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Structures Related to Right Duo Factor Rings

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Abstract. We study the structure of rings whose factor rings modulo nonzero proper ideals are right duo; such rings are called *right FD*. We first see that this new ring property is not left-right symmetric. We prove for a non-prime right FD ring R that R is a subdirect product of subdirectly irreducible right FD rings; and that $R/N_*(R)$ is a subdirect product of right duo domains, and $R/J(R)$ is a subdirect product of division rings, where $N_*(R)$ ($J(R)$) is the prime (Jacobson) radical of R. We study the relation among right FD rings, division rings, commutative rings, right duo rings and simple rings, in relation to matrix rings, polynomial rings and direct products. We prove that if a ring R is right FD and $0 \neq e^2 = e \in R$ then eRe is also right FD, examining that the class of right FD rings is not closed under subrings.

1. Introduction

Throughout this note every ring is an associative ring with identity unless otherwise stated. Let R be a ring. We use $N(R)$, $J(R)$, $N_*(R)$, $N^*(R)$, and $W(R)$ to denote the set of all nilpotent elements, Jacobson radical, lower nilradical (i.e., prime radical), upper nilradical (i.e., the sum of all nil ideals), and the Wedder-

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burn radical (i.e., the sum of all nilpotent ideals) of R, respectively. The center of R is denoted by $Z(R)$. It is well-known that $W(R) \subseteq N_*(R) \subseteq N^*(R) \subseteq N(R)$ and $N^*(R) \subseteq J(R)$. $U(R)$ denotes the group of all units in R. The polynomial (resp., power series) ring with an indeterminate x over R is denoted by $R[x]$ (resp., $R[[x]]$). $\mathbb{Z}$ ($\mathbb{Z}_n$) denotes the ring of integers (modulo n). Denote the n by n ($n \geq 2$) full (resp., upper triangular) matrix ring over R by $Mat_n(R)$ (resp., $T_n(R)$). Write $D_n(R) = \{(a_{ij}) \in T_n(R) \mid a_{11} = \cdots = a_{nn}\}$. Use E_{ij} for the matrix with (i,j)-entry 1 and zeros elsewhere. I_n denotes the identity matrix in $Mat_n(R)$. $\prod$ means the direct product. Use $|S|$ to denote the cardinality of a given set S. The characteristic of R is written by $ch(R)$. An element u of R is called right (resp., left) regular if $ur = 0$ (resp., $ru = 0$) for $r \in R$ implies $r = 0$. An element is regular if it is both left and right regular. The monoid of all regular elements in R is denoted by $C(R)$.

This article is motivated by the results in [9]. In Section 2 we study the structure of right FD rings, focusing on the relation among right FD rings, commutative rings and simple rings. We investigate that in several kinds of ring extensions that play important roles in ring theory. In Section 3 we examine the right FD property of polynomial rings, subrings and direct products for given right FD rings.

A ring is called *Abelian* if every idempotent is central. Following Feller [2], a ring is called *right duo* if every right ideal is two-sided. Left duo rings are defined similarly. A ring is called *duo* if it is both left and right duo. Right (left) duo rings are easily shown to be Abelian. A ring is usually called *reduced* if it has no nonzero nilpotents. It is easily checked that a ring R is reduced if and only if $a^2 = 0$ for $a \in R$ implies $a = 0$. Reduced rings are clearly Abelian, but not conversely by [6, Lemma 2].

2. When Factor Rings are Right Duo

In this section we are concerned with the class of rings whose factor rings modulo nonzero proper ideals are right duo. A ring R shall be called *right FD* if R is simple, or else R/I is a right duo ring for every nonzero proper ideal I of R. A left FD ring is can be defined similarly. A ring is called *FD* if it is both right and left FD. There exist many non-simple FD rings as we will see. We first examine the following basic results about right duo rings.

Lemma 2.1.

