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On the Hyers–Ulam Stability of Polynomial Equations in Dislocated Quasi–metric Spaces

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Abstract. This paper primarily discusses and proves the Hyers–Ulam stability of three types of polynomial equations: $x^n+a_1x+a_0=0$, $a_nx^n+\cdots+a_1x+a_0=0$, and the infinite series equation: $\sum_{i=0}^{\infty} a_ix^i = 0$, in dislocated quasi–metric spaces under certain conditions by constructing contraction mappings and using fixed–point methods. We present an example to illustrate that the Hyers–Ulam stability of polynomial equations in dislocated quasi–metric spaces do not work when the constant term is not equal to zero.

1. Introduction and Preliminaries

"When is it true that a function which approximately satisfies a function equation ε must be somehow close to an exact solution of ε" is a classical question in the theory of functional equations. Such a problem was formulated by Ulam [17] in 1940 and solved by Hyers [5] for the Cauchy function equation in the next year. It has given rise to the stability theory for functional equations. Since then, the stability of functional equations has been extensively investigated by many mathematicians [2, 6, 7, 9, 10, 11, 12, 13, 14, 15]. Recently, Li and Hua [8] investigated the Hyers–Ulam stability of the following polynomial equation:

$$x^n + \alpha x + \beta = 0,$$

where $x \in [-1, 1]$, and $\alpha, \beta \in R$. Later in 2010, Bikhdam, Soleiman Mezerji, and Eshaghi Gordji [1] proved that if $|a_1|$ is large and $|a_0|$ is small enough, then the

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following polynomial equation of degree n:

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0,$$

where $x \in [-1, 1]$, and $a_i \in R, (i = 0, 1, 2, \cdots, n)$, have Hyers–Ulam stability. They also established the Hyers–Ulam stability of power series equations, and investigated their generalized Hyers–Ulam stability.

Operator equations in Banach spaces are of great significance in pure and applied mathematics, and Banach spaces are the generalization of $[-1, 1]$, There have been some results concerning the Hyers–Ulam stability of equations in Banach spaces, Banach algebras, and quasi–Banach algebras. In 2016, Eghbali [3] discussed and proved the Hyers–Ulam stability of the generalized polynomial equation over Banach algebras with real coefficients. In 2019, Weizheng Zhang, Liubin Hua, and Yongjin Li [18] investigated and proved the Hyers–Ulam stability of polynomial equations in quasi–Banach algebras. Whether Banach algebras or quasi–Banach algebras are metric spaces, considering that dislocated quasi–metric spaces are the generalization of metric spaces, is of great importance to investigate equations in dislocated quasi–metric spaces. In this paper, motivated by the results mentioned above, we investigate the Hyers–Ulam stability of three typical polynomial equations in dislocated quasi–metric spaces.

Definition 1.1. Let X be a Banach space and $F(x) = 0$ be an equation in X. If a constant $C > 0$ exists with the following property: for every $\varepsilon > 0$ and $y \in X$, if $\|F(y)\| \leq \varepsilon$, then some $x_0 \in X$ exists satisfying $F(x_0) = 0$, such that $\|y - x_0\| < C\varepsilon$. Such C is called a *Hyers–Ulam stability constant* for equation $F(x) = 0$. Thus, we say that the equation $F(x) = 0$ has *Hyers–Ulam stability* in X.

The concept of dislocated quasi–metric spaces was introduced in 2001 by Hitzler [7] in which the self–distance of points need not be zero.

Definition 1.2.([4]) Let X be a nonempty set and $d : X \times X \to [0, \infty)$ be a given function that satisfies

(1) $d(x, y) = d(y, x) = 0$ implies that $x = y$;

(2) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then, d is called a *dislocated quasi metric* on X, and the pair (X, d) is called a *dislocated quasi–metric space*.

