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Delta Moves and Arrow Polynomials of Virtual Knots

Myeong–Ju Jeong*
Department of Mathematics, Korea Science Academy 111 Baekyang Gwanmun–Ro, Busanjin–Gu, Busan 614–822, Korea
e-mail : mjjeong@kaist.ac.kr

Chan–Young Park
Department of Mathematics, College of Natural Sciences, Kyungpook National University, Daegu 702–701, Korea
e-mail : chnypark@knu.ac.kr

Abstract. Δ-moves are closely related with a Vassiliev invariant of degree 2. For classical knots, M. Okada showed that the second coefficients of the Conway polynomials of two knots differ by 1 if the two knots are related by a single Δ-move. The first author extended the Okada's result for virtual knots by using a Vassiliev invariant of virtual knots of type 2 which is induced from the Kauffman polynomial of a virtual knot. The arrow polynomial is a generalization of the Kauffman polynomial. We will generalize this result by using Vassiliev invariants of type 2 induced from the arrow polynomial of a virtual knot and give a lower bound for the number of Δ-moves transforming K_1 to K_2 if two virtual knots K_1 and K_2 are related by a finite sequence of Δ-moves.

1. Introduction

We will generalize a theorem on Δ-moves of virtual knots by using the arrow polynomial. Any knot can be unknotted by a finite sequence of Δ-moves [20] and the second coefficient of the Conway polynomial gives a lower bound for the number of Δ-moves in the sequence [26]. The second coefficient of the Conway polynomial is a Vassiliev invariant and we can extend the result to virtual knots by using a Vassiliev invariant induced from the Kauffman polynomial of virtual knots [11]. The Kauffman polynomial of virtual knots can be generalized to an arrow polynomial of virtual knots [2] and we give a necessary condition for two virtual knots to be

* Corresponding Author.
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related by a Δ-move by using numerical Vassiliev invariants induced from the arrow polynomial of virtual knots.

In 1989 H. Murakami and Y. Nakanishi [20] introduced the Δ-*move* for link diagrams as illustrated in Figure 1. The Δ-*Gordian distance* $d_G^{\Delta}(L, L')$ of links L and L' is defined to be the minimal number of the Δ-moves to transform a diagram of L to a diagram of L'. They showed that two oriented ordered n-component links L and L' can be related by a finite sequence of Δ-moves if L and L' are link-homologous. In particular Δ-moves unknot all knots. The Δ-*unknotting number* $u^{\Delta}(K)$ for a knot K is defined to be $d_G^{\Delta}(K, O)$, where O denotes the trivial knot. If two links are related by a finite sequence of Δ-moves then they are said to be Δ-*homotopic*. We will see that the Δ-move is not an unknotting operation for virtual knots.

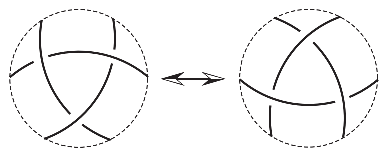

Figure 1: A Δ-move.

A *virtual link diagram* is closed curves generically immersed in the 2–dimensional Euclidean space. A double point of the curve is either a (classical) crossing or a virtual crossing. A virtual crossing is denoted by an encircled singular point. In particular if the virtual link diagram has one component then it is called a *virtual knot diagram*. See Figure 2 for a virtual knot diagram with six crossings and two virtual crossings.

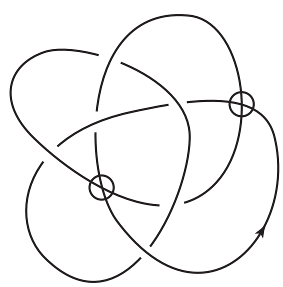

Figure 2:

Two virtual link diagrams are said to be *equivalent* if there is a finite sequence of Reidemeister moves and virtual moves transforming one diagram to the other diagram. See Figure 3 for *Reidemeister moves* and Figure 4 for *virtual moves.* A *virtual link* is defined to be an equivalence class of a virtual link diagram under the equivalence relation. A virtual link with one circle component is called a *virtual knot*. If two knot diagrams are equivalent then they can be related by a finite sequence of Reidemeister moves [3, 15].