(1) *Every simple (or right primitive) right duo ring is a division ring.*

(2) *Every prime right (left) duo ring is a domain.*

(3) *The class of right (left) duo rings is closed under factor rings and direct products.*

(4) *If R is a division ring then $D_2(R)$ is a duo ring; but $D_n(A)$ is neither right nor left duo for all $n \geq 3$ over any ring A.*

(5) *Let A be any ring and $n \geq 3$. Then $T_n(A)$ is neither right nor left FD.*

(6) *Let A be any ring and $n \geq 4$. Then $D_n(A)$ is neither right nor left FD.*

(7) *The class of right (left) FD rings is closed under factor rings.*

Proof. (1) Let R be a simple right duo ring and $0 \neq a \in R$. Then $aR = RaR = R$, so that $a \in U(R)$. Thus R is a division ring. Every right primitive right duo ring is a division ring through a simple computation.

(2) Let R be a prime right duo ring and suppose that $ab = 0$ for $a, b \in R$. Then $aRb \subseteq abR = 0$, so that $a = 0$ or $b = 0$. Thus R is a domain.

(3) is obvious.

(4) Take $0 \neq (a_{ij}) \in D_2(R)$. If $a_{ii} \neq 0$ then $(a_{ij}) \in U(D_2(R))$, so that $D_2(R)(a_{ij}) = D_2(R) = (a_{ij})D_2(R)$. Assume $a_{ii} = 0$. Then $a_{12} \neq 0$ and so $D_2(R)(a_{ij}) = \begin{pmatrix} 0 & R \\ 0 & 0 \end{pmatrix} = D_2(R)(a_{ij})D_2(R) = (a_{ij})D_2(R)$. So $D_2(R)$ is duo. Next consider $D_n(A)$ for $n \geq 3$ over any ring A. Then $AE_{(n-1)n}$ is a right ideal of $D_n(A)$, but not two-sided; and AE_{12} is a left ideal of $D_n(A)$, but not two-sided. So $D_n(A)$ is neither right nor left duo.

(5) Note that $T_n(A)$ is not simple. Consider the proper ideal $I = AE_{1n} + AE_{2n} + \cdots + AE_{(n-1)n} + AE_{nn}$ of $T_n(A)$. Then $T_n(A)/I$ is isomorphic to $T_{n-1}(A)$ that is non-Abelian (hence not right duo) since $n \geq 3$. Thus $T_n(A)$ is not right FD.

(6) Note that $D_n(A)$ is not simple. Consider the proper ideal $J = AE_{1n} + AE_{2n} + \cdots + AE_{(n-1)n}$ of $D_n(A)$. Then $D_n(A)/J$ is isomorphic to $D_{n-1}(A)$ that is not right duo by (4) since $n - 1 \geq 3$. Thus $D_n(A)$ is not right FD.

(7) Let R be a right FD ring and I be a proper ideal of R. Consider R/I. If $I = 0$ then $R/0$ is right FD. So assume $I \neq 0$. Let J/I be a proper nonzero ideal of R/I. Then J is a nonzero proper ideal of R. So $(R/I)/(J/I) \cong R/J$ is right duo. Thus R/I is right FD.

The proofs for the left cases of (1)–(7) are similar. □

Right duo rings are right FD by Lemma 2.1(3); but the converse is not true in general by the following. Note that $Mat_n(A)$ cannot be Abelian (hence neither right nor left duo) for $n \geq 2$ over any ring A. If A is simple then $Mat_n(A)$ is simple (hence FD). In the following we consider the right FD property of $T_n(R)$ and $D_n(R)$ for $n = 2$ and $n = 3$, respectively, based on Lemma 2.1(5, 6).

Theorem 2.2. *Let R be a ring and $n \geq 2$.*

(1) *R is simple if and only if $Mat_n(R)$ is right FD if and only if $Mat_n(R)$ is simple.*

(2) *The following conditions are equivalent:*

  (i) *R is a division ring;*

  (ii) *$T_2(R)$ is a right (left) FD ring;*

  (iii) *$D_3(R)$ is a right (left) FD ring.*

(3) *Let R be simple. Then R is a division ring if and only if $D_2(R)$ is right (left) FD.*

Proof. (1) It suffices to show that if $Mat_n(R)$ is right FD then R is simple. Let R be non-simple. Consider a nonzero proper ideal I of R. Then $Mat_n(R)/Mat_n(I)$ is isomorphic to $Mat_n(R/I)$ that is not right duo. So $Mat_n(R)$ is not right FD.

