf{Ab}$ the category of abelian groups. A functor F from $\mathbf{C}^*$ to $\mathbf{Ab}$ is called *stable* if $F(\varphi)$ is an isomorphism for every G-equivariant corner embedding $\varphi : A \to A \otimes \mathcal{K}$, where A is an G-C^*-algebra, $\mathcal{K}$ denotes the compacts on some separable Hilbert space, and $A \otimes \mathcal{K}$ is a G-algebra where the G-action may be arbitrary and not necessarily diagonal. Similarly, F is called *homotopy invariant* if any two homotopic morphisms $\varphi_0, \varphi_1 : A \to B$ in $\mathbf{C}^*$ satisfy $F(\varphi_0) = F(\varphi_1)$, and *split exact* if F turns every short split exact sequence in $\mathbf{C}^*$ canonically into a short split exact sequence in $\mathbf{Ab}$.

Let $\mathbf{K^G}$ the denote the category consisting of separable G-equivariant C^*-algebras as object class and the G-equivariant KK-theory group $KK^G(A,B)$ as the morphism set between two objects A and B; thereby, composition of morphisms is given by the Kasparov product ($g \circ f := f \otimes_B g$ for $f \in KK^G(A,B)$ and $g \in KK^G(B,C)$). The one-element of the ring $KK^G(A,A)$ is denoted by 1_A.

$\mathbf{K^G}$ is an additive category $\mathbf{A}$, that means, the homomorphism set $\mathbf{A}(A,B)$ is an abelian group and the composition law is bilinear. We call a functor F from $\mathbf{C}^*$ into an additive categtory $\mathbf{A}$ homotopy invariant, stable and split exact if the functor $B \mapsto \mathbf{A}(F(A), F(B))$ enjoys these properties for every object A. This notion is justified by the fact that this property is equivalent to saying that F itself satisfies these properties in the sense introduced above for $\mathbf{Ab}$, see the properties (i)-(iii) in Higson [4], page 269, or Lemma 12.2.

Let κ denote the functor from $\mathbf{C}^*$ to $\mathbf{K^G}$ which is identical on objects and maps a morphism $g : A \to B$ to the morphism $g_*(1_A) \in KK^G(A,B)$. Then we have the following proposition, theorems and corollary.

Proposition 1.1. *Let G be a countable unital inverse semigroup. Let A be an object in $\mathbf{C}^*$. Then $B \mapsto KK^G(A,B)$ is a homotopy invariant, stable and split exact covariant functor from $\mathbf{C}^*$ to $\mathbf{Ab}$.*

Theorem 1.2. *Let G be a countable unital inverse semigroup. Let F be a homotopy invariant, stable and split exact covariant functor from $\mathbf{C}^*$ to $\mathbf{Ab}$. Let A be an object in $\mathbf{C}^*$ and d an element in $F(A)$. Then there exists a unique natural transformation ξ from the functor $B \mapsto KK^G(A,B)$ to the functor F such that $\xi_A(1_A) = d$.*

Theorem 1.3. *Let G be a countable unital inverse semigroup. Let F be a homotopy invariant, stable and split exact covariant functor from $\mathbf{C}^*$ to an additive category $\mathbf{A}$. Then there exists a unique functor $\hat{F}$ from $\mathbf{K^G}$ to $\mathbf{A}$ such that $F = \hat{F} \circ \kappa$, where $\kappa : \mathbf{C}^* \to \mathbf{K^G}$ denotes the canonical functor.*

Corollary 1.4. *The above results are also valid for countable non-unital inverse semigroups G and for countable discrete groupoids G.*

A very brief overview of this note is as follows. (We give further short summaries at the beginning of each section).

In Section 2 we briefly recall the definitions of inverse semigroup equivariant KK-theory. In Section 4 we prove Proposition 1.1. In Section 8 we establish a Cuntz picture of KK-theory. The core of how to associate homomorphisms in the image of a homotopy invariant, stable and split-exact functor F to Kasparov elements is explained in Section 9. Finally Theorem 1.2, Theorem 1.3 and Corollary 1.4 are proved in Sections 11, 12 and 13, respectively.

Sections 4–11 present an adaption of the content of Thomsen's paper [12]; this goes mostly without saying. Section 12 is essentially taken from Higson's paper [4].

We finally would like to clarify the implications of our results to groupoid equivariant KK-theory. Note that there exists a natural transition between inverse semigroups and groupoids by Paterson [10]. Usually these groupoids are non-Hausdorff topological spaces, and thus our results cover a subclass of non-Hausdorff topological groupoid equivariant KK-theory. The equivariant KK-theory of topological, not necessarily Hausdorff groupoids is defined in Le Gall [8]. The G-actions used in this paper, see Def. 2.2, correspond exactly to the G-actions used by Le Gall, and are based on $C_0(X)$-algebras, which go back to Kasparov and his $C_0(X)$-G-equivariant KK-theory $\mathcal{R}KK^G$ in [7]. See also [2] for more details on this translation.

Actually, all our results of this paper work for all topological, not necessarily Hausdorff groupoids equivariant KK-theory groups by minor and easy adption.

Indeed, let $\mathcal{G}$ be a locally compact groupoid with base space X. At first we may consider it as a discrete inverse semigroup S by adjoining a zero element to $\mathcal{G}$, i.e. set $S := \mathcal{G} \cup \{0\}$.

A $\mathcal{G}$-action α on A is then fiber-wise just like an inverse semigroup S-action on A (the zero element $0 \in S$ acts always as zero), with the additional property that it is continuous in the sense that it forms a map $\alpha : s^*A \to r^*A$. We cannot, as in inverse semigroup theory, say that $\alpha_{ss^{-1}}(A)$ is a subalgebra of A ($s \in S$), because this instead we would interpret as a fiber $A_{ss^{-1}}$ of A. But all computations done for inverse semigroups would be the same if we did it for a groupoid on fibers. That is why we need only take care that every introduced $\mathcal{G}$-action is continuous.

But the introduced actions, or similar constructions are just:

(1) Cocycles: The Definition 5.1 has to be replaced by the analogous Definition 1.5 below.

(2) Unitization: One replaces Definition 3.3 by the corresponding definition of [8].

(3) Direct sum, internal, external tensor product: It is clear that these constructions are also continuous for groupoids.

(4) For an element $[T, \mathcal{E}] \in KK^{\mathcal{G}}(A, B)$ one has the condition that the bundle $g \mapsto g(T_{s(g)}) - T_{r(g)} \in \mathcal{K}(\mathcal{E}_{r(g)})$ is in $r^*\mathcal{K}(\mathcal{E})$. Here one has also additionally to check continuity.

(5) One has also to consider $C_0(X)$-structure of the base space X of the groupoid, which recall is homomorphims $C_0(X) \to \mathcal{ZM}(A)$ (center of the multiplier

algebra of A). Any possible necessary computations one makes analogously as we do it for the restricted G-action $E \to \mathcal{ZM}(A)$ to the idempotent elements E of the inverse semigroup by replacing $e \in E$ by $f \in C_0(X)$.

Definition 1.5. Let (A, α) be a $\mathcal{G}$-algebra. Set the $\mathcal{G}$-action on $\mathcal{L}_A(A) \cong \mathcal{M}(A)$ to be $\overline{\alpha} := \mathrm{Ad}(\alpha)$. An α-cocycle is a unitary u in $r^*\big(\mathcal{M}(A)\big)$ such that

$$u_{gh} = \overline{\alpha}_g(u_h)$$

in $\mathcal{M}(A)_{r(g)}$ for all $g, h \in \mathcal{G}$ with $s(g) = r(h)$.

In this way it is (almost) clear that the results of this paper hold also in the locally compact, not necessarily Hausdorff groupoid equivariant setting.

Corollary 1.6. *Let $\mathcal{G}$ be a locally compact, not necessarily Hausdorff groupoid. Then $KK^{\mathcal{G}}$ is the universal stable, homotopy invariant and split exact category deduced from the category of $\mathcal{G}$-equivariant, separable, ungraded C^*-algebras.*

2. Equivariant KK-theory

Our reference for inverse semigroup equivariant KK-theory is [1]. We shall however exclusively work with its slightly adapted variant called compatible K-theory, as in [2], but denote it by KK^G rather than $\widehat{KK^G}$ as in [2]. All (exterior) tensor products are however ordinary as in [1] and are not forced to be $C_0(X)$-balanced as in [2]! (The internal tensor products are automatically $C_0(X)$-balanced.) For convenience of the reader we completely recall the basic definitions.

Definition 2.1. Let G denote a countable unital inverse semigroup. The involution in G is denoted by $g \mapsto g^{-1}$ (determined by $gg^{-1}g = g$ and $g^{-1}gg^{-1} = g^{-1}$). A semigroup homomorphism is said to be *unital* if it preserves the identity $1 \in G$. To include also semigroups with a zero element, we insist that a semigroup homomorphism preserves also the zero element $0 \in G$ if it exists.

We denote the set of idempotent elements of G by E.

Definition 2.2. A G-algebra (A, α) is a $\mathbb{Z}/2$-graded C^*-algebra A with a unital semigroup homomorphism $\alpha : G \to \mathrm{End}(A)$ such that α_g respects the grading and $\alpha_{gg^{-1}}(x)y = x\alpha_{gg^{-1}}(y)$ for all $x, y \in A$ and $g \in G$.

Throughout we shall identify the C^*-algebras $\mathcal{L}_A(A)$ (adjoint-able operators on the Hilbert A-module A) and $\mathcal{M}(A)$ (multiplier algebra of A) by a well-known $*$-isomorphism. We denote the set of idempotent elements of G by E.

Definition 2.3. A G-Hilbert B-module $\mathcal{E}$ is a $\mathbb{Z}/2$-graded Hilbert module over a G-algebra (B, β) endowed with a unital semigroup homomorphism $G \to \mathrm{Lin}(\mathcal{E})$ (linear maps on $\mathcal{E}$) such that U_g respects the grading and

(a) $\langle U_g(\xi), U_g(\eta)\rangle = \beta_g(\langle \xi, \eta\rangle)$

(b) $U_g(\xi b) = U_g(\xi)\beta_g(b)$

(c) $U_{gg^{-1}}(\xi)b = \xi\beta_{gg^{-1}}(b)$

for all $g \in G, \xi, \eta \in \mathcal{E}$ and $b \in B$.