Rahman, and Sarwar [16] defined the convergence and completeness in dislocated quasi–metric spaces as follows:

Definition 1.3.([16]) Let (X, d) be a dislocated quasi–metric space, $\{x_n\}$ be a sequence in X, and $x \in X$. Then, the sequence $\{x_n\}$ *converges* to x if and only if

$$\lim_{n\to\infty} d(x_n, x) = \lim_{n\to\infty} d(x, x_n) = 0.$$

Definition 1.4.([16]) Let (X, d) be a dislocated quasi–metric space and $\{x_n\}$ be a sequence in X. We say that the sequence $\{x_n\}$ is a *left-Cauchy sequence* if and only if for every $\varepsilon > 0$, a positive integer $N = N(\varepsilon)$ exists such that $d(x_n, x_m) \leq \varepsilon$ for all $n \geq m > N$.

Definition 1.5.([16]) Let (X, d) be a dislocated quasi–metric space and $\{x_n\}$ be a sequence in X. We say that the sequence $\{x_n\}$ is a *right-Cauchy sequence* if and only if for every $\varepsilon > 0$, a positive integer $N = N(\varepsilon)$ exists such that $d(x_n, x_m) \leq \varepsilon$ for all $m \geq n > N$.

Definition 1.6.([16]) Let (X, d) be a dislocated quasi–metric space and $\{x_n\}$ be a sequence in X. We say that the sequence $\{x_n\}$ is a *Cauchy sequence* if and only if for every $\varepsilon > 0$, a positive integer $N = N(\varepsilon)$ exists such that $d(x_n, x_m) \leq \varepsilon$ for all $n, m > N$.

Definition 1.7.([16]) Let (X, d) be a dislocated quasi–metric space. We say that

(1) (X, d) is *left-complete* if and only if every left-Cauchy sequence in X is convergent;

(2) (X, d) is *right-complete* if and only if every right-Cauchy sequence in X is convergent;

(3) (X, d) is *complete* if and only if every Cauchy sequence in X is convergent.

Therefore, a dislocated quasi–metric space that is left-complete may not be right-complete.

Example 1.1.([12]) Let $X = R$, and define the metric in R as follows:

$$d(x_m, x_n) = \begin{cases} |x_m - x_n|, & \text{if} \quad m \leq n; \\ 1, & \text{if} \quad m > n. \end{cases}$$

Then, (R, d) is right-complete but not left-complete.

Remark 1.1. Every metric space is a dislocated quasi–metric space, but the converse is not necessarily true. Thus, it is interesting to investigate the difference of completeness between metric spaces and dislocated quasi–metric spaces.

Theorem 1.1. *Let $(X, \|\cdot\|)$ be a normed space, for all $x, y \in X$. We define the metric space in X as $\rho(x, y) = \|x - y\|$, and define the dislocated quasi metric in X as $d(x, y) = \|x - y\| + \|x\|$. Then, the completeness of X under the metric ρ is equivalent to the completeness under the dislocated quasi metric d if and only if every Cauchy sequence with metric ρ converges to* 0.

Proof. (*Sufficiency*) Suppose that the completeness of X under the metric ρ is equivalent to the completeness under the dislocated quasi metric d, and $\{x_n\}$ is a Cauchy sequence under the metric ρ, then $\{x_n\}$ is also a Cauchy sequence under the dislocated quasi metric d. Thus, for any positive integer N, where $n, m > N$, we have the following:

$$\lim_{n,m\to\infty} d(x_n, x_m) = \lim_{n,m\to\infty} \|x_n - x_m\| + \lim_{n\to\infty} \|x_n\| = 0.$$

Therefore,

$$\lim_{n\to\infty} \|x_n\| = 0.$$

That is, every Cauchy sequence under the metric ρ converges to 0.
(*Necessity*) Suppose that any Cauchy sequence of X under the metric ρ converges to 0, then X is complete with the metric ρ. Suppose $\{x_n\}$ is a Cauchy sequence of X under the dislocated quasi metric d. Then, for any positive integer N, where $n, m > N$, we have the following:

$$\lim_{n,m\to\infty} d(x_n, x_m) = \lim_{n,m\to\infty} \|x_n - x_m\| + \lim_{n\to\infty} \|x_n\| = 0.$$

That is,

$$\lim_{n,m\to\infty} \|x_n - x_m\| = 0, \lim_{n\to\infty} \|x_n\| = 0.$$

Thus, $\{x_n\}$ is also a Cauchy sequence under the metric ρ, and $x_n \to 0 \in X(n \to \infty)$. Moreover, X is complete under the dislocated quasi metric d. □

Using the fixed–point theorem is a key point to prove the Hyers–Ulam stability of polynomial equations. In 2014, Rahman, and Sarwar [16] investigated and proved some fixed–point theorems in dislocated quasi–metric spaces.