(2) We apply the proof of [9, Theorem 1.10(3)]. (i) $\Rightarrow$ (ii). Let F be a division ring and $R = T_2(F)$. A nonzero proper ideal of R is one of the following: $I_1 = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$, $I_2 = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$, and $I_3 = \begin{pmatrix} 0 & F \\ 0 & F \end{pmatrix}$. Then $R/I_1 \cong F \times F$ and $R/I_2 \cong F \cong R/I_3$, hence they are duo. So R is FD.

(ii) $\Rightarrow$ (i). Suppose that $T_2(R)$ is right FD. Assume that R is not simple. Then $T_2(R)/T_2(M)$ is isomorphic to the non-Abelian (hence not right duo) ring $T_2(R/M)$ for each maximal ideal M of R, entailing that $T_2(R)$ is not right FD. Thus R must be simple. Next consider the proper ideal $I = \begin{pmatrix} 0 & R \\ 0 & R \end{pmatrix}$ of $T_2(R)$. Then $T_2(R)/I$ is isomorphic to R, and hence R is right duo because $T_2(R)$ is right FD. Summarizing, R is a division ring by Lemma 2.1(1). The proof for the left case is similar.

(i) $\Rightarrow$ (iii). Let R be a division ring. Then it is easy to check that each nonzero proper ideal of $D_3(R)$ is one of the following: $\begin{pmatrix} 0 & 0 & R \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & R & R \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 & R \\ 0 & 0 & R \\ 0 & 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & R & R \\ 0 & 0 & R \\ 0 & 0 & 0 \end{pmatrix}$. So the factor rings modulo by these ideals are isomorphic to

$$R' = \left\{ \begin{pmatrix} a & b & 0 \\ 0 & a & c \\ 0 & 0 & a \end{pmatrix} \mid a, b, c \in R \text{ and } bc = 0 \right\}, D_2(R), D_2(R) \text{ and } R,$$

respectively. Note that $D_2(R)$ is duo by Lemma 2.1(4) and R is clearly duo. Next R' is isomorphic to the subring

$$\left\{ \left(\begin{pmatrix} a & b & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix}, \begin{pmatrix} a & 0 & 0 \\ 0 & a & c \\ 0 & 0 & a \end{pmatrix} \right) \mid a, b, c \in R \right\}$$

of $D_3(R) \times D_3(R)$, which is also isomorphic to the subring

$$\left\{ \left(\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}, \begin{pmatrix} a & c \\ 0 & a \end{pmatrix} \right) \mid a, b, c \in R \right\}$$

of $D_2(R) \times D_2(R)$. This ring is duo by the proof of Lemma 2.1(4), noting that every non-invertible element is of the form $\left(\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix} \right)$. Therefore $D_3(R)$ is FD.

(iii) $\Rightarrow$ (i). Suppose that $D_3(R)$ is right FD. Assume that R is not simple. Then $D_3(R)/D_3(M)$ is isomorphic to the noncommutative ring $D_3(R/M)$, that is not right duo by Lemma 2.1(4), for a maximal ideal M of R. So $D_3(R)$ is not right FD. Thus R must be simple. Next consider the proper ideal $I = \begin{pmatrix} 0 & R & R \\ 0 & 0 & R \\ 0 & 0 & 0 \end{pmatrix}$ of $D_3(R)$. Then $D_3(R)/I$ is isomorphic to R, and hence R is right duo because $D_3(R)$ is right FD. Summarizing, R is a division ring by Lemma 2.1(1). The proof for the left case is similar.