Lemma 2.4. *In the last definition, automatically $U_{gg^{-1}}$ is a self-adjoint projection in the center of the algebra $\mathcal{L}(\mathcal{E})$.*

Proof. For a positive approximate unit $(b_i) \subseteq B$ we compute

$$\langle U_{gg^{-1}}\xi, \eta\rangle \approx \langle \xi\beta_{gg^{-1}}(b_i), \eta\rangle = \beta_{gg^{-1}}(b_i)\langle \xi, \eta\rangle \approx \beta_{gg^{-1}}\langle \xi, \eta\rangle = \langle \xi, U_{gg^{-1}}\eta\rangle,$$

so that $U_{gg^{-1}}$ is seen to be self-adjoint. This operator is in the center because

$$(U_{gg^{-1}}T(\xi))b = T(\xi)\beta_{gg^{-1}}(b) = T(\xi\beta_{gg^{-1}}(b)) = (TU_{gg^{-1}}(\xi))b$$

for all $T \in \mathcal{L}(\mathcal{E}), \xi \in \mathcal{E}$ and $b \in B$. □

Definition 2.5. Given a G-Hilbert B-module $(\mathcal{E}, U)$ we turn the C^*-algebra $\mathcal{L}_B(\mathcal{E})$ to a G-algebra under the action $g(T) := U_g T U_{g^{-1}}$ for all $g \in G$ and $T \in \mathcal{L}(\mathcal{E})$.

It is useful to notice that every G-algebra (A, α) is a G-Hilbert module over itself under the inner product $\langle a, b\rangle = a^*b$; so we have all the identities of Definition 2.3 for $U := \beta := \alpha$. Actually Definitions 2.2 and 2.3 are equivalent for C^*-algebras. Hence we note that

Lemma 2.6. *Let (A, α) be a G-algebra. Every $\alpha_e \in \mathcal{L}_A(A) = \mathcal{M}(A)$ for $e \in E$ is a self-adjoint projection in the center of $\mathcal{M}(A)$. The application of α_e is given by multiplication, that is, $\alpha_e(a) = a\alpha_e$ in $\mathcal{M}(A)$.*

Definition 2.7. A $*$-homomorphism $\varphi : (A, \alpha) \to (B, \beta)$ between G-algebras is called *G-equivariant* if $\varphi(\alpha_g(a)) = \beta_g(\varphi(a))$ for all $a \in A, g \in G$.

Definition 2.8. A *G-Hilbert A, B-bimodule* over G-algebras A and B is a G-Hilbert B-module $\mathcal{E}$ equipped with a G-equivariant $*$-homomorphism $\pi : A \to \mathcal{L}(\mathcal{E})$ (the left module multiplication operator).

Definition 2.9. Let A and B be G-algebras. We define a G-equivariant Kasparov A, B-cycle to be an ordinary Kasparov cycle $(\mathcal{E}, T)$ without G-action (see [6, 7]) such that however $\mathcal{E}$ is a G-Hilbert A, B-bimodule and the operator $T \in \mathcal{L}(\mathcal{E})$ satisfies

$$U_g T U_{g^{-1}} - T U_{gg^{-1}} \in \{S \in \mathcal{L}(\mathcal{E}) \,|\, aS, Sa \in \mathcal{K}(\mathcal{E}) \text{ for all } a \in A\} \tag{2.1}$$

for all $g \in G$. The Kasparov group $KK^G(A, B)$ is defined to be the collection of all G-equivariant Kasparov A, B-cycles divided by homotopy induced by G-equivariant Kasparov $A, B[0, 1]$-cycles. (Throughout, $B[0, 1] := B \otimes C([0, 1])$.)

We equip the multiplier algebra of a G-algebra with a G-action as described in Definition 2.5. This is also the continuous extension of the G-action on A to $\mathcal{M}(A)$ in the strict topology. We redundantly emphasize this again:

Definition 2.10. Given a G-algebra (A, α), $\mathcal{L}_A(A) = \mathcal{M}(A)$ becomes a G-algebra under the G-action $\overline{\alpha} : G \to \mathrm{End}(\mathcal{L}_A(A))$ given by $\overline{\alpha_g}(T) := \alpha_g \circ T \circ \alpha_{g^{-1}}$ for all $g \in G$ and $T \in \mathcal{L}_A(A)$.

We also write $\overline{\alpha} : \mathcal{M}(A) \to \mathcal{M}(B)$ for the strictly continuous extension of a $*$-homomomorphism $\alpha : A \to B$ of C^*-algebras.

Definition 2.11. Write $\mathcal{K}$ for the compact operators on a separable Hilbert space. Call a G-algebra (B, β) *stable* if there is a G-equivariant $*$-isomorphism $(B, \beta) \to (B \otimes \mathcal{K}, \beta \otimes \text{trivial})$.

Since every G-algebra can be stabilized, we use the last definition as a convenient way to avoid the cumbersome notation $B \otimes \mathcal{K}$.

Definition 2.12. Let $(\mathcal{E}, U)$ be a G-Hilbert A-module. An operator T in $\mathcal{L}(\mathcal{E})$ is called *G-invariant* if T commutes with the operator $U_g : \mathcal{E} \to \mathcal{E}$ (that is, $T \circ U_g = U_g \circ T$) for all $g \in G$. (Equivalently: $U_g T U_{g^{-1}} = T U_{gg^{-1}}$.)

Note that then T^* automatically commutes also with U_g.

Lemma 2.13. *Let (B, β) be a stable G-algebra. Then there exist G-invariant isometries V_1 and V_2 in $\mathcal{L}_B(B) = \mathcal{M}(B)$ such that $V_1 V_1^* + V_2 V_2^* = 1$.*

Proof. Let $\mathcal{K}$ be the compact operators on $\ell^2(\mathbb{N})$. Write $e_{i,j} \in \mathcal{K}$ for the standard matrix units. We define the operators $V_1, V_2 \in \mathcal{L}_{B \otimes \mathcal{K}}(B \otimes \mathcal{K})$ for example by

$$V_k(b \otimes e_{i,j}) = b \otimes e_{2i+k-2,j}$$

for all $i, j \in \mathbb{N}$, $b \in B$ and $k = 1, 2$. $\square$

Lemma 2.14. *Let (B, β) be stable. A G-invariant unitary $U \in \mathcal{M}(B)$ can be connected to $1 \in \mathcal{M}(B)$ by a G-invariant, strictly continuous unitary path in $\mathcal{M}(B)$.*

Proof. In the non-equivariant case this is for example [5, Lemma 1.3.7]. Its canonical proof works equivariantly without modification. $\square$

Let us point out that we have all the necessary techniques for inverse semigroup equivariant KK-theory that we shall need available: KK^G allows an associative Kasparov product, and $KK^G(A, B)$ is functorial in A and B (see [1]).

From now on all C^*-algebras are assumed to be trivially graded and separable!

3. The Unitization of a G-algebra

We shall later need a unitization of a G-algebra. To this end we cannot simply add a single unit but need to adjoin the whole G-algebra $C^*(E)$ to a given G-algebra. This section is dedicated to describe this.

Definition 3.1. Let $C^*(E)$ denote the universal *abelian* C^*-algebra generated by the free set E of commuting self-adjoint projections. This algebra is endowed with the G-action τ induced by $\tau_g(e) := geg^{-1} \in E$ for $g \in G, e \in E$ which turns it to a G-algebra.

Lemma 3.2. *Let (A, α) be a G-algebra. Given $a \in A$ and $z \in C^*(E)$ write*

$$az := za := \gamma_z(a),$$

where $\gamma : C^*(E) \to \mathcal{M}(A)$ denotes the canonical $*$-homomorphism such that $\gamma_e(a) = \alpha_e(a)$ for all $e \in E, a \in A$.

Then the linear direct sum $A \oplus C^*(E)$ turns to a G-algebra under the operations

$$(a \oplus z)^* := a^* \oplus z^*,$$
$$(a \oplus z) \cdot (b \oplus w) := ab + zb + aw \oplus zw$$

for all $a, b \in A, z, w \in C^*(E)$ and under the diagonal G-action $\alpha \oplus \tau$.

Proof. In this proof, $\oplus$ indicates only a linear sum, and $\oplus^{(C^*)}$ a C^*-direct sum. Put $Z := \mathrm{span}(E) \subseteq C^*(E)$. Of course, Z is a dense $*$-subalgebra of $C^*(E)$.

We leave it to the reader to show that $A \oplus C^*(E)$ is a $*$-algebra. We claim that $A \oplus Z \subseteq A \oplus C^*(E)$ can be equipped with a C^*-norm. Given a finite subset $F \subseteq E$, write $Z_F \subseteq Z \subseteq C^*(E)$ for the finite-dimensional $*$-subalgebra of $C^*(E)$ generated by F. It is sufficient to define a C^*-norm for each single $A \oplus Z_F$, since $A \oplus Z$ is the directed union over all such $*$-algebras $A \oplus Z_F$, and each direct sum $A \oplus Z_F$ must then be topologically closed as Z_F is finite-dimensional.

Since Z_F is a finite-dimensional commutative C^*-algebra, there is an isomorphism $\psi : \mathbb{C}^n \to Z_F$ of C^*-algebras. Assume by induction hypothesis for $0 \le k < n$ that $A \oplus \psi(\mathbb{C}^k) \subseteq A \oplus C^*(E)$ is a C^*-algebra. Let $z = \psi(e_{k+1})$, where $e_{k+1} = 0 \oplus \cdots \oplus 0 \oplus 1_{\mathbb{C}} \in \mathbb{C}^{k+1}$. Multiplication in $A \oplus \psi(\mathbb{C}^{k+1})$ is as follows:

$$(x + \lambda z)(y + \mu z) = xy + \lambda zy + \mu xz + \lambda\mu z$$

for $x, y \in A \oplus \psi(\mathbb{C}^k) \subseteq A \oplus \psi(\mathbb{C}^{k+1})$ and $\lambda, \mu \in \mathbb{C}$. If $\gamma_z \in A$ then

$$\varphi : A \oplus \psi(\mathbb{C}^{k+1}) \to (A \oplus \psi(\mathbb{C}^k)) \oplus^{(C^*)} \mathbb{C} : \varphi(x + \lambda z) = x + \lambda\gamma_z \oplus \lambda$$

defines a $*$-isomorphism ($x \in A \oplus \psi(\mathbb{C}^k), \lambda \in \mathbb{C}$), whence $A \oplus \psi(\mathbb{C}^{k+1})$ is a C^*-algebra.

If $\gamma_z \notin A$ then consider the operator $P \in \mathcal{M}(A \oplus \psi(\mathbb{C}^k))$ defined by $P(x) = xz$. Observe that $P \notin A \oplus \mathbb{C}^k$ (as otherwise P and so γ_z would be in A, since $\psi(\mathbb{C}^k)z = 0$). The injective $*$-homomorphism

$$\varphi : A \oplus \psi(\mathbb{C}^{k+1}) \to \mathcal{M}(A \oplus \psi(\mathbb{C}^k)) : \varphi(x + \lambda z) = x + \lambda P$$

($x \in A \oplus \psi(\mathbb{C}^k), \lambda \in \mathbb{C}$) shows that $A \oplus \psi(\mathbb{C}^{k+1})$ is a C^*-algebra. This completes induction. Thus $A \oplus \psi(\mathbb{C}^n) \cong A \oplus Z_F$ is a C^*-algebra.