Definition 1.8.([16]) Let (X, d) be a metric space. A self–mapping $T : X \to X$ is called a *generalized contraction* if and only if, for all $x, y \in X$, c_1, c_2, c_3, c_4 exist such that $\sup\{c_1 + c_2 + c_3 + 2c_4\} < 1$ and

$$d(Tx, Ty) \leq c_1 d(x, y) + c_2 d(x, Tx) + c_3 d(y, Ty) + c_4[d(x, Ty) + d(y, Tx)].$$

Theorem 1.2.([16]) *Let (X, d) be a complete dislocated quasi metric space, and $T : X \to X$ be a continuous self–mapping satisfying the following condition:*

$$\begin{aligned} d(Tx, Ty) \leq{}& \alpha d(x, y) + \beta \frac{d(x, Ty)d(y, Ty)}{d(x, y) + d(y, Ty)} \\ &+ \gamma \frac{d(x, Tx)d(y, Ty)}{1 + d(x, y)} + \mu \frac{d(x, Tx)d(x, Ty)}{1 + d(x, y)} \end{aligned}$$

for all $x, y \in X$ and $\alpha, \beta, \gamma, \mu \geq 0$ with $\alpha + \beta + \gamma + 2\mu < 1$. Then, T has a unique fixed point.

Let $X = [-1, 1]$. We define the metric in X as follows:

$$d(x, y) = |x - y| + |x|, x, y \in X.$$

This paper primarily investigates the following three operator equations in X:

$$x^n + a_1 x + a_0 = 0, \tag{1.1}$$

$$\sum_{i=0}^{n} a_i x^i = 0, \tag{1.2}$$

$$\sum_{i=0}^{\infty} a_i x^i = 0, \tag{1.3}$$

where $a_i \in R(i = 0, 1, 2, \cdots, n, \cdots)$ are the coefficient operators of the corresponding equations. This paper proves that the above equations have Hyers–Ulam stability on (X, d) under some conditions of their coefficient operators.

2. Main Results in Dislocated Quasi-metric Spaces

Because we already have the basic knowledge, we can investigate the Hyers–Ulam stability of the polynomial equations in dislocated quasi–metric spaces.

Lemma 2.1. *Let $X = [-1, 1]$. We define the dislocated quasi metric in X as $d(x, y) = |x - y| + |x|$. Then, X is a complete quasi–metric space.*

Proof. X is a dislocated quasi–metric space under the dislocated quasi metric d. Let $\{x_n\}$ be a Cauchy sequence of X. Then, for any positive integer N, where $n, m > N$, we have the following:

$$\lim_{n,m\to\infty} d(x_n, x_m) = \lim_{n,m\to\infty} |x_n - x_m| + \lim_{n\to\infty} |x_n| = 0.$$

Therefore,

$$\lim_{n\to\infty} |x_n| = 0, \lim_{n\to\infty} x_n = 0 \in X.$$

That is, any Cauchy sequence in X converges to X. Thus, (X, d) is a complete dislocated quasi–metric space. □

2.1. Hyers–Ulam Stability of Equation (1.1) in Dislocated Quasi–metric Spaces

Theorem 2.1. *If $|a_1| > n, a_0 = 0$, and $y \in X$ satisfies the inequality*

$$|y^n + a_1 y + a_0| \leq \varepsilon,$$

then a solution $x_0 \in X$ exists for Equation (1.1) such that,

$$\max\{d(x_0, y), d(y, x_0)\} < \frac{2\varepsilon}{(1 - \alpha)|a_1|},$$

where $\alpha = \frac{n}{|a_1|}$.