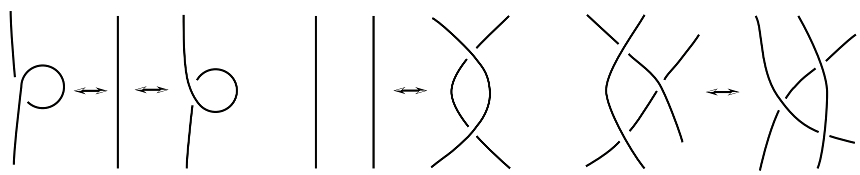

Figure 3: Reidemeister moves.

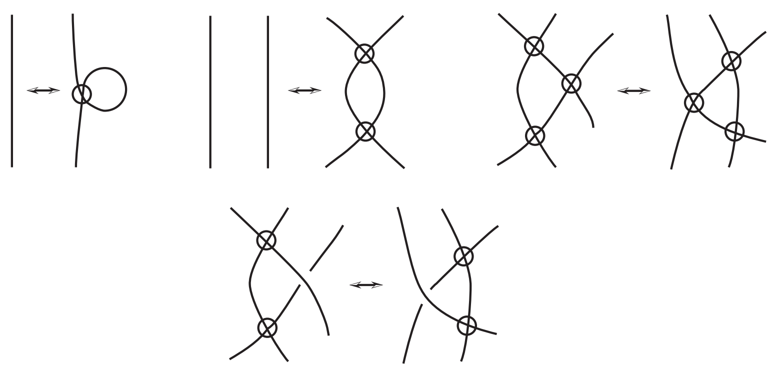

Figure 4: Virtual moves.

In this paper all virtual links are assumed to be oriented. It is well-known that any Δ-move on an oriented diagram can be realized by the move as shown in Figure 5. We extend Δ-moves, Δ-homotopy, Δ-Gordian distance and Δ-unknotting number for virtual knots and links naturally.

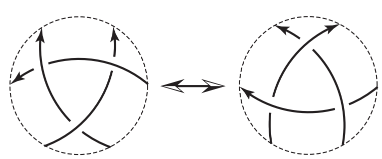

Figure 5:

H. A. Dye and L. H. Kauffman introduced the arrow polynomial of a virtual link. They used the state expansion of a crossing as shown in Figure 6. If we resolve a crossing in a virtual link diagram via the state expansion then it may have some nodal cusps. By replacing the nodal cusps with poles as shown in Figure 7, we get a diagram with poles.

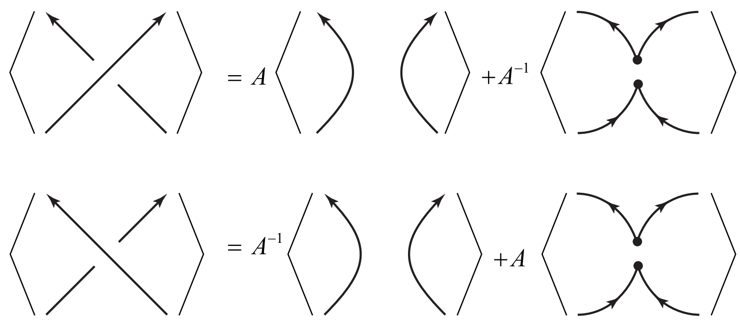

Figure 6: State expansion.

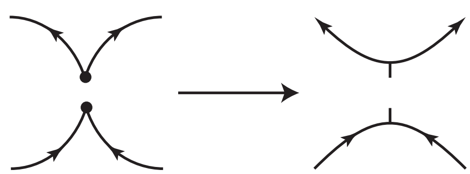

Figure 7:

Y. Miyazwa introduced a multi-variable polynomial invariant of virtual links by using decorated virtual magnetic graph diagram [17, 18]. A. Ishii simplified the polynomial by using pole diagrams [8]. A *pole* on a strand of a virtual link diagram is a unit normal vector with the initial point on the strand and it is denoted by a small line segment attached on the diagram. A virtual link diagram allowed to have poles is called a *pole diagram* or a *polar link diagram.* The two fragments of the diagram attached to a pole are assumed to be oriented either inward to the pole or outward from the pole as shown in Figure 8. See Figure 9 for polar link diagrams.