Note that the canonical projection $A \oplus Z_F \to Z_F \subseteq C^*(E)$ is a $*$-homomorphism of C^*-algebras and thus contractive. Thus by taking the direct limit we obtain a canonical contractive projection $\overline{A \oplus Z} \to C^*(E)$. Since $A \oplus C^*(E) \subseteq \overline{A \oplus Z}$, we have a contractive projection $A \oplus C^*(E) \to C^*(E)$. By a standard result in the theory of topological vector spaces, $A \oplus C^*(E)$ is complete. Thus $A \oplus C^*(E) = \overline{A \oplus Z}$ is a C^*-algebra.

Finally, a straightforward check shows that $A \oplus C^*(E)$ is a G-algebra. □

Definition 3.3. For a G-algebra (A, α) we define its *unitization* to be the G-algebra $(A^+, \alpha^+) := (A \oplus C^*(E), \alpha \oplus \tau)$ as described in Lemma 3.2.

Because G has a unit, A^+ is unital. Actually, this is the only reason and place where we need a unit in G.

4. The Split-exactness, Stability and Homotopy Invariance of KK^G

The aim of this section is the proof of Proposition 1.1.

Lemma 4.1. *Let D be a G-algebra and*

$$0 \longrightarrow A \xrightarrow{j} B \xrightarrow{f} C \longrightarrow 0$$

an exact sequence of G-algebras. If $[\varphi, \mathcal{E}, T] \in KK^G(D, B)$ such that $f_*(\varphi, \mathcal{E}, T)$ is a degenerate Kasparov cycle in $KK^G(D, C)$, then it is of the form

$$[\varphi, \mathcal{E}, T] = j_*[\varphi', \mathcal{E}', T'],$$

where $[\varphi', \mathcal{E}', T'] \in KK^G(D, A)$ with $\mathcal{E}' = \{\xi \in \mathcal{E} | \langle \xi, \xi \rangle \in j(A)\} \subseteq \mathcal{E}$, $\varphi' = \varphi(\cdot)|_{\mathcal{E}'}$ and $T' = T|_{\mathcal{E}'}$.

Proof. The canonical proof of [11, Lemma 3.2] works verbatim also G-equivariantly. □

Definition 4.2. We recall that $F(-) = KK^G(A, -)$ denotes the functor $F : \mathbf{C}^* \to \mathbf{Ab}$ with $F(B) = KK^G(A, B)$ and $F(f) = f_*(z) = z \otimes_B f_*(1_B)$ for $f \in \mathbf{C}^*(B, C)$ and $z \in KK^G(A, B)$.

Lemma 4.3. *The functor F given by $F(B) = KK^G(A, B)$ from $\mathbf{C}^*$ to the abelian groups is stable. That is, for any G-algebra $(B \otimes \mathcal{K}, \gamma)$ and any minimal projection $e \in \mathcal{K}$ such that the associated corner embedding $\varphi : B \to B \otimes \mathcal{K}$ with $\varphi(b) = b \otimes e$ happens to be a G-equivariant $*$-homomorphism, the map $F(\varphi)$ is invertible.*

Proof. Notice that $F(\varphi) = (\cdot) \otimes_B [\varphi]$, where

$$[\varphi] := \varphi_*(1_B) = [\mathrm{id}_B, B \otimes_\varphi B \otimes \mathcal{K}, 0] = [\mathrm{id}_B, B \otimes e\mathcal{K}, 0] \in KK^G(B, B \otimes \mathcal{K}),$$

where the G-action on $B \otimes e\mathcal{K}$ is given by γ. We propose an inverse element for $[\varphi]$ by

$$z := [m, B \otimes \mathcal{K}e, 0] \in KK^G(B \otimes \mathcal{K}, B),$$

where m is the multiplication operator, the G-action on $B \otimes \mathcal{K}e$ is given by γ, and the B-valued inner product on $B \otimes \mathcal{K}e$ is defined by $\langle x, y\rangle = \varphi^{-1}(x^*y)$. We are done when showing that $[\varphi] \otimes_{B\otimes\mathcal{K}} z = 1_B$ and $z \otimes_B [\varphi] = 1_{B\otimes\mathcal{K}}$ in KK^G. We have

$$[\varphi] \otimes_{B\otimes\mathcal{K}} z = [\mathrm{id}_B, (B \otimes e\mathcal{K}) \otimes_m (B \otimes \mathcal{K}e), 0] = [\mathrm{id}_B, B, 0] = 1_B$$

by the G-Hilbert B, B-module isomorphism determined by

$$b_1 \otimes ek_1 \otimes b_2 \otimes k_2e \mapsto \varphi^{-1}(b_1b_2 \otimes ek_1k_2e)$$

for all $b_i \in B, k_i \in \mathcal{K}$. Similarly, $z \otimes_B [\varphi] = 1_{B\otimes\mathcal{K}}$ is computed by the G-Hilbert $B \otimes \mathcal{K}, B \otimes \mathcal{K}$-module isomorphism given by

$$b_1 \otimes k_1e \otimes b_2 \otimes ek_2 \mapsto b_1b_2 \otimes k_1ek_2. \qquad \square$$

Lemma 4.4. *The functor F given by $F(B) = KK^G(D, B)$ from $\mathbf{C}^*$ to the abelian groups is split-exact. That is, given a split exact sequence*

$$0 \longrightarrow A \xrightarrow{\ j\ } B \underset{s}{\overset{f}{\rightleftarrows}} C \longrightarrow 0$$

its image under F is canonically split-exact.

Proof. Let $\pi : B \to \mathcal{M}(A) = \mathcal{L}_A(A)$ be the standard $*$-homomorphism $\pi(b)(a) = j^{-1}(bj(a))$ associated to the exact sequence and notice that it is G-equivariant. Consider the G-Hilbert A-module $A \oplus A$ with grading $\epsilon(x, y) = (x, -y)$. We have an element $\{j\}^{-1} := [\varphi, A \oplus A, F] \in KK^G(B, A)$, where F is the flip automorphism and $\varphi : B \to \mathcal{L}_A(A \oplus A)$ is given by

$$\varphi(b)(x, y) = (\pi(b)x, (\pi \circ s \circ f)(b)y).$$

Similarly, there is an element $\{s \circ f\}^{\perp} := [\psi, B \oplus B, F'] \in KK^G(B, B)$, where F' is the flip automorphism and $\psi : B \to \mathcal{L}_B(B \otimes B)$ is determined by

$$\psi(b)(x, y) = (bx, (s \circ f)(b)y).$$

It was checked in [7] that $\{s \circ f\}^{\perp} = 1_B - (s \circ f)_*(1_B)$. Lemma 4.1 shows that $\{s \circ f\}^{\perp} = j_*(\{j\}^{-1})$.

Let D be another G-algebra, and $x \in KK^G(D, B)$ be in the kernel of f_*. Then

$$\begin{aligned} x &= x - (s \circ f)_*(x) = x \otimes_B (1_B - (s \circ f)_*(1_B)) \\ &= x \otimes_B j_*(\{j\}^{-1}) = j_*(x \otimes_B \{j\}^{-1}), \end{aligned}$$

which is in the image of j_*. Thus the sequence

$$0 \longrightarrow KK^G(D, A) \xrightarrow{\ j_*\ } KK^G(D, B) \underset{s_*}{\overset{f_*}{\rightleftarrows}} KK^G(D, C) \longrightarrow 0$$

is split exact. □

By Lemmas 4.3 and 4.4 and the evidence of homotopy invariance we obtain the main result of this section:

Corollary 4.5. *Proposition 1.1 is true.*

5. Cocycles

When we shall later introduce a Cuntz picture of Kasparov theory, the corresponding transformation produces a G-action S on a G-Hilbert A-module A, which will be - synthetically - written as $S_g = u_g \circ \alpha_g$, where α denotes the C^*-action on A. This $u_g = S_g \circ \alpha_{g^{-1}}$ will be defined next:

Definition 5.1. Let (A, α) be a G-algebra. An *α-cocycle* is a map $u : G \to \mathcal{M}(A)$ such that the identities

$$\alpha_{gg^{-1}} = u_g^* u_g, \quad u_{gg^{-1}} = u_g u_g^*, \quad u_{gh} = u_g \overline{\alpha_g}(u_h) \tag{5.1}$$

hold in $\mathcal{M}(A)$ for all g and h in G.

Lemma 5.2. *Let u be an α-cocycle.*

(a) *Then we have*

$$\alpha_{gg^{-1}} = u_g^* u_g = u_g u_g^* = u_{gg^{-1}} \in \mathcal{M}(A).$$

In particular, every u_g is a partial isometry and the source and range projection of u_g both agree with $\alpha_{gg^{-1}}$ and are in the center of $\mathcal{M}(A)$.

(b) *In particular, every $u_e = \alpha_e$ is a self-adjoint projection in the center of $\mathcal{M}(A)$ for all $e \in E$.*

(c) *We may replace the second identity of (5.1) by the identity*

$$\overline{\alpha_g}(u_{g^{-1}}) = u_g^*$$

without changing the definition of a cocycle.

Proof. Note that $\alpha_{gg^{-1}}$ is a projection of the center of $\mathcal{M}(A)$. Hence, u_g is a partial isometry by the first identity of (5.1). Using only the identities (5.1), we have

$$\overline{\alpha_g}(u_{g^{-1}}) = u_g^* u_g \overline{\alpha_g}(u_{g^{-1}}) = u_g^* u_{gg^{-1}} = u_g^* u_g u_g^* = u_g^*,$$

which checks Lemma 5.2(c). The second identity of (5.1) is on the other hand easily obtained from this new identity. The identity $u_g^* u_g = u_g u_g^*$ follows now from the first identity of (5.1) and the identity of Lemma 5.2(c) through

$$\begin{aligned} u_g^* u_g &= \alpha_g \alpha_{g^{-1}g} \alpha_{g^{-1}} = \alpha_g u_{g^{-1}}^* u_{g^{-1}} \alpha_{g^{-1}} = \alpha_g \overline{\alpha_{g^{-1}}}(u_g) u_{g^{-1}} \alpha_{g^{-1}} \\ &= \alpha_g \circ \alpha_{g^{-1}} \circ u_g \circ \alpha_g \circ u_{g^{-1}} \circ \alpha_{g^{-1}} = u_g \overline{\alpha}_g(u_{g^{-1}}) = u_g u_g^*. \end{aligned}$$

□

Definition 5.3. Given an α-cocycle u we write $u\alpha u^*$ for the G-action $(u\alpha u^*)_g(a) = u_g\alpha_g(a)u_g^*$ on A.

Definition 5.4. For an α-cocycle u we introduce a G-action δ^u on $M_2(A)$ under the formula

$$\delta_g^u \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha_g(a) & \alpha_g(b)u_g^* \\ u_g\alpha_g(c) & u_g\alpha_g(d)u_g^* \end{pmatrix}.$$

Notice that $\alpha_e(a)u_e^* = \alpha_e(a)\alpha_e = \alpha_e(a)$ for every $e \in E$ by Lemmas 5.2 and 2.6, such that $\delta_e^u = \mathrm{id}_{M_2} \otimes \alpha_e$. With that and Lemma 5.2 it is straighforward to check that δ^u is indeed a G-action.