Proof. Let $\varepsilon > 0$ and $y \in X$ satisfy the inequality $|y^n + a_1 y + a_0| \leq \varepsilon$. We prove that an $x_0 \in X$ that satisfies Equation (1.1) also satisfies $\max\{d(x_0, y), d(y, x_0)\} < \frac{2\varepsilon}{(1-\alpha)|a_1|}$. First, we define a new operator on X:

$$g(x) = -\frac{1}{a_1}(a_0 + x^n), x \in [-1, 1],$$

and we have the following inequality:

$$|g(x)| = |\frac{1}{a_1}(a_0 + x^n)| \leq 1.$$

Thus, as a function of x, $g(x)$ is a continuous linear operator and maps X to X. Next, we prove that g satisfies the condition of Theorem 1.2. For any $x, y \in X$ and $x \neq y$, we have the following:

$$|g(x)| = |\frac{1}{a_1}(a_0 + x^n)| = \frac{|x^n|}{|a_1|} \leq \frac{n|x|}{|a_1|}.$$

Thus,

$$\begin{aligned}
d(g(x), g(y)) &= |g(x) - g(y)| + |g(x)| \\
&= |\frac{1}{a_1}(x^n - y^n)| + |g(x)| \\
&\leq \frac{1}{|a_1|}|x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1}||x - y| + |g(x)| \\
&\leq \frac{n}{|a_1|}|x - y| + \frac{n}{|a_1|}|x| \\
&\leq \frac{n}{|a_1|}(|x - y| + |x|) = \frac{n}{|a_1|}d(x, y).
\end{aligned}$$

Let $\alpha = \frac{n}{|a_1|}, \beta = 0, \gamma = 0$, and $\mu = 0$, then we have $\alpha \in [0, 1)$, and

$$\begin{aligned}
d(g(x), g(y)) &\leq \alpha d(x, y) \\
&\leq \alpha d(x, y) + \beta \frac{d(x, Ty)d(y, Ty)}{d(x, y) + d(y, Ty)} \\
&\quad + \gamma \frac{d(x, Tx)d(y, Ty)}{1 + d(x, y)} + \mu \frac{d(x, Tx)d(x, Ty)}{1 + d(x, y)}.
\end{aligned}$$

Thus, g satisfies the condition of Theorem 1.2. By Theorem 1.2, unique $x_0 \in X$ exist, such that $g(x_0) = x_0$. Hence, Equation (1.1) has a solution on X.
Finally, we demonstrate that Equation (1.1) has Hyers–Ulam stability. On one hand,

$$\begin{aligned}
d(x_0, y) &= |x_0 - y| + |x_0| \\
&= |x_0 - g(y) + g(y) - y| + |g(x_0)| \\
&\leq d(g(x_0), g(y)) + \frac{\varepsilon}{|a_1|} \\
&< \alpha d(x_0, y) + \frac{\varepsilon}{|a_1|}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}d(y,x_0) &= |y-x_0|+|y| \\ &= |y-g(y)+g(y)-x_0|+|y-g(y)+g(y)| \\ &\le |y-g(y)|+|g(y)-g(x_0)|+|g(y)|+|y-g(y)| \\ &= 2|y-g(y)|+d(g(y),g(x_0)) \\ &< 2|y-g(y)|+\alpha d(y,x_0).\end{aligned}$$

Thus, we derive the following inequality:

$$\max\{d(x_0,y),d(y,x_0)\} < \frac{2\varepsilon}{(1-\alpha)|a_1|},$$

which completes the proof. □

2.2. Hyers–Ulam Stability of Equation (1.2) in Dislocated Quasi–metric Spaces

Using a similar method, we obtain the result regarding the Hyers–Ulam stability of Equation (1.2) in X.

Theorem 2.2. *If $a_0 = 0$, $|a_1| > 2|a_2| + 3|a_3| + \cdots + n|a_n|$, and $y \in X$ satisfies the inequality*

$$|a_n y^n + a_{n-1}y^{n-1} + \cdots + a_1 y + a_0| \le \varepsilon,$$

then a solution $x_0 \in X$ of Equation (1.2) exists such that

$$\max\{d(x_0,y),d(y,x_0)\} < \frac{2\varepsilon}{(1-\alpha)|a_1|},$$

where $\alpha = \frac{2|a_2|+3|a_3|+\cdots+n|a_n|}{|a_1|}$.