(3) It suffices to show the sufficiency by Lemma 2.1(4). Let $D_2(R)$ is right FD. Then $R \cong \frac{D_2(R)}{I}$ is right duo, and hence R is a division ring by Lemma 2.1(1), where $I = RE_{12}$. The proof for the left case is similar. □

Following [9], a ring R is called *FC* if R is simple, or else R/I is a commutative ring for every nonzero proper ideal I of R. FC rings are clearly FD, but the converse need not hold by Theorem 2.2(2). Indeed, letting R be a noncommutative division ring, $T_2(R)$ is right FD by Theorem 2.2(2), but not FC by [9, Theorem 1.10(3)].

Following Birkhoff [1], a ring R is called *subdirectly irreducible* if the intersection of all nonzero ideals in R is nonzero. It is obvious that a ring R is subdirectly irreducible if and only if for every set of nonzero proper ideals of R, $\{K_l \mid l \in L\}$ say, we have $\cap_{l \in L} K_l \neq 0$. We will use this fact freely.

Lemma 2.3. *Let R be a non-prime right (resp., left) FD ring. Then each of the following holds.*

(1) *R is a subdirect product of subdirectly irreducible right (resp., left) FD rings.*

(2) *$R/N_*(R)$ is a subdirect product of right (resp., left) duo domains, and $R/J(R)$ is a subdirect product of division rings.*

(3) *If R is semiprime then R is a subdirect product of right (resp., left) duo domains (hence reduced).*

Proof. (1) It is proved by Birkhoff [1] that any ring is a subdirect product of subdirectly irreducible rings. We apply the proof of [10, Theorem 4.12.3]. For any $0 \neq a \in R$, there exists a proper ideal M_a that is maximal with respect to the property that $a \notin M_a$. Then $\cap_{0 \neq a \in R} M_a = 0$, and R/M_a is subdirectly irreducible since every nonzero ideal of R/M_a contains $a + M_a$ by the maximality of M_a. Moreover R/M_a is right FD by Lemma 2.1(7). Therefore R is a subdirect product of subdirectly irreducible right FD rings R/M_a. The left case can be proved similarly.

(2) Let P_i ($i \in I$) be all prime ideals of R. Then every P_i is nonzero because R is not prime. So R/P_i is right duo since R is right FD. Moreover R/P_i is a right duo domain by Lemma 2.1(2). Thus $R/N_*(R)$ is a subdirect product of right duo domains. The remainder is proved similarly by Lemma 2.1(1).

(3) is an immediate consequence of (2). The proofs of (1), (2) and (3) for the left case are similar. □

There exist non-prime FD rings which are not subdirectly irreducible. In fact, each of $\mathbb{Z} \times \mathbb{Z}$ and $\mathbb{Z}_{pq}$ is not subdirectly irreducible because $J(\mathbb{Z} \times \mathbb{Z}) = 0$ and $p\mathbb{Z}_{pq} \cap q\mathbb{Z}_{pq} = 0$, where p and q are distinct prime numbers. This elaborates on Lemma 1.3(1). The condition "non-prime" is not superfluous in Lemma 1.3(3) as can be seen by the simple ring $Mat_n(A)$ for $n \geq 2$ over a simple ring A.

Based on Lemma 1.3, one may ask whether a ring R is right FD if every right primitive factor ring of R is a simple domain. But the answer is negative as follows. There exists a semiprime ring R for which $J(R) \neq 0$ and $R/J(R)$ is a simple domain, but R is neither right nor left FD.

Example 2.4. We refer to the construction and argument in [7, Example 1.2] and [8, Theorem 2.2(2)]. Let K be a simple domain that is neither right nor left duo (e.g., the first Weyl algebra over a field of characteristic zero). Let $R_n = D_{2^n}(K)$ for $n \geq 1$ with the function $\sigma : R_n \to R_{n+1}$ by $A \mapsto \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$. Next set $R = \bigcup_{n=1}^{\infty} R_n$, noting that R_n can be considered as a subring of R_{n+1} via σ. Then R is a semiprime ring by [8, Theorem 2.2(2)]. But

$$J(R) = N^*(R) = \{(a_{ij}) \in R \mid a_{ii} = 0 \text{ for all } i\} \text{ and } R/J(R) \cong K.$$

This implies that $J(R)$ is maximal (hence primitive), entailing that every primitive factor ring of R is a simple domain. But R is neither right nor left FD since $R/J(R)$ is neither right nor left duo.