Figure 8:

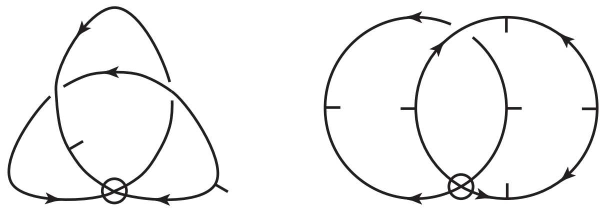

Figure 9: Polar link diagrams.

We can naturally extend Reidemeister moves and virtual moves of virtual link diagrams to pole diagrams. The local moves on polar link diagrams as shown in Figure 10 are called *polar moves.* Two polar link diagrams are said to be *equivalent* if they are related by a finite sequence of Reidemeister moves, virtual moves and polar moves. A *polar link* is an equivalence class of a polar link diagram under Reidemeister moves, virtual moves and polar moves.

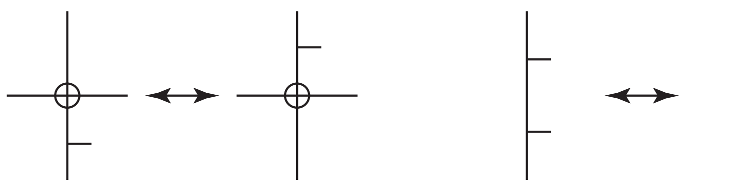

Figure 10:

Forbidden moves unknot all virtual knots. There are two kinds of *forbidden moves*, an F_H-move and an F_T-move, on virtual knot diagrams as shown in Figure 11. By representing forbidden moves in Gauss diagrams, S. Nelson showed that any virtual knot can be unknotted by applying finitely many forbidden moves [24]. T. Kanenobu showed that any classical crossing of a virtual knot diagram can be transformed to a virtual crossing by using the forbidden moves, the Reidemeister moves and the virtual moves [9]. Since a virtual knot diagram with only virtual crossing represents the trivial knot [15], we see that the family of forbidden moves is an unknotting operation.

T. Kanenobu also showed that a Δ-move can be realized by a finite sequence of the Reidemeister moves, the virtual moves and the forbidden moves [9]. Y. Nakanishi and T. Shibuya studied Δ link homotopy and gave a necessary condition for two links to be Δ link homotopic in terms of Conway polynomials [21, 22].

It can be easily proved that any long virtual knot can be unknotted by a finite sequence of forbidden moves by using Gauss diagrams. In [3], M. Goussarov, M. Polyak and O. Viro introduced finite type invariants of virtual knots via semi-

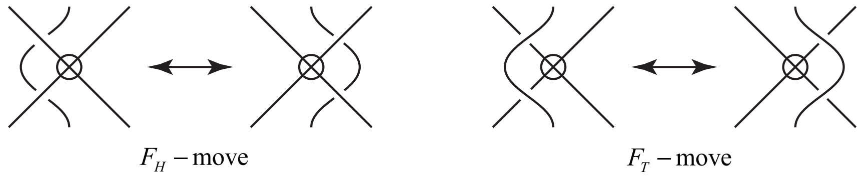

Figure 11: Forbidden moves.

virtual crossings. Then they gave combinatorial representations of several finite type invariants of low degrees by using Gauss diagram formulae. Every finite type invariant is a Vassiliev invariant and there is no non-trivial finite type invariant of degree ≤ 2 for virtual knots. But there is two independent finite type invariants of degree 2 for long virtual knots.

M. Sakurai and the first author independently calculated the difference of the values of a finite type invariant of degree 2 for two long virtual knots K and K' which are related by a forbidden move and showed that it is 0 or 1 [10, 27]. From this we can get a lower bound for the number of forbidden moves to unknot a long virtual knot by using the two finite type invariants of degree 2.

K. Habiro [7] introduced C_n-moves and gave relationships between C_n-moves and Vassiliev invariants of knots. For each positive integer n, a C_n*-move* is a local move of knots as shown in Figure 12. He showed that two knots are related by a finite sequence of C_n-moves if and only if they take the same values for all Vassiliev invariants of degree $< n$. M. N. Goussarov proved similar results independently [4, 5, 6]. In particular a C_2-move is equivalent to a Δ-move. Since any knot can be unknotted by a finite sequence of Δ-moves, we see that all Vassiliev invariants of degree < 2 are constants.