Proof. Let $\varepsilon > 0$, and let $y \in X$ satisfy the following inequality

$$|a_n y^n + a_{n-1}y^{n-1} + \cdots + a_1 y + a_0| \le \varepsilon.$$

We demonstrate that a solution $x_0 \in X$ of Equation (1.2) exists such that $\max\{d(x_0,y),d(y,x_0)\} < \frac{2\varepsilon}{(1-\alpha)|a_1|}$. First, we define a new operator on X:

$$g(x) = -\frac{1}{a_1}(a_0 + a_2x^2 + \cdots + a_nx^n), x \in [-1,1],$$

and we have the following inequality:

$$\begin{aligned}|g(x)| &= \frac{|a_2x^2 + \cdots + a_nx^n|}{|a_1|} \\ &= \frac{|a_2x + \cdots + a_nx^{n-1}||x|}{|a_1|} \\ &\leq \frac{|a_2||x| + \cdots + |a_n||x|^{n-1}}{|a_1|}|x| \\ &< \frac{2|a_2| + 3|a_3| + \cdots + n|a_n|}{|a_1|}|x| \\ &< \frac{|a_1|}{|a_1|}|x| < |x| \leq 1.\end{aligned}$$

Thus, as a function of x, $g(x)$ is a continuous linear operator and maps X to X. Next, we prove that g satisfies the condition of Theorem 1.2. For any $x, y \in X$ and $x \neq y$, we have

$$|g(x)| = \frac{|a_2x^2 + \cdots + a_nx^n|}{|a_1|} < \frac{2|a_2| + 3|a_3| + \cdots + n|a_n|}{|a_1|}|x|$$

Thus,

$$\begin{aligned}d(g(x), g(y)) &= |g(x) - g(y)| + |g(x)| \\ &= |-\frac{1}{a_1}(a_0 + a_2x^2 + \cdots + a_nx^n) + \frac{1}{a_1}(a_0 + a_2y^2 + \cdots + a_ny^n)| \\ &\quad + |g(x)| \\ &= \frac{1}{|a_1|}|a_2(x^2 - y^2) + \cdots + a_n(x^n - y^n)| + |g(x)| \\ &\leq \frac{1}{|a_1|}(|a_2||x + y||x - y| + \cdots + |a_n||x^{n-1} \\ &\quad + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1}||x - y|) + |g(x)| \\ &\leq \frac{2|a_2| + 3|a_3| + \cdots + n|a_n|}{|a_1|}|x - y| + |g(x)| \\ &\leq \frac{2|a_2| + 3|a_3| + \cdots + n|a_n|}{|a_1|}(|x - y| + |x|) \\ &= \frac{2|a_2| + 3|a_3| + \cdots + n|a_n|}{|a_1|}d(x, y).\end{aligned}$$

Let $\alpha = \frac{2|a_2|+3|a_3|+\cdots+n|a_n|}{|a_1|}$, and $\beta = \gamma = \mu = 0$. Thus, we have $\alpha \in [0,1)$, and

$$\begin{aligned} d(g(x), g(y)) &\leq \alpha d(x,y) \\ &\leq \alpha d(x,y) + \beta \frac{d(x,Ty)d(y,Ty)}{d(x,y)+d(y,Ty)} \\ &+ \gamma \frac{d(x,Tx)d(y,Ty)}{1+d(x,y)} + \mu \frac{d(x,Tx)d(x,Ty)}{1+d(x,y)}. \end{aligned}$$