6. The Isomorphism $u_\#$

In this section we shall see that the objects (A, α) and $(A, u\alpha u^*)$ are isomorphic under a stable functor, where u denotes an α-cocycle.

Definition 6.1. Consider the two corner embeddings $S_A(a) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ and $T_A(a) = \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}$ which define G-equivariant $*$-homomorphisms $S_A : (A, \alpha) \to (M_2(A), \delta^u)$ and $T_A : (A, u\alpha u^*) \to (M_2(A), \delta^u)$.

Definition 6.2. Let F be a stable functor from $\mathbf{C}^*$ to the abelian groups. Then define an isomorphism

$$u_\# := F(T_A)^{-1} \circ F(S_A) : F(A, \alpha) \to F(A, u\alpha u^*).$$

That is, under a stable functor 'the actions α and $u\alpha u^*$ are isomorphic' and we can switch between them via $u_\#$ as we like.

Lemma 6.3. *Consider the stable functor F from $\mathbf{C}^*$ to the abelian groups defined by $F(A) = KK^G(D, A)$. Then the map $u_\#$ from the last definition and its inverse map $u_\#^{-1}$ can be realized by multiplication with the following Kasparov elements:*

$$z := (id_A, (A, \alpha u^*), 0) \in KK^G\big((A, \alpha), (A, u\alpha u^*)\big),$$
$$z^{-1} := (id_A, (A, u\alpha), 0) \in KK^G\big((A, u\alpha u^*), (A, \alpha)\big),$$

respectively, where the occurring Hilbert A-modules are trivially graded and $(\alpha u)_g(a) = \alpha_g(a)u_g$ and $(\alpha u^*)_g(a) = \alpha_g(a)u_g^*$ denote their G-actions, respectively.

Proof. Since $u_\#^{-1} = (S_A)_*^{-1} \circ (T_A)_*$ the claim is that $z^{-1} = [T_A] \otimes_{M_2(A)} [S_A]^{-1}$. The KK^G-inverse $[S_A]^{-1}$ may be represented by

$$[m, M_2(A) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, 0] \in KK^G(M_2(A), A),$$

where m denotes the multiplication operator. On the other hand $[T_A] = [T_A, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} M_2(A), 0] \in KK^G(A, M_2(A))$. Here, the Hilbert modules have trivial grading and the G-actions are given by restriction of δ^u. We have an isomorphism

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} M_2(A) \otimes_{M_2(A)} M_2(A) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} : x \otimes y \mapsto xy$$

of G-Hilbert A, A-bimodules, where the image is also equipped with the restricted δ^u-action. This proves the claim. The case $u_\#$ is proven similarly. □

Lemma 6.4. *Let $\varphi : (A, \alpha) \to (B, \beta)$ be an equivariant $*$-homomorphism. Let u be an α-cocycle and v a β-cocycle such that $v_g \varphi(a) = \varphi(u_g a)$ for all $g \in G$ and $a \in A$. Let F be a stable functor from* $\mathbf{C}^*$ *to* $\mathbf{Ab}$. *Then φ is also an equivariant $*$-homomorphism $\varphi : (A, u\alpha u^*) \to (B, v\beta v^*)$ such that*

$$v_\# \circ F(\varphi) = F(\varphi) \circ u_\# \; : \; F(A, \alpha) \to F(B, v\beta v^*).$$

Proof. Just note that $\mathrm{id}_{M_2} \otimes \varphi : (M_2(A), \delta^u) \to (M_2(B), \delta^v)$ is an equivariant $*$-homomorphism satisfying $(\mathrm{id}_{M_2} \otimes \varphi) \circ S_A = S_B \circ \varphi$ and $(\mathrm{id}_{M_2} \otimes \varphi) \circ T_A = T_B \circ \varphi$. □

7. The Cocyle Set $\mathbb{E}^G(A, B)$

Until Section 10 assume that B is stable (i.e. $B \cong B \otimes \mathcal{K}$)!

The A, B-cocycles defined next will serve as a Cuntz-picture of Kasparov cycles. We shall prove in the next section that they may substitute Kasparov theory. Confer Remark 8.4 for a motivation of the following definition:

Definition 7.1. Let (A, α) and (B, β) be G-algebras (where B is stable). An A, B-*cocycle* is a quadruple

$$(\varphi_\pm, u_\pm) := (\varphi_+, \varphi_-, u_+, u_-) \in \big(\mathrm{Hom}(A, \mathcal{M}(B))\big)^2 \times \big(\mathcal{M}(B)^G\big)^2$$

where

(a) u_+ and u_- denote β-cocycles $G \to \mathcal{M}(B)$,

(b) $\varphi_\pm$ denote G-equivariant $*$-homomorphisms $A \to \big(\mathcal{M}(B), u_\pm \overline{\beta} u_\pm^*\big)$, respectively,

(c) $\varphi_+(a) - \varphi_-(a) \in B$,

(d) $u_{+g} - u_{-g} \in B$

for all a in A and g in G.

Definition 7.2. Two A, B-cocycles $(\varphi_\pm, u_\pm)$ and $(\psi_\pm, v_\pm)$ are *isomorphic* when there exists a G-equivariant automorphism $\gamma \in \mathrm{Aut}(B, \beta)$ such that

$$\big(\overline{\gamma} \circ \varphi_\pm, \overline{\gamma}(u_\pm)\big) = (\psi_\pm, v_\pm).$$

In the rest of the paper we shall identify isomorphic A, B-cocycles.

Definition 7.3. The set of isomorphism classes of A, B-cocycles is denoted by $\mathbb{E}^G(A, B)$.

Definition 7.4. An A, B-cocycles $(\varphi_\pm, u_\pm)$ is called *degenerate* if $\varphi_\pm = 0$. The set of degenerate A, B-cocycles is denoted by $\mathbb{D}^G(A, B)$.

Definition 7.5. Two A, B-cocycles $(\varphi^t_\pm, u^t_\pm)$ $(t = 0, 1)$ are called *homotopic* if there exists an $A, B[0, 1]$-cocycle $(\varphi_\pm, u_\pm)$ such that

$$\big(\overline{\pi_t} \circ \varphi_\pm, \overline{\pi_t}(u_\pm)\big) = \big(\varphi^t_\pm, u^t_\pm\big),$$

where $\pi_t : B[0, 1] \to B$ denotes evaluation at time $t = 0, 1$.

Definition 7.6. For $(\varphi_\pm, u_\pm), (\psi_\pm, v_\pm) \in \mathbb{E}^G(A, B)$ define their *sum* to be

$$(\varphi_\pm, u_\pm) + (\psi_\pm, v_\pm) := (V_1\varphi_\pm V_1^* + V_2\psi_\pm V_2^*,\ V_1u_\pm V_1^* + V_2v_\pm V_2^*) \in \mathbb{E}^G(A, B),$$

where $V_1, V_2 \in \mathcal{M}(B)$ are G-invariant isometries such that $V_1V_1^* + V_2V_2^* = 1$ (see Lemma 2.13).

Lemma 7.7. *Up to homotopy of A, B-cocycles, the last definition of sum of A, B-cocycles does not depend on the choice of the isometries V_1, V_2.*

Proof. Let $W_1, W_2 \in \mathcal{M}(B)$ be another pair of G-invariant isometries such that $W_1W_1^* + W_2W_2^* = 1$. Then $U = W_1V_1^* + W_2V_2^*$ defines a G-invariant unitary in $\mathcal{M}(B)$ such that $UV_i = W_i$ $(i = 0, 1)$. By Lemma 2.14, U may be connected to 1 by a G-invariant, strictly continuous path $(U_t)_t$ in $\mathcal{M}(B)$. Then the cocycle

$$(U_tV_1\varphi_\pm V_1^*U_t^* + U_tV_2\psi_\pm V_2^*U_t^*,\ U_tV_1u_\pm V_1^*U_t^* + U_tV_2v_\pm V_2^*U_t^*)_{t\in[0,1]}$$

in $\mathbb{E}^G(A, B[0, 1])$ yields the desired homotopy. □

Definition 7.8. Let $\mathbb{F}^G(A, B)$ denote the quotient $\mathbb{E}^G(A, B)/\sim$ under the equivalence relation $\sim$ defined by $x_1 \sim x_2$ for $x_1, x_2 \in \mathbb{E}^G(A, B)$ if and only if there exists degenerate $d_1, d_2 \in \mathbb{D}^G(A, B)$ such that $x_1 + d_1$ is homotopic to $x_2 + d_2$. We equip $\mathbb{F}^G(A, B)$ with the addition $[x_1] + [x_2] := [x_1 + x_2]$.

8. The Isomorphism Φ

In this section we shall isomorphically substitute Kasparov theory by its Cuntz picture in form of A, B-cocycles. This transition is given as follows:

Definition 8.1. There is a map

$$\Delta : \mathbb{E}^G(A, B) \to KK^G(A, B), \quad \Delta(\varphi_\pm, u_\pm) := [\varphi, B \oplus B, F],$$

where the G-Hilbert B-module $B \oplus B$ is equipped with the grading $\epsilon(x, y) = (x, -y)$ and the G-action

$$(u_+\beta, u_-\beta)_g(x, y) = (u_{+g}\beta_g(x), u_{-g}\beta_g(y)),$$

F is the flip automorphism, and the operator $\varphi : A \to \mathcal{L}_B(B \oplus B)$ is defined by

$$\varphi(a)(x, y) = (\varphi_+(a)x, \varphi_-(a)y).$$

Lemma 8.2. *The just defined $\Delta(\varphi_\pm, u_\pm)$ is indeed a Kasparov group element.*

Proof. Denoting $u = (u_+, u_-)$, $\gamma = (\beta, \beta)$ and the indicated G-action $(u_+\beta, u_-\beta)$ on $B \oplus B$ by $W = u\gamma$, one has

$$W_{gh} = u_{gh}\gamma_{gh} = u_g\overline{\gamma_g}(u_h)\gamma_{gh} = u_g\gamma_g u_h\gamma_{g^{-1}}\gamma_g\gamma_h = W_g W_h$$

because of the third identity of (5.1) and because $\beta_{g^{-1}g}$ is in the center of $\mathcal{M}(B)$. We have

$$\langle W_g(x, y), W_g(a, b)\rangle = \langle \gamma_g(x, y), u_g^* u_g \gamma_g(a, b)\rangle = \gamma_g\langle (x, y), (a, b)\rangle,$$

because $\gamma_{gg^{-1}} = u_g^* u_g$ by the first identity of the cocycle axioms (5.1). We have $\overline{\gamma_g}(u_{g^{-1}}) = u_g^*$ by Lemma 5.2(c), and thus $W_g W_{g^{-1}} = u_g\gamma_g u_{g^{-1}}\gamma_{g^{-1}} = u_g u_g^*$, which is a self-adjoint projection in $\mathcal{L}_B(B\oplus B)$. By a similar argument, and with condition (b) of Definition 7.1 we get

$$\varphi(\alpha_g(a)) = u_g\overline{\gamma_g}(\varphi(a))u_g^* = u_g\gamma_g\varphi(a)\gamma_{g^{-1}}\gamma_g u_{g^{-1}}\gamma_{g^{-1}} = W_g\varphi_g(a)W_{g^{-1}}.$$

The B-module multiplication on $B \oplus B$ is E-compatible, in other words

$$W_e(\xi)b = u_e\gamma_e(\xi)b = \gamma_e(\xi)b = \xi\gamma_e(b)$$

for $\xi \in B \oplus B, b \in B$ and $e \in E$, because $\gamma_e = u_e$ by Lemma 5.2. A straightforward computation shows that the operator $W_g F W_{g^{-1}} - W_g W_{g^{-1}} F$ is in $B \oplus B$ because $u_{+g} - u_{-g}$ is in the ideal B by Definition 7.1. This verifies Definition 2.9 of a Kasparov cycle. □

Proposition 8.3. *Every element of $KK^G(A, B)$ may be represented in the form $[\varphi, B \oplus B, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}]$ for a certain G-action $S = (S_+, S_-)$ on the Hilbert B-module $B \oplus B$ with grading $\epsilon(x, y) = (x, -y)$, where $S_\pm$ are G-actions on the ungraded Hilbert B-module B. Moreover,*

$$S_g \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} S_{g^{-1}} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} S_g S_{g^{-1}} \in \mathcal{K}(\mathcal{L}_B(B \oplus B)) \cong M_2(B) \tag{8.1}$$

for all $g \in G$.