Let $R' = R \times R$. Then $J(R') = J(R) \times J(R)$ and all right (left) primitive factor ring of R' is $M_1 = R \times J(R)$ and $M_2 = J(R) \times R$. Note $R'/J(R') \cong K \times K$ and $R'/M_i \cong K$. But R' is also neither right nor left FD by Lemma 2.1(7).

Following Neumann [12], a ring R is said to be *regular* if for each $a \in R$ there exists $b \in R$ such that $a = aba$. Such a ring is also called *von Neumann regular* by Goodearl [3]. It is shown that R is regular if and only if every principal right (left) ideal of R is generated by an idempotent in [3, Theorem 1.1]. From this fact we can easily conclude that every regular ring R is clearly semiprimitive (i.e., $J(R) = 0$).

Proposition 2.5. *Let R be a non-prime regular ring. Then the following conditions are equivalent:*

(1) *R is right FD;*

(2) *R is reduced;*

(3) *R is right duo;*

(4) *R is left duo;*

(5) *R is left FD;*

(6) *Every right primitive factor ring of R is a division ring;*

(7) *R is a subdirect product of division ring;*

(8) *R is a subdirect product of domains.*

Proof. Since R is regular, $J(R) = 0$ and hence $N_*(R) = 0$. Thus R is reduced by Lemma 2.3(3), showing (1) $\Rightarrow$ (2) and (5) $\Rightarrow$ (2). (2) $\Rightarrow$ (3) and (2) $\Rightarrow$ (4) are proved by [3, Theorem 3.2]. (3) $\Rightarrow$ (1), (4) $\Rightarrow$ (5), (7) $\Rightarrow$ (8) and (8) $\Rightarrow$ (2) are obvious.

(1) $\Rightarrow$ (6) is obtained from Lemma 2.1(1) because R is right FD. (6) $\Rightarrow$ (7) is obvious since R is semiprimitive. $\square$

The condition "non-prime" is not superfluous in Proposition 2.5 as can be seen by the regular ring $Mat_n(A)$ for $n \geq 2$ over a division ring A (refer to [3, Lemma 1.6]). Indeed, $Mat_n(A)$ is simple (hence FD) but not reduced.

Following [14], a ring R is called *right* (resp., *left*) *quasi-duo* if every maximal right (resp., left) ideal of R is two-sided. It is obvious that a ring R is right quasi-duo if and only if $R/J(R)$ is right quasi-duo. Right duo rings are clearly right quasi-duo but not conversely. It is proved by [5, Proposition 1] that a ring R is right quasi-duo if and only if every right primitive factor ring of R is a division ring.

Proposition 2.6.

(1) *Every non-prime right FD ring is right quasi-duo.*

(2) *If R is a non-prime right FD ring then $R/J(R)$ is a reduced right quasi-duo ring.*

Proof. (1) Let R be a non-prime right FD ring. Since R is not prime, every right primitive ideal of R is nonzero. Then since R is right FD, R/P is right duo for every right primitive ideal P of R. Hence R/P is a division ring by Lemma 2.1(1). So R is right quasi-duo by [5, Proposition 1].

(2) is obtained from (1) and Lemma 2.3(2). $\square$

The following elaborates upon Proposition 2.6.

Remark 2.7.

(1) Simple (hence FD) rings need not be quasi-duo by the existence of simple domains which are not division rings (e.g., the first Weyl algebra over a field of characteristic zero), which is compared with Proposition 2.6(1). Indeed this domain is neither right nor left quasi-duo.

(2) There exist non-prime noncommutative FD rings as can be seen by $T_2(R)$ and $D_3(R)$ over a division ring R (see Theorem 2.2(2)). This provides examples to Proposition 2.6.

(3) Based on Proposition 2.6(1), one may ask whether a non-prime right quasi-duo ring is right FD. But the answer is negative. Let A be a right quasi-duo ring and $R = T_n(A)$ for $n \geq 3$. Then R is right quasi-duo by [14, Proposition 2.1]. Let $I = AE_{1n}$. Then R/I is non-Abelian (hence not right duo), and so R is not right FD.

Next we will show that the FD property is not left-right symmetric.

Example 2.8. Consider a skewed trivial extension in [13, Definition 1.3] as follows. Let R be a commutative ring with an endomorphism σ and M be an R-module. For $R \oplus M$, the addition and multiplication are given by $(r_1, m_1) + (r_2, m_2) = (r_1 + r_2, m_1 + m_2)$ and $(r_1, m_1)(r_2, m_2) = (r_1 r_2, \sigma(r_1)m_2 + m_1 r_2)$. Then this construction forms a ring. Following the literature, this extension is called the *skew-trivial extension* of R by M, denoted by $R \propto M$. Note that $R \propto R$ is isomorphic to $R[x;\sigma]/(x^2)$ via the corresponding $(r, m) \mapsto r + \bar{x}m$, where $R[x;\sigma]$ is the skew polynomial ring, with the coefficients written on the right, only subject to $ax = x\sigma(a)$ for $a \in R$ and (x^2) is the ideal of $R[x;\sigma]$ generated by x^2.

Now let K be a field with a monomorphism σ and M be a K-module. Suppose that σ is not surjective. Then $K \propto M$ is a right duo ring that is not left duo by [11, Theorem 2.5]. Next set $E = R_1 \times R_2$ with $R_1 = K$ and $R_2 = K \propto M$. Then E is right duo (hence right FD) but not left duo, by Lemma 2.1(3). Let $I = K \times 0$. Then E/I is isomorphic to the ring $K \propto M$ that is not left duo. Thus E is not left FD.

3. Subrings, Polynomial Rings and Direct Products

In this section we study the right FD property of polynomial rings, subrings and direct products of given right FD rings. We consider first the polynomial ring case.

Theorem 3.1. *The following conditions are equivalent for a given ring R:*

(1) *$R[x]$ is right (left) FD;*

(2) *R is commutative;*

(3) *$R[x]$ is commutative.*

Proof. It suffices to prove that if $R[x]$ is right FD then R is commutative. Let $n \geq 3$ and suppose that $R[x]$ is right FD. We first obtain

$$R + Rx + \cdots + Rx^{n-1} \cong \frac{R[x]}{x^n R[x]}$$

from the nonzero proper ideal $x^n R[x]$ of $R[x]$. Then since $R[x]$ is right FD, the ring $R + Rx + \cdots + Rx^{n-1}$ is right duo. This implies that for any $a, b \in R$, $b(a + x) = (a + x)f(x)$ for some $f(x) \in R$. Comparing the degrees of both sides, we must get $f(x) \in R$. c say. It then follows that $b = c$ and $ba = ac = ab$. The commuting of a and b can be shown also by [4, Lemma 3]. Thus R is commutative. The proof of the left case is similar. □

We can show, by help of Theorem 3.1, that the right FD property does not pass to polynomial rings.

We can write the following by help of Theorem 3.1 and Lemma 2.1(3): For a ring R, $R[x]$ is right (resp., left) FD if and only if $R[x]$ is right (resp., left) duo.

We next argue about subrings of right (left) FD rings.

Example 3.2.

(1) Let R be the first Weyl algebra over a field of characteristic zero. Consider $R[x]$. Since $R[x]$ is right Noetherian domain, there exists the quotient division ring, Q say. Q is clearly FD. But $R[x]$ is noncommutative, hence $R[x]$ is neither right nor left FD by Theorem 3.1.

(2) We extend (1). Let D be any right Noetherian domain that is not a division ring. Let Q be the quotient division ring. Then $T_2(Q)$ is FD by Theorem 2.2(2). But the subring $T_2(D)$ is neither right nor left FD by Theorem 2.2(2) because D is not a division ring.

In the following we find a kind of subring which inherits the right FD property.

Theorem 3.3.

(1) *Let R be a ring and $0 \neq e^2 = e \in R$. If R is right (resp., left) duo then eRe is right (resp., left) duo.*

(2) *Let R be a ring and $0 \neq e^2 = e \in R$. If R is right (resp., left) FD then eRe is right (resp., left) FD.*

Proof. (1) Let R be a right duo ring and $0 \neq e^2 = e \in R$. Consider $eae, ebe \in eRe$. Since R is right duo, $ebeae = eaec$ for some $c \in R$. This yields $eaec = eaece = eaeece$. Thus eRe is right duo. The proof for the left case is similar.

(2) We apply the proof of [9, Theorem 1.12]. Suppose that R is simple. Then eRe is simple (hence FD) by the proof of [9, Theorem 1.12].

Suppose that R is FC and eRe is non-simple. Then R is non-simple by the preceding argument. Let J be a nonzero proper ideal of eRe. Then, by the proof of [9, Theorem 1.12], $J = eReJeRe = eIe$ where $I = ReJeR$ is a nonzero proper ideal of R. Since R is right FD, R/I is right duo.

Write $\bar{R} = R/I$ and $\bar{r} = r + I$ for $r \in R$. Note that $e \notin J$ implies $e \notin I$. So $\bar{e} \neq 0$ in $\bar{R}$. Next consider the epimorphism $f : eRe \to \bar{e}\bar{R}\bar{e}$ defined by $f(ere) = \bar{e}\bar{r}\bar{e}$.

Since $\bar{R}$ is right duo, the subring $\bar{e}\bar{R}\bar{e}$ of $\bar{R}$ is also right duo by (1). So $\frac{eRe}{Ker(f)} (\cong \bar{e}\bar{R}\bar{e})$ is right duo, where $Ker(f)$ is the kernel of f. But $Ker(f) = J$ by the proof of [9, Theorem 1.12], so that $(eRe)/J$ is right duo. Therefore eRe is right FD. The proof for the left case is similar. □

Let A be a simple ring, B be a noncommutative ring and set $R = A \times B[x]$. Then letting $e = (1, 0)$, we get $eRe \cong A$ is simple (hence FD); but $R/(A \times 0) \cong B[x]$ is not right FD by Theorem 3.1. Whence R is not right FD, showing that the converse of Theorem 3.3(2) need not hold.

Next let A be a simple ring and $R = A \times A$. Then for $e = (1, 0) \in R$, $eRe \cong A$ is simple, but R is not simple; which shows that the converse of the first part of the proof of Theorem 3.3(2) need not hold.

Recall that right duo rings are right FD. In contrast to Lemma 2.1(3), one may ask whether the direct product of right FD rings is also right FD. But the answer is negative as follows. Let A be a simple ring and $n \geq 2$. Then $Mat_n(A)$ is simple (hence FD). Set $R = Mat_n(A) \times Mat_n(A) \times Mat_n(A)$ and $I = Mat_n(A) \times 0 \times 0$. Then I is a nonzero proper ideal of R and R/I is isomorphic to $Mat_n(A) \times Mat_n(A)$ that is not right duo. Thus R is not right FD.

In the following we see an equivalent condition for direct products of right FD rings to be right FD.

Theorem 3.4. *Let R_i be rings for all $i \in I$, and $R = \prod_{i \in I} R_i$, where $|I| \geq 2$. The following conditions are equivalent:*

(1) *R is right FD;*

(2) *R_i is right duo for all $i \in I$;*

(3) *R is right duo.*

Proof. (1) $\Rightarrow$ (2). Suppose R is right FD. Let $j \in I$ and $I_j = \{(a_i)_{i \in I} \in R \mid a_j = 0\}$. Then I_j is a nonzero proper ideal of R such that R/I_j is isomorphic to R_j. Since R is right FD, R_j is right duo.

(3) $\Rightarrow$ (1) is obvious, and (2) $\Rightarrow$ (3) is shown by Lemma 2.1(3). □

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