Therefore, g satisfies the condition of Theorem 1.2. By Theorem 1.2, a unique $x_0 \in X$ exists, such that $g(x_0) = x_0$. Hence Equation (1.2) has a solution on X. Finally, we demonstrate that Equation (1.2) has the Hyers–Ulam stability. On one hand,

$$\begin{aligned} d(x_0, y) = |x_0 - y| + |x_0| &= |x_0 - g(y) + g(y) - y| + |g(x_0)| \\ &\leq d(g(x_0), g(y)) + \frac{\varepsilon}{|a_1|} \\ &< \alpha d(x_0, y) + \frac{\varepsilon}{|a_1|}. \end{aligned}$$

On the other hand,

$$\begin{aligned} d(y, x_0) &= |y - x_0| + |y| = |y - g(y) + g(y) - x_0| + |y - g(y) + g(y)| \\ &\leq |y - g(y)| + |g(y) - g(x_0)| + |g(y)| + |y - g(y)| \\ &= 2|y - g(y)| + d(g(y), g(x_0)) \\ &< 2|y - g(y)| + \alpha d(y, x_0). \end{aligned}$$

Thus, we derive the following inequality:

$$\max\{d(x_0, y), d(y, x_0)\} < \frac{2\varepsilon}{(1-\alpha)|a_1|},$$

which completes the proof. □

2.3. Hyers–Ulam Stability of Equation (1.3) in Dislocated Quasi–metric Spaces

Using a similar method, we obtain the result regarding the Hyers–Ulam stability of Equation (1.3) in X.

Theorem 2.3. *If $a_0 = 0$, $|a_1| > \sum_{i=2}^{\infty} i|a_i|$, and $y \in X$ satisfies the following inequality:*

$$|\sum_{i=0}^{\infty} a_i y^i| \leq \varepsilon,$$

then a solution $x_0 \in X$ *of Equation (1.3) exists such that*

$$\max\{d(x_0, y), d(y, x_0)\} < \frac{2\varepsilon}{(1-\alpha)|a_1|},$$

where $\alpha = \frac{\sum_{i=2}^{\infty} i|a_i|}{|a_1|}$.

Proof. Let $\varepsilon > 0$ and $y \in X$ satisfies the following inequality:

$$|\sum_{i=0}^{\infty} a_i y^i| \leq \varepsilon.$$

We demonstrate that a solution $x_0 \in X$ satisfying Equation (1.3) exists such that $\max\{d(x_0, y), d(y, x_0)\} < \frac{2\varepsilon}{(1-\alpha)|a_1|}$. First, we define a new operator on X:

$$g(x) = -\frac{1}{a_1}(\sum_{i=0, i\neq 1}^{\infty} a_i x^i), x \in [-1, 1],$$

and we have the following inequality:

$$\begin{aligned}
|g(x)| &= \frac{|\sum_{i=2}^{\infty} a_i x^i|}{|a_1|} \\
&= \frac{|\sum_{i=2}^{\infty} a_i x^{i-1}||x|}{|a_1|} \\
&\leq \frac{\sum_{i=2}^{\infty} |a_i||x|^{i-1}}{|a_1|}|x| \\
&< \frac{\sum_{i=2}^{\infty} i|a_i|}{|a_1|}|x| \\
&< \frac{|a_1|}{|a_1|}|x| < |x| \leq 1.
\end{aligned}$$

Thus, as a function of x, $g(x)$ is a continuous linear operator and it maps X to X. Next, we prove that g satisfies the condition of Theorem 1.2. For any $x, y \in X$ and $x \neq y$, we have the following:

$$|g(x)| = \frac{|\sum_{i=2}^{\infty} a_i x^i|}{|a_1|} < \frac{\sum_{i=2}^{\infty} i|a_i|}{|a_1|}|x|.$$

Thus,

$$\begin{aligned}
d(g(x), g(y)) &= |g(x) - g(y)| + |g(x)| \\
&= |-\frac{1}{a_1}(\sum_{i=2}^{\infty} a_i x^i) + \frac{1}{a_1}(\sum_{i=2}^{\infty} a_i y^i)| + |g(x)| \\
&= \frac{1}{|a_1|}|\sum_{i=2}^{\infty} a_i (x^i - y^i)| + |g(x)| \\
&\leq \frac{1}{|a_1|}(\sum_{i=2}^{\infty} |a_i||\sum_{j=0}^{i-1} x^j y^{i-1-j}||x-y|) + |g(x)| \\
&\leq \frac{\sum\limits_{i=2}^{\infty} i|a_i|}{|a_1|}|x-y| + |g(x)| \\
&\leq \frac{\sum\limits_{i=2}^{\infty} i|a_i|}{|a_1|}(|x-y| + |x|) \\
&= \frac{\sum\limits_{i=2}^{\infty} i|a_i|}{|a_1|} d(x,y).
\end{aligned}$$