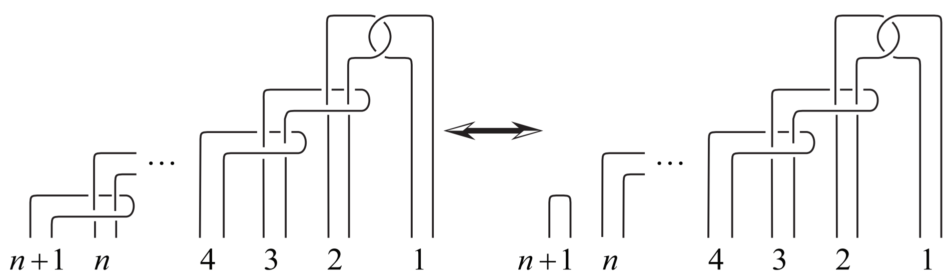

Figure 12: A C_n-move

If a Δ-move on a link happens on the same component then it is called a *self-delta move*. T. Kanenobu and R. Nikkuni studied self-delta moves and gave several relationships among the values of Vassiliev invariants induced from the HOMFLY polynomials of a delta skein quadruple [16].

In [23] Y. Nakanishi and Y. Ohyama studied C_k-distances of knots and showed that, for a given knot K, there are infinitely many knots which take the same value for any Vassiliev invariant of degree $< n$ and have some properties for C_k-distance.

The Jones-Kauffman polynomial of knots can be extended to virtual knots but so far we do not have a virtual knot invariant whose restriction to classical knots is the Conway polynomial. But the second coefficient of the Conway polynomial is a Vassiliev invariant of type 2 and there are infinitely many independent Vassiliev invariants of type 2 for virtual knots. In particular we may obtain many numerical Vassiliev invariants from the arrow polynomial. We show that the Okada's result on Δ-moves can be extended to virtual knots by using numerical Vassiliev invariants of type 2.

In Section 2, we treat Vassiliev invariants and the arrow polynomial and induce numerical Vassiliev invariants of type n from the arrow polynomial for each $n \in \mathbb{N}$. In Section 3, we represent a Δ-move of virtual knots in terms of singular virtual knots. Let $n_0, n_1, \ldots$ be integers such that $n_0 \neq 0$. We prove that if two virtual knots K_1 and K_2 are related by a single Δ-move then $h''_{K_1}(1) - h''_{K_2}(1) = \pm 96$, where $h_K(A) = f_K(A^{n_0}, A^{n_1}, \ldots)$ is the polynomial obtained from the arrow polynomial $f_K(A, Y_1, \ldots)$ of a virtual knot K by change of variables. We give examples for Δ-homotopy and the Δ-Gordian distance of virtual knots.

2. Numerical Vassiliev Invariants induced from the Arrow Polynomial

A (virtual) link invariant v which takes values in an abelian group can be extended to a singular (virtual) link invariant by using the Vassiliev skein relation: $v(L_\times) = v(L_+) - v(L_-)$, where $L_\times$, L_+ and L_- are singular (virtual) link diagrams which are identical except the indicated local parts as illustrated in Figure 13.

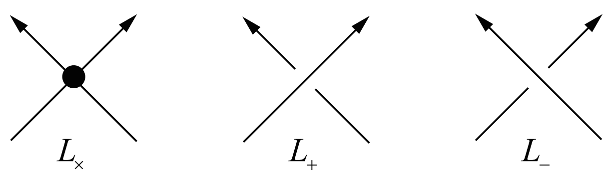

Figure 13:

A (virtual) link invariant v is called a *Vassiliev invariant of type* n if v vanishes on singular (virtual) links with more than n singular points. The smallest such nonnegative integer n is called the *degree* of v.

We may get Vassiliev invariants from the quantum polynomial invariants of knots. For example the coefficient $a_n(K)$ of z^n in the Conway polynomial $\nabla_K(z)$ of a knot K is a Vassiliev invariant of type n [1]. In particular the Vassiliev invariant

$a_2(K)$ of type 2 gives a lower bound for the Δ-Gordian distance of two knots. M. Okada showed that $a_2(K) - a_2(K') = \pm 1$, if two knots K and K' are related by a Δ-move [26]. Immediately, we see that

$$d_G^{\Delta}(K, K') \geq |a_2(K) - a_2(K')|.$$

This is useful to determine the Δ-Gordian distance of knots. In particular the Δ-unknotting numbers for torus knots, some positive knots and positive closed 3-braids are determined by the second coefficient of the Conway polynomial [25].

Let $X_K(A)$ be the Kauffman polynomial of a virtual knot K. If we substitute A with e^x and expand it in the Maclaurin series then the coefficient of x^n in the series is a Vassiliev invariant of type n [15]. Let K_2 be a virtual knot obtained from a virtual knot K_1 by applying a single Δ-move. Then the coefficient of x^2 in the Maclaurin series of $X_{K_1}(e^x)$ and that of $X_{K_2}(e^x)$ differ by 48 [11].

L. H. Kauffman introduced state models of the Jones polynomial for classical knots and links [12, 14] and then extended it for virtual knots and links [15]. Y. Miyazawa gave several polynomial invariants of virtual knots. In particular he constructed a multi-variable polynomial invariant of virtual links, which generalize the Kauffman polynomial [18].

H. A. Dye and L. H. Kauffman defined the arrow polynomial of a virtual link which is a generalization of the Kauffman polynomial and showed that we can get a lower bound for the virtual crossing number from the arrow polynomial [2].

The multi-variable Miyazawa polynomial and the arrow polynomial were found independently but basically are the same. Actually we may get the arrow polynomial from the multi-variable Miyazawa polynomial and vice versa by a suitable change of variables. Recall that all virtual links are assumed to be oriented in this paper. Although we have interest on virtual links, we will define the arrow bracket polynomial $\langle L \rangle$ of a polar link diagram L. This extension will be useful to prove some lemmas.

We modify the original definition of the arrow polynomial given by H. A. Dye and L. H. Kauffman a little bit. We define the arrow bracket polynomial $\langle L \rangle$ of a polar link diagram L by using the following relations.

1. $\langle L_+ \rangle = A\langle L_0 \rangle + A^{-1}\langle L_\infty \rangle$ and $\langle L_- \rangle = A^{-1}\langle L_0 \rangle + A\langle L_\infty \rangle$, where $L_+,\ L_-,\ L_0$, and L_∞ are polar link diagrams as shown in Figure 14.
2. $\langle L_1 \rangle = \langle L_2 \rangle$, if L_1 and L_2 can be related by a polar move or a virtual move.
3. $\langle O \rangle = 1$, $\langle O_m \rangle = Y_m$, $\langle L' \cup O \rangle = (-A^2 - A^{-2})\langle L' \rangle$ and $\langle L' \cup O_m \rangle = (-A^2 - A^{-2})Y_m\langle L' \rangle$ for any polar link diagram L', where O is the trivial knot diagram and O_m is the polar link diagram with $2m$ poles as shown in Figure 15.

If a function v from the set of all polar link diagrams to a set takes the same value for any pair of equivalent polar link diagrams, then it is called an *invariant* of polar links. We define the *sign* of a crossing of a polar link diagram as shown in Figure 16. The *writhe* $w(L)$ of a pole diagram L is defined to be the sum of signs of all crossings of L.

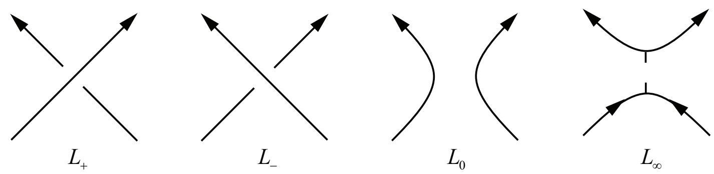

Figure 14:

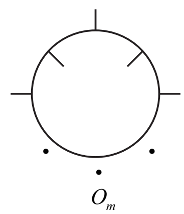

Figure 15:

H. A. Dye and L. H. Kauffman showed that the polynomial $\langle L\rangle$ of a virtual link is invariant under the second Reidemeister moves, the third Reidemeister moves and the virtual moves. Note that $w(L)$ is invariant under all Reidemeister moves and virtual moves except for the first Reidemeister move. They obtained an invariant $f_L(A, Y_1, \ldots)