Proof. Let $(\varphi, \mathcal{E}, T) \in KK^G(A, B)$ be given. Denote the G-action on $\mathcal{E}$ by U. We may assume that $U_g T U_{g^{-1}} - U_{gg^{-1}} T \in \mathcal{K}(\mathcal{E})$. (If G were a group then this would be by Remark 2 on page 156 of Kasparov's paper [7]. But this works also in our setting by a similar proof as suggested by Kasparov but with $\varphi(g) = U_g T U_{g^{-1}} - T U_{gg^{-1}}$

rather than $\varphi(g) = U_g T U_g - T$, and applied to the technical Theorem 1 in [1] rather than the technical Theorem 1.4 in [7].)

Let B_2 denote the G-Hilbert B-module $B \oplus B$ with grading $\epsilon(x, y) = (x, -y)$ and G-action (β, β). Since $(0, B_2, 0) \in KK^G(A, B)$ is degenerate,

$$(\varphi, \mathcal{E}, T) \oplus (0, B_2, 0)$$

is homotopic in KK^G-theory to $(\varphi, \mathcal{E}, T)$. By Kasparov's stabilization theorem (the graded version), there is an isomorphism $\Lambda : \mathcal{E} \oplus B_2 \to B_2$ of graded Hilbert B-modules; we use here the fact that B is stable, and thus $H_B \cong B$, see [5, Lemma 1.3.2]. We define the G-action on B_2 in the image of Λ in such a way that Λ becomes G-equivariant, and denote this new B_2 by B_2'. Hence we may write $[\varphi, \mathcal{E}, T] = [\psi, B_2', T_1]$.

Since the G-action W on B_2' is grading preserving, it must be of the form

$$W_g(x, y) = (S_g x, V_g y),$$

where S and V are G-actions on the ungraded homogeneous parts (B-parts) of B_2'. Hence

$$W_g'(x, y) = (V_g x, S_g y)$$

is another G-action on B_2, and we denote this new B_2 by B_2''. Using $[\psi, B_2', T_1] = [\psi, B_2', T_1] + [0, B_2'', 0]$ (degenerate), and using isomorphisms $B \oplus B \cong B$ on the respective homogeneous parts by Kasparov's stabilization theorem, we may assume that the G-action on B_2' is of the form

$$W_g(x, y) = (S_g x, S_g y),$$

where S is a G-action on the homogeneous B-parts of B_2'.

Identifying $\mathcal{L}_B(B_2') \cong M_2(\mathcal{L}_B(B))$, T_1 takes on the form $T_1 = \begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix}$.

By considering the same homotopies as in the non-equivariant case, see [5, p. 125] (notice that $U_g T_1^n U_{g^{-1}} = (U_g T_1 U_{g^{-1}})^n = T_1^n U_g U_{g^{-1}}$ by Lemma 2.4 and (2.1)), we may assume that $x = y^*$ and $\|x\| \leq 1$. Also, by adding on the degenerate cycle $[0, B_2', 0]$, and performing the same homotopy as in the non-equivariant case, see [5, p. 126], we may assume that $T_1 = \begin{pmatrix} 0 & u \\ u^* & 0 \end{pmatrix}$ for some unitary $u \in \mathcal{M}(B)$.

Define an automorphism $\Theta : B_2 \to B_2$ of Hilbert B-modules by

$$\Theta(x, y) = (u^* x, y),$$

and define a G-action on its image B_2, then denoted by B_2''', in such a way that $\Theta : B_2' \to B_2'''$ becomes G-equivariant. Hence $[\psi, B_2', T_1] = [\vartheta, B_2''', \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}]$. □

Remark 8.4. We use the last proposition as a basis for a Cuntz picture of KK-theory. The S_+-action appearing there we shall define (in the next theorem) to be written as $S_{+g} = u_{+g} \circ \beta_g$ (exactly the G-Hilbert module action appearing in Definition 8.1 and in Definition 7.1(b)), or in other words, we define $S_{+g} \circ \beta_{g^{-1}} =: u_{+g}$, and u_+ turns out to be a β-cocycle. In other words, u_+ encodes the difference between the C^*-action β on B and the Hilbert module action S_+ on B. That is the function of β-cocycles.

Theorem 8.5. *The set $\mathbb{F}^G(A,B)$ is an abelian group, and the map Δ of Definition 8.1 canonically induces an abelian group isomorphism*

$$\Phi : \mathbb{F}^G(A,B) \to KK^G(A,B)$$

by $\Phi([x]) := \Delta(x)$.

Proof. It is clear that Φ is a well-defined map which preserves addition. It will thus be sufficient to show that Φ is bijective.

We are going to show that Φ is surjective. Let us be given an element z in $KK^G(A,B)$ as indicated in Proposition 8.3. Since φ respects grading, it is of the form $\varphi = (\varphi_+, \varphi_-)$. We claim that $\Phi([\varphi_\pm, u_\pm]) = z$, where the β-cocylces $u_\pm$ are defined by $u_{\pm g} := S_{\pm g} \circ \beta_{g^{-1}}$.

To check that u_+ (and similarly u_-) is a β-cocycle, we compute

$$\begin{aligned}\langle u_{+g}x, y\rangle &= \langle S_{+g}\beta_{g^{-1}}x, y\rangle = \langle S_{+g}\beta_{g^{-1}}x, S_{+g}S_{+g^{-1}}y\rangle = \beta_g(\langle \beta_{g^{-1}}x, S_{+g^{-1}}y\rangle)\\ &= \beta_g\beta_{g^{-1}}(x^*) \cdot \beta_g S_{+g^{-1}}(y) = x^* \cdot \beta_g S_{+g^{-1}}(y) = \langle x, \beta_g S_{+g^{-1}}y\rangle\end{aligned}$$

for all x and y in B by Lemma 2.4 and Definitions 2.2 and 2.3, so that

$$u_{\pm g}{}^* = \beta_g \circ S_{\pm g^{-1}}. \tag{8.2}$$

For an idempotent $e \in E$ and all $x, y \in B$ we have $S_{+e}(x)y = x\beta_e(y) = \beta_e(x)y$ by Definitions 2.2 and 2.3, so that we obtain

$$S_{+e} = \beta_e. \tag{8.3}$$

This shows that

$$u_{+g}{}^* u_{+g} = \beta_g S_{+g^{-1}} S_{+g}\beta_{g^{-1}} = \beta_g\beta_{g^{-1}g}\beta_{g^{-1}} = \beta_g\beta_{g^{-1}},$$

the first identity of (5.1). Similarly we get the second identity and the third one computes as

$$u_{+gh} = S_{+gh}\beta_{h^{-1}g^{-1}} = S_{+g}\beta_{g^{-1}}\beta_g S_{+h}\beta_{h^{-1}}\beta_{g^{-1}} = u_{+g}\overline{\beta}_g(u_{+h}).$$

We note that, since $S_{+g^{-1}g} = \beta_{g^{-1}g}$, we have, with $S := (S_+, S_-)$, (8.3) and (8.1),

$$\begin{aligned}&S_g \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} S_{g^{-1}} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} S_g S_{g^{-1}}\\ &= \begin{pmatrix} 0 & S_{+g}\beta_{g^{-1}g}S_{-g^{-1}} - S_{-g}\beta_{g^{-1}g}S_{-g^{-1}} \\ x & 0 \end{pmatrix} = \begin{pmatrix} 0 & (u_{+g} - u_{-g})u_{-g}{}^* \\ x & 0 \end{pmatrix}\end{aligned}$$

is in $M_2(B)$ for a certain obvious but irrelevant x, and thus

$$(u_{+g} - u_{-g})u_{-g}{}^* u_{-g} = (u_{+g} - u_{-g})\beta_{gg^{-1}} = u_{+g} - u_{-g}$$

is in B as required by item (d) of Definition 7.1.

Since φ is G-equivariant, we have $\varphi_\pm(\alpha_g(a)) = S_{\pm g}\varphi_\pm(a)S_{\pm g^{-1}}$. Thus, by (8.2) and (8.3),

$$\begin{aligned}&(u_+\overline{\beta}u_+{}^*)_g\big(\varphi_+(a)\big) = u_{+g}\overline{\beta}_g\big(\varphi_+(a)\big)u_{+g}^*\\&= S_{+g}\circ\beta_{g^{-1}}\circ\beta_g\circ\varphi_+(a)\circ\beta_{g^{-1}}\circ\beta_g\circ S_{+g^{-1}} = S_{+g}\circ\varphi_+(a)\circ S_{+g^{-1}}\\&= \varphi_+(\alpha_g(a)),\end{aligned}$$

which verifies item (b) of Definition 7.1.

Now notice that indeed $\Delta(\varphi_\pm, u_\pm) = z$ (see Definition 8.1), since $u_\pm\beta = S_\pm$ by (8.3). This proves surjectivity of Φ.

We are going to prove injectivity of Φ. Let $(\varphi^i_\pm, u^i_\pm) \in \mathbb{E}^G(A,B)$ for $i = 0,1$. Assume that $\Phi([\varphi^0_\pm, u^0_\pm]) = \Phi([\varphi^1_\pm, u^1_\pm])$. Then there exists a Kasparov cycle $(\sigma, \mathcal{E}, T)$ in $KK^G(A, B[0,1])$ connecting the two cycles $\Delta(\varphi^i_\pm, u^i_\pm)$. We apply the procedure described in the surjectivity proof of Φ (the construction of the preimage of an element) to the cycle $(\sigma, \mathcal{E}, T)$, and end up with an element $(\psi_\pm, v_\pm) \in \mathbb{E}^G(A, B[0,1])$. This is also a homotopy in the sense of Definition 7.5.

Because at the endpoints of the cycle $(\sigma, \mathcal{E}, T)$ we have already the nice form of Definition 8.1, all operations that we perform in Proposition 8.3 for $(\sigma, \mathcal{E}, T)$ are empty at the endpoints, except adding on degenerate cycles and application of the Kasparov stabilization theorem. Thus, at the endpoints of the homotopy $(\psi_\pm, v_\pm)$ we have the following situation. Let $\pi_t : B[0,1] \to B$ be the evaluation map for $t \in [0,1]$. There is a degenerate A,B-cocycle $(0,0,z_\pm) \in \mathbb{D}^G(A,B)$ and an isomorphism $\Lambda : B \oplus B \to B$ of Hilbert B-modules such that

$$\begin{aligned}\overline{\pi}_i\circ\psi_\pm(\cdot) &= \Lambda(\varphi^i_\pm(\cdot)\oplus 0)\Lambda^{-1}\\ \overline{\pi}_i(v_{\pm g})\circ\beta_g &= \Lambda(u^i_{\pm g}\circ\beta_g \oplus z_{\pm g}\circ\beta_g)\Lambda^{-1}\end{aligned}$$

for $i = 0,1$.

Our next goal is to make Λ G-equivariant by multiplying it with some unitary path. Define isometries $W_1, W_2 \in \mathcal{M}(B)$ by $W_1(x) = \Lambda(x,0)$ and $W_2(y) = \Lambda(0,y)$, so that

$$\Lambda(x,y) = W_1(x) + W_2(y)$$

and $W_1W_1^* + W_2W_2^* = 1$. Choose G-invariant isometries $V_1, V_2 \in \mathcal{M}(B)$ as in Lemma 2.13. Consider the unitary $U = V_1W_1^* + V_2W_2^* \in \mathcal{M}(B)$ and - as the unitary group of $\mathcal{M}(B)$ is connected by Lemma 2.14 - connect it to $1 \in \mathcal{M}(B)$ by a unitary path $(U_t)_{t\in[0,1]}$ in $\mathcal{M}(B)$. Then

$$\big(\, U_t(\overline{\pi}_i\circ\psi_\pm(\cdot))U_t^*,\; U_t\circ\overline{\pi}_i(v_g)\circ\beta_g\circ U_t^*\circ\beta_{g^{-1}}\,\big)_{t\in[0,1]} \in \mathbb{E}^G(A, B[0,1])$$

defines a path of A, B-cocycles which connects the two elements

$$\big(\overline{\pi}_i \circ \psi_\pm, \overline{\pi}_i(v)\big), \qquad \big(\varphi^i_\pm, u^i_\pm\big) + \big(0,0,z_\pm\big)$$

in $\mathbb{E}^G(A,B)$ by Definitions 7.5 and 7.6. Together with the homotopy $(\psi_\pm, v_\pm)$ and Definition 7.8 this shows that $[\varphi^0_\pm, u^0_{\pm g}] = [\varphi^1_\pm, u^1_{\pm g}]$. □

Definition 8.6. For an equivariant $*$-homomorphism $\lambda : B \to C$ (where B and C are stable) define an abelian group homomorphism

$$\lambda_* : \mathbb{F}^G(A,B) \to \mathbb{F}^G(A,C) : \lambda_*[x] = \Phi^{-1}(\lambda_* \Phi([x])).$$

Lemma 8.7. *Let $\lambda_1 : (B_1, \beta_1) \to (C_1, \gamma_1)$ be a unital $*$-homomorphism of unital G-algebras B_1, C_1. Let $\lambda := \lambda_1 \otimes id : (B, \beta) := (B_1 \otimes \mathcal{K}, \beta_1 \otimes id) \to (C, \gamma) := (C_1 \otimes \mathcal{K}, \gamma_1 \otimes id)$. Then one has*

$$\lambda_*[\varphi_\pm, u_\pm] = [\overline{\lambda} \circ \varphi_\pm, \overline{\lambda}(u_\pm)].$$

Proof. (Sketch) We have $\overline{\lambda}(\beta_{gg^{-1}}) = \overline{\lambda}(\overline{\beta}_g(1)) = \overline{\gamma}_g \overline{\lambda}(1) = \gamma_{gg^{-1}}$, so that it is easy to see that $\overline{\lambda}(u_\pm)$ are γ-cocycles.

By unitality of λ_1, $B \otimes_B C \cong C$ as G-Hilbert C-modules via $x \otimes y \mapsto \lambda(x)y$. Under this isomorphism, $\varphi_\pm \otimes \mathrm{id}_C : A \to \mathcal{L}_C(B \otimes_B C)$ turns to $\overline{\lambda} \circ \varphi_\pm$, and the G-Hilbert C-module actions $u_\pm \beta \otimes \gamma$ on $B \otimes_B C$ turn to $\overline{\lambda}(u_\pm)\gamma$. □

9. The Map Ψ

In this section we shall see how elements of Kasparov theory $KK^G(A,B)$ - in its form of A, B-cocylces (Cuntz picture cycles) in $\mathbb{E}^G(A,B)$ by the isomorphism Φ if we like - induce homomorphisms in $\mathrm{Hom}(F(A), F(B))$ for every split exact, homotopy invariant stable functor F from $\mathbf{C}^*$ to $\mathbf{Ab}$. This goes back to Cuntz in its core, see [3].

Definition 9.1. Given an A, B-cocycle $x = (\varphi_\pm, u_\pm) \in \mathbb{E}^G(A,B)$ we define a C^*-algebra

$$A_x := \{(a,m) \in A \oplus \mathcal{M}(B) | \, \varphi_+(a) = m \text{ modulo } B\}$$

with two G-actions ($+$ and $-$)

$$\Gamma^\pm = (\alpha, u_\pm \overline{\beta} u^*_\pm).$$

A Γ^+-cocycle u for (A_x, Γ^+) is given by

$$u_g(a,m) = (\alpha_{gg^{-1}}(a), u_{-g} u_{+g}^* m)$$

for $a \in A, m \in \mathcal{M}(B)$.

Definition 9.2. We sloppily use $\Gamma^\pm$ also to denote the G-action on B by restricting $\Gamma^\pm$ to B, that is, $\Gamma^\pm := u_\pm \overline{\beta} u^*_\pm$ on B.

Lemma 9.3. *Definition 9.1 is valid.*

Proof. We show that u is a Γ^+-cocycle. By Lemma 5.2, and since ${u_-}_g^* {u_-}_g$ is in the center of $\mathcal{M}(B)$, we have

$$\begin{aligned} &u_g^* u_g(a,m) = (\alpha_{gg^{-1}}(a), {u_+}_g {u_-}_g^* {u_-}_g {u_+}_g^* m) \\ &= (\alpha_{gg^{-1}}(a), {u_+}_g {u_+}_g^* {u_-}_g^* {u_-}_g m) = (\alpha_{gg^{-1}}(a), \beta_{gg^{-1}} m \beta_{gg^{-1}}) \\ &= (\alpha_{gg^{-1}}(a), {u_+}_{gg^{-1}} \overline{\beta}_{gg^{-1}}(m) {u_+}_{gg^{-1}}{}^*) = \Gamma^+_{gg^{-1}}(a,m). \end{aligned}$$

This shows that $u_g^* u_g = \Gamma^+_{gg^{-1}}$, and so the first identity of (5.1). The second identity of (5.1) is left to the reader and the third one is computed as follows:

$$\begin{aligned} &u_g \overline{\Gamma}^+_g(u_h)(a,m) = u_g \circ \Gamma^+_g \circ u_h \circ \Gamma^+_{g^{-1}}(a,m) \\ &= \big(\alpha_{gg^{-1}ghh^{-1}g^{-1}}(a),\ {u_-}_g {u_+}_g^* {u_+}_g \overline{\beta}_g\big(\ {u_-}_h {u_+}_h^* {u_+}_{g^{-1}} \overline{\beta}_{g^{-1}}(m) {u_+}_{g^{-1}}^*\ \big) {u_+}_g^*\ \big) \\ &= \big(\alpha_{ghh^{-1}g^{-1}}(a),\ {u_-}_{gh} \overline{\beta}_g\big(\ {u_+}_h^* {u_+}_{g^{-1}} \overline{\beta}_{g^{-1}}(m) {u_+}_{g^{-1}}^*\ \big) {u_+}_g^*\ \big) \\ &= \big(\alpha_{ghh^{-1}g^{-1}}(a),\ {u_-}_{gh} \overline{\beta}_g({u_+}_h)^* {u_+}_g^* \overline{\beta}_{gg^{-1}}(m) {u_+}_g {u_+}_g^*\ \big) \\ &= \big(\alpha_{ghh^{-1}g^{-1}}(a),\ {u_-}_{gh} {u_+}_{gh}^*(m)\ \big) \\ &= u_{gh}(a,m) \end{aligned}$$

with the usual center properties, identities (5.1), and the identity of Lemma 5.2(c).

We show that A_x is invariant under the G-action u. Let $(a,m) \in A_x$, so $\varphi_+(a) - m \in B$. By items (b) and (d) of Definition 7.1 and the identity $\beta_{gg^{-1}} = {u_+}_{gg^{-1}}$ of (5.1) we get modulo B

$$\begin{aligned} &\varphi_+(\alpha_{gg^{-1}}(a)) = {u_+}_{gg^{-1}} \overline{\beta}_{gg^{-1}}\big(\varphi_+(a)\big) {u_+}_{gg^{-1}}^* \equiv \overline{\beta}_{gg^{-1}}(m) \\ &= \beta_{gg^{-1}} m = {u_-}_g {u_-}_g^* m \equiv {u_-}_g {u_+}_g^* m. \end{aligned}$$

This proves that $u_g(a,m)$ is in A_x. $\square$

Definition 9.4. Let $x = (\varphi_\pm, u_\pm) \in \mathbb{E}^G(A,B)$. We have two split exact sequences ($+$ and $-$)

$$0 \longrightarrow (B, \Gamma^\pm) \xrightarrow{\ j\ } (A_x, \Gamma^\pm) \underset{s^\pm}{\overset{p}{\rightleftarrows}} (A, \alpha) \longrightarrow 0, \tag{9.1}$$

where $j(b) = (0,b)$, $p(a,m) = a$ and $s^\pm(a) = (a, \varphi_\pm(a))$.

Let F be a stable, homotopy invariant, split-exact functor from $\mathbf{C}^*$ to $\mathbf{Ab}$. Define an abelian group homomorphism

$$\Psi_x : F(A,\alpha) \to F(B,\beta)$$

by

$$\Psi_x = {u_-}_\#^{-1} \circ F(j)^{-1} \circ \big(u_\# \circ F(s^+) - F(s^-)\big).$$

Notice here that the occurrence of $F(j)^{-1}$ is valid as $u_\#$ alters only the G-action whence $F(p) \circ (u_\# \circ F(s^+) - F(s^-)) = 0$. Observe that $u_\# \circ F(s^+)$ maps into (A_x, Γ^-).

Lemma 9.5. *The definition of Ψ_x is insensitive against homotopy equivalence of x.*

Proof. Let $x = (\varphi_\pm, u_\pm) \in \mathbb{E}^G(A, B[0,1])$ be a homotopy. Let $\pi_t : B[0,1] \to B$ be evaluation at time $t = 0, 1$. Define two end points

$$x_t := \big(\varphi_\pm^{(t)}, u_\pm^{(t)}\big) := \big(\overline{\pi}_t \circ \varphi_\pm, \overline{\pi}_t(u_\pm)\big) \in \mathbb{E}^G(A, B)$$

$(t = 0, 1)$. The exact sequence (9.1) produces a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \big(B, \Gamma^{\pm(t)}\big) \arrow[r, "j^{(t)}"] & \big(A_{x_t}, \Gamma^{\pm(t)}\big) \arrow[r, shift left, "p^{(t)}"] & (A, \alpha) \arrow[l, shift left, "s^{\pm(t)}"] \arrow[r] & 0 \\ 0 \arrow[r] & (B[0,1], \Gamma^\pm) \arrow[r, "j"] \arrow[u, "\pi_t"] & (A_x, \Gamma^\pm) \arrow[r, shift left, "p"] \arrow[u, "\lambda_t"] & (A, \alpha) \arrow[l, shift left, "s^\pm"] \arrow[r] \arrow[u, equal] & 0, \end{tikzcd} \tag{9.2}$$

where $\lambda_t := (\mathrm{id}_A, \overline{\pi}_t)$. Note that

$$\overline{\lambda}_t(u_g) = \big(\alpha_{gg^{-1}}, \overline{\pi}_t(u_- u_+^*)\big) = u_g^{(t)},$$

whence Lemma 6.4 applies to λ_t and the Γ^+-cocycle u. Also, Lemma 6.4 applies to $\overline{\pi}_t(u_\pm) = u_\pm^{(t)}$. Thus by Lemma 6.4 and diagram (9.2) we get

$$\begin{aligned} F(\pi_t) \circ \Psi_x &= F(\pi_t) \circ u_{-\#}^{-1} \circ F(j)^{-1} \circ \big(u_\# \circ F(s^+) - F(s^-)\big) \\ &= u_{-\#}^{(t)-1} \circ F(\pi_t) \circ F(j)^{-1} \circ \big(u_\# \circ F(s^+) - F(s^-)\big) \\ &= u_{-\#}^{(t)-1} \circ F(j^{(t)})^{-1} \circ F(\lambda_t) \circ \big(u_\# \circ F(s^+) - F(s^-)\big) \\ &= u_{-\#}^{(t)-1} \circ F(j^{(t)})^{-1} \circ \big(u_\#^{(t)} \circ F(\lambda_t) \circ F(s^+) - F(\lambda_t) \circ F(s^-)\big) \\ &= u_{-\#}^{(t)-1} \circ F(j^{(t)})^{-1} \circ \big(u_\#^{(t)} \circ F(s^{+(t)}) - F(s^{-(t)})\big) \;=\; \Psi_{x_t}. \end{aligned}$$

By homotopy invariance of F we have $F(\pi_0) = F(\pi_1)$ and thus $\Psi_{x_0} = \Psi_{x_1}$. □

Lemma 9.6. *Let $x, d \in \mathbb{E}^G(A, B)$ where d is degenerate. Then $\Psi_{x+d} = \Psi_x$.*

Proof. Like the proof of Lemma 9.5 the proof is rather insensitive between the group and inverse semigroup case, and it is also similar to the proof of Lemma 9.5, so we omit the details and refer to Thomsen's paper [12]. □

We may summarize Lemmas 9.5 and 9.6 as follows.

Corollary 9.7. *The map Ψ canonically induces a map*

$$\Psi : \mathbb{F}^G(A, B) \to \mathit{Hom}(F(A), F(B))$$

by $\Psi_{[x]} := \Psi_x$ *for* $x \in \mathbb{E}^G(A,B)$.

Lemma 9.8. *For every unital* $*$*-homomorphism* $\lambda : (C,\gamma) \to (D,\delta)$ *(where* C *and* D *are unital) one has*

$$F(\lambda \otimes id_{\mathcal{K}}) \circ \Psi_{[x]} = \Psi_{(\lambda \otimes id_{\mathcal{K}})_*[x]}$$

for $[x] \in \mathbb{E}^G\big((A,\alpha),(C \otimes \mathcal{K}, \gamma \otimes triv)\big)$.

Proof. The proof is similar to the proof of Lemma 9.5, and rather insensitive between the group and inverse semigroup case, and thus we refer to Thomsen's paper [12]. One uses Lemma 8.7. □

10. The Abelian Group Homomorphism Ψ'

In this section we shall define a variation Ψ' of the map Ψ so as that the functoriality of Lemma 9.8 holds also in the non-unital case. It will then follow that Ψ' is an abelian group homomorphism. We shall also remove the stability restriction on B. Also Φ^{-1} will be implemented in order to switch from $\mathbb{E}^G$ to KK^G.

From now on B need not longer be stable!

Definition 10.1. Fix a one-dimensional projection e in $(\mathcal{K}, \mathrm{triv})$ and define $c_A : A \to A \otimes \mathcal{K}$ to be the corner embedding $c_A(a) = a \otimes e$ for all objects A in $\mathbf{C}^*$.

Definition 10.2. Consider the canonical split exact sequence

$$0 \longrightarrow \big(B \otimes \mathcal{K}, \beta \otimes \mathrm{triv}\big) \xrightarrow{j_B} \big(B^+ \otimes \mathcal{K}, \beta^+ \otimes \mathrm{triv}\big) \xrightarrow{p_B} \big(C^*(E) \otimes \mathcal{K}, \tau \otimes \mathrm{triv}\big) \longrightarrow 0,$$

where (B^+, β^+) denotes the G-equivariant unitization of (B,β), see Definition 3.3. Let F be a stable, homotopy invariant, split-exact functor from $\mathbf{C}^*$ to $\mathbf{Ab}$. For every $z \in KK^G(A,B)$ we define an abelian group homomorphism

$$\Psi'_z : F(A,\alpha) \to F(B,\beta)$$

by

$$\Psi'_z = F(c_B)^{-1} \circ F(j_B)^{-1} \circ \Psi_{j_{B*} c_{B*} \Phi^{-1}(z)}.$$

The occurrence of $F(j_B)^{-1}$ is here valid, as

$$F(p_B) \circ \Psi_{j_{B*} c_{B*} \Phi^{-1}(z)} = \Psi_{p_{B*} j_{B*} c_{B*} \Phi^{-1}(z)} = \Psi_{[0]} = 0$$

by Lemma 9.8 and $F(j_B)$ is injective by split-exactness of F.

Lemma 10.3. *For any* $*$*-homomorphism* $\lambda : (B,\beta) \to (C,\gamma)$ *one has*

$$\Psi'_{\lambda_*(z)} = F(\lambda) \circ \Psi'_z.$$

Proof. By Definition 8.6, λ_* commutes with Φ^{-1}, and by Lemma 9.8 we get

$$\begin{aligned}\Psi'_{\lambda_*(z)} &= F(c_C)^{-1} \circ F(j_C)^{-1} \circ \Psi_{j_{C*} c_{C*} \lambda_* \Phi^{-1}(z)} \\ &= F(c_C)^{-1} \circ F(j_C)^{-1} \circ \Psi_{(\lambda^+ \otimes id_{\mathcal{K}})_* j_{B*} c_{B*} \Phi^{-1}(z)} = F(\lambda) \circ \Psi'_z.\end{aligned}$$

□

Lemma 10.4. *The map*

$$\Psi' : KK^G(A,B) \to Hom(F(A,\alpha), F(B,\beta))$$

is an abelian group homomorphism.

Proof. If Ψ' was additive then by the functoriality of Lemma 10.3 the collection of the maps Ψ' would form a natural transformation from the functor $KK^G(A,-) : \mathbf{C}^* \to \mathbf{Ab}$ to the functor $\mathrm{Hom}(F(A), F(-)) : \mathbf{C}^* \to \mathbf{Ab}$. Both functors are stable, homotopy invariant and split-exact by Corollary 4.5. Then [4, Lemma 3.2] states in the non-equivariant case that in such a situation the map Ψ' is automatically additive. The general proof works verbatim also G-equivariantly in our setting. □

Lemma 10.5. *Let (A,α) be a G-algebra. Then*

$$\Psi'_{1_A} = id_{F(A,\alpha)}.$$

Proof. By Definition 8.6 one has

$$j_{A*}c_{A*}\Phi^{-1}(1_A) = \Phi^{-1}(j_{A*}c_{A*}1_A) = \Phi^{-1}([j_A c_A, A^+ \otimes \mathcal{K}, 0]) = [j_A c_A, 0, \nu, \nu]$$

in $\mathbb{E}^G(A, A^+ \otimes \mathcal{K})$, where the last identity may be chosen by Definition 8.1, and where $\nu : G \to \mathcal{M}(A^+ \otimes \mathcal{K})$ is the cocycle $\nu_g := \alpha^+_{gg^{-1}} \otimes \mathrm{id}$.

Consider now Definition 9.4 with respect to $(\varphi_\pm, u_\pm) := (j_A c_A, 0, \nu, \nu)$. We have $\Gamma^+ = \Gamma^- = \alpha^+_{gg^{-1}} \otimes \mathrm{id}$, $u = u_+ = u_- = \nu$ and $u_{-\#} = \mathrm{id}$, $u_\# = \mathrm{id}$. Hence

$$\begin{aligned}
\Psi'_{1_A} &= F(c_A)^{-1} \circ F(j_A)^{-1} \circ \Psi_{[j_A c_A, 0, \nu, \nu]} \\
&= F(c_A)^{-1} \circ F(j_A)^{-1} \circ u_{-\#}^{-1} \circ F(j)^{-1} \circ \big(u_\# \circ F(s^+) - F(s^-)\big) \\
&= F(c_A)^{-1} \circ F(j_A)^{-1} \circ F(j)^{-1} \circ \big(F(s^+) - F(s^-)\big) \;=\; \mathrm{id}_{F(A)}
\end{aligned}$$

as the difference $s^+ - s^- = j \circ j_A \circ c_A$ happens to be a $*$-homomorphism and thus $F(s^+) - F(s^-) = F(s^+ - s^-)$. □

11. The Natural Transformation ξ

In this section we shall show Theorem 1.2.

Definition 11.1. Let F be a stable, homotopy invariant, split-exact functor from $\mathbf{C}^*$ to $\mathbf{Ab}$. Let $d \in F(A,\alpha)$. There is a natural transformation

$$\xi : KK(A,-) \to F(-)$$

defined by

$$\xi_B(z) = \Psi'_z(d)$$

for $z \in KK^G(A,B)$.

That ξ is a natural transformation follows from Definition 8.6, and Lemmas 10.3 and 10.4.

Lemma 11.2. *Consider the maps* Ψ *and*

$$\Psi' : KK^G(A,B) \to Hom\big(KK^G(A,A), KK^G(A,B)\big)$$

developed in Definitions 9.4 and 10.2, respectively, for the homotopy invariant, stable, split-exact functor $F(-) = KK^G(A,-)$ *from* $\mathbf{C}^*$ *to* $\mathbf{Ab}$*. Then*

$$\Psi'_z(1_A) = z$$

for all $z \in KK^G(A,B)$*.*

Proof. Let $x = (\varphi_\pm, u_\pm) \in \mathbb{E}^G(A,B)$, where B is stable. Then we compute

$$\begin{aligned}
&\Psi_x(1_A)\\
&= u_{-\#}^{-1} \circ F(j)^{-1} \circ \big(u_\# \circ F(s^+) - F(s^-)\big)(1_A)\\
&= u_{-\#}^{-1} \circ F(j)^{-1}\big(u_\#[(id_A, \varphi_+), (A_x, \Gamma^+), 0] - [(id_A, \varphi_-), (A_x, \Gamma^-), 0]\big)\\
&= u_{-\#}^{-1} \circ F(j)^{-1}\big([(id_A, \varphi_+), (A_x, \Gamma^+ u^*), 0] - [(id_A, \varphi_-), (A_x, \Gamma^-), 0]\big)\\
&= u_{-\#}^{-1}\big((\varphi_+, \varphi_-), (B \oplus B, (u_+\beta u_-^*, u_-\beta u_-^*)), T\big)\\
&= \big((\varphi_+, \varphi_-), (B \oplus B, (u_+\beta, u_-\beta)), T\big)\\
&= \Delta(x) = \Phi([x]),
\end{aligned}$$

where $B \oplus B$ is equipped with the grading $\epsilon(x,y) = (x,-y)$ and T is the flip operator. Then, with Definitions 8.6 and 10.2,

$$\begin{aligned}
\Psi'_z(1_A) &= F(c_B)^{-1} \circ F(j_B)^{-1} \circ \Psi_{j_{B*}c_{B*}\Phi^{-1}(z)}(1_A)\\
&= F(c_B)^{-1} \circ F(j_B)^{-1} \circ \Phi\big(j_{B*}c_{B*}\Phi^{-1}(z)\big) = z.
\end{aligned}$$

□

Proposition 11.3. *Given* F *and* d *as in Definition 11.1,* ξ *is the only existing natural transformation from* $KK^G(A,-)$ *to* $F(-)$ *such that*

$$\xi_A(1_A) = d.$$

Proof. That $\xi_A(1_A) = d$ follows from Lemma 10.5. It remains to prove uniqueness of ξ.

Now consider another natural transformation $\eta : KK(A,-) \to F(-)$ such that $\eta_A(1_A) = d$. Define $K(-) = KK^G(A,-)$. Denote the Ψ for K by $\Psi^{(K)}$ for clearity.

We have a commuting diagram

$$\begin{array}{ccc}
KK^G(A,C) & \xrightarrow{K(f)} & KK^G(A,D)\\
\downarrow{\scriptstyle \eta_C} & & \downarrow{\scriptstyle \eta_D}\\
F(C) & \xrightarrow{F(f)} & F(D)
\end{array}$$

for all homomorphisms $f \in \mathbf{C}^*(C,D)$. Since $\Psi_{[x]}^{(K)}$ and $\Psi_z^{(K)'}$ are only compositions of such maps $K(f)$, we also have

$$\eta_B \circ \Psi_z^{(K)'} = \Psi'_z \circ \eta_A. \tag{11.1}$$

Thus

$$\eta_B(z) = \eta_B\big(\Psi_z^{(K)'}(1_A)\big) = \Psi'_z\big(\eta_A(1_A)\big) = \Psi'_z(d) = \xi_B(z)$$

by Lemma 11.2. □

Definition 11.1 and Proposition 11.3 sum then up to:

Corollary 11.4. *Theorem 1.2 is true.*

12. The Universality Theorem

In this section we shall deduce Theorem 1.3 as described in [4, Theorem 4.5].

Definition 12.1. A functor $F : \mathbf{C}^* \to \mathbf{A}$ into an additive category is called split-exact, homotopy invariant and stable if the functor $H^A(-) = \mathrm{Hom}(F(A), F(-))$ from $\mathbf{C}^*$ to the abelian groups has these properties for all objects A in $\mathbf{C}^*$.

For convenience of the reader we recall another characterization of split-exact, homotopy invariant, stable functors into additive categories, see [4, p. 269].

Lemma 12.2. *A functor $F : \mathbf{C}^* \to \mathbf{A}$ into an additive category $\mathbf{A}$ is stable, homotopy invariant and split-exact if and only if*

(a) *$F(f)$ is invertible for every corner embedding $f \in \mathbf{C}^*(A, A \otimes \mathcal{K})$,*

(b) *$F(f) = F(g)$ for all homotopic $f, g \in \mathbf{C}^*(A,B)$, and*

(c) *for every split exact sequence*

$$0 \longrightarrow A \xrightarrow{\;j\;} D \underset{s}{\overset{p}{\rightleftarrows}} B \longrightarrow 0$$

the map $F(A) \oplus F(B) \to F(D)$ defined by

$$F(j) \circ p_1 + F(s) \circ p_2$$

is an isomorphism, where p_1, p_2 denotes the projection maps.

Lemma 12.3. *Consider Ψ' for the functor $F(-) = KK^G(A,-)$. Then*

$$\Psi'_w(z) = z \otimes_B w$$

for all $w \in KK^G(B,C)$ and $z \in KK^G(A,B)$.

Proof. By the functoriality of the Kasparov product and Lemma 11.2 we have

$$\begin{aligned}\Psi'_w(z) &= F(c_C)^{-1} \circ F(j_C)^{-1} \circ \Psi_{j_{C*}c_{C*}\Phi^{-1}(z)}(z \otimes_B 1_B)\\ &= F(c_C)^{-1} \circ F(j_C)^{-1} \circ u_{-\#}^{-1} \circ F(j)^{-1} \circ \big(u_\# \circ F(s^+) - F(s^-)\big)(z \otimes_B 1_B)\\ &= z \otimes_B \Psi'_w(1_B) = z \otimes_B w.\end{aligned}$$

Actually, the last Ψ' refers to the functor $F(-) = KK^G(B,-)$. □

Theorem 12.4. *Theorem 1.3 is true.*

Proof. Consider a functor F as in Definition 12.1. Let A be an object in $\mathbf{C}^*$. Apply Definition 11.1 to the split-exact, stable, homotopy invariant functor $H^A : \mathbf{C}^* \to \mathbf{Ab}$ defined by

$$H^A(-) = \mathrm{Hom}(F(A), F(-))$$

and the element $d = 1_{F(A)} \in H^A(A)$.

We obtain a natural transformation

$$\xi^A : KK^G(A,-) \to \mathrm{Hom}(F(A), F(-)). \tag{12.1}$$

Define the functor $\hat{F} : \mathbf{K^G} \to \mathbf{A}$ by

$$\hat{F}(z) = \xi^A_B(z) \tag{12.2}$$

for all $z \in KK^G(A,B)$. By Proposition 11.3,

$$\hat{F}(1_A) = \xi^A_A(1_A) = 1_{F(A)}.$$

Since by definition

$$H^A(f)(w) = F(f) \circ w$$

for $f \in \mathbf{C}^*(B,C)$ and $w \in \mathrm{Hom}(F(A), F(B))$, and Ψ'_z is just a composition of such $H^A(f)$s, notice that

$$\hat{F}(z) = \Psi'_z(1_{F(A)}) = \Psi'_z \circ 1_{F(A)} = \Psi'_z \ \in \ \mathrm{Hom}(F(A), F(B)). \tag{12.3}$$

We compute the functoriality of $\hat{F}$ as follows. Consider $z \in KK^G(A,B)$ and $w \in KK^G(B,C)$. Then with Lemma 12.3, identity (11.1) and (12.3) we compute

$$\begin{aligned}\xi^A_C\big(z \otimes_B w\big) &= \xi^A_C\big(\Psi'_w(z)\big) = \Psi'_w\big(\xi^A_B(z)\big) = \Psi'_w\big(\hat{F}(z)\big)\\ &= \Psi'_w \circ \hat{F}(z) = \Psi'_w(1_{F(A)}) \circ \hat{F}(z) = \hat{F}(w) \circ \hat{F}(z).\end{aligned}$$

Let $f \in \mathbf{C}^*(A,B)$. By (12.3), Definition 8.7 and Lemma 10.3 we have

$$\hat{F}(\kappa(f)) = \hat{F}(f_*(1_A)) = \Psi'_{f_*(1_A)} = F(f) \circ \Psi'_{1_A} = F(f) \circ \hat{F}(1_A) = F(f).$$

We are going to show uniqueness of $\hat{F}$. Let now $\hat{F} : \mathbf{C}^* \to \mathbf{A}$ be any given functor with $\hat{F} \circ \kappa = F$. For $z \in KK^G(A,B)$ and $f \in \mathbf{C}^*(B,C)$ we then have

$$\hat{F}(f_*(z)) = \hat{F}(z \otimes_B f_*(1_B)) = F(f) \circ \hat{F}(z) = H^A(f)\big(\hat{F}(z)\big).$$

Hence (12.2) defines a natural transformation (12.1), which by Proposition 11.3 is uniquely determined. Hence $\hat{F}$ is uniquely determined. □

13. Non-unital Inverse Semigroups

Corollary 13.1 *Corollary 1.4 is true.*

Proof. If G is declared to be a non-unital inverse semigroup (even it may have a unit), then we define G-algebras and KK^G-theory as before, with the only difference that the G-action $\alpha : G \to \mathrm{End}(A)$ on a C^*-algebra is not required to be unital, and similar so for Hilbert modules. Then we adjoin unconditionally a unit 1 to G to obtain $G^+ := G \sqcup \{1\}$ and regard it as a unital inverse semigroup. Then every non-unital G-action can be extended to a unital G^+-action, and every unital G^+-action can be restricted to a non-unital G-action. This one-to-one correspondence shows that the C^*-categories $\mathbf{C}^*_{\mathbf{G}}$ and $\mathbf{C}^*_{\mathbf{G}^+}$ and the KK-theories $\mathbf{K}^{\mathbf{G}}$ and $\mathbf{K}^{\mathbf{G}^+}$ are the same in a trivial way. Corollary 1.4 follows then by applying Proposition 1.1 and Theorems 1.2 and 1.3 to G^+.

Similarly we may view a countable discrete groupoid $\mathcal{G}$ as an inverse semigroup $G := \mathcal{G} \sqcup \{0\}$ by adjoing a zero element 0, which always has to act as the zero operator on C^*-algebras and Hilbert modules as already noted, and where $gh := 0$ in G if $g, h \in \mathcal{G}$ are incomposable. □

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