Let $\alpha = \frac{\sum_{i=2}^{\infty} i|a_i|}{|a_1|}$, and $\beta = \gamma = \mu = 0$. Then, we have $\alpha \in [0,1)$, and

$$\begin{aligned}
d(g(x), g(y)) &\leq \alpha d(x,y) \\
&\leq \alpha d(x,y) + \beta \frac{d(x,Ty)d(y,Ty)}{d(x,y) + d(y,Ty)} \\
&\quad + \gamma \frac{d(x,Tx)d(y,Ty)}{1 + d(x,y)} + \mu \frac{d(x,Tx)d(x,Ty)}{1 + d(x,y)}.
\end{aligned}$$

Therefore, g satisfies the condition of Theorem 1.2. By Theorem 1.2, a unique $x_0 \in X$ exists, such that $g(x_0) = x_0$. Hence, Equation (1.3) has a solution on X. Finally, we show that Equation (1.3) has the Hyers–Ulam stability. On one hand,

$$\begin{aligned}
d(x_0, y) &= |x_0 - y| + |x_0| \\
&= |x_0 - g(y) + g(y) - y| + |g(x_0)| \\
&\leq d(g(x_0), g(y)) + \frac{\varepsilon}{|a_1|} \\
&< \alpha d(x_0, y) + \frac{\varepsilon}{|a_1|}.
\end{aligned}$$

On the other hand,

$$\begin{aligned} d(y,x_0) &= |y-x_0|+|y| \\ &= |y-g(y)+g(y)-x_0|+|y-g(y)+g(y)| \\ &\le |y-g(y)|+|g(y)-g(x_0)|+|g(y)|+|y-g(y)| \\ &= 2|y-g(y)|+d(g(y),g(x_0)) \\ &< 2|y-g(y)|+\alpha d(y,x_0). \end{aligned}$$

Thus, we derive the following inequality:

$$\max\{d(x_0,y),d(y,x_0)\} < \frac{2\varepsilon}{(1-\alpha)|a_1|},$$

which completes the proof. □

It is easy to see that, using a similar method, we can discuss the Hyers–Ulam stability of the polynomial equation on any complete dislocated quasi–metric spaces. However, the results above work only under the condition in which the constant term of the polynomial equation equals to 0, and the Hyers–Ulam stability of polynomial equations in dislocated quasi–metric spaces does not work when the constant term is not equal to 0.

Example 2.1. Let $X=[-1,1]$. We define the dislocated quasi metric of X as: $d(x,y)=|x-y|+|x|$. In the following equation:

$$x^2-2x+1=0, \tag{2.1}$$

we take $\varepsilon=\frac{1}{n}$, where $n\ge 2, n\in N^+$. Therefore, $x=1$ is a solution to Equation (2.1). Then, suppose $y=1-\frac{1}{n}$, and we have the following:

$$\begin{aligned} |y^2-2y+1| &= |(1-\frac{1}{n})^2-2(1-\frac{1}{n})+1| \\ &= |(\frac{1}{n})^2| = \frac{1}{n^2} \\ &< \frac{1}{n} = \varepsilon. \end{aligned}$$

However,

$$d(x,y)=|x-y|+|x|=|1-1+\frac{1}{n}|+|1|=1+\frac{1}{n}>\frac{1}{n}=\varepsilon,$$

and

$$d(y,x)=|y-x|+|y|=|1-\frac{1}{n}-1|+|1-\frac{1}{n}|\ge 1>\varepsilon.$$

Thus, the equation $x^2-2x+1=0$ has no Hyers-Ulam stability in (X,d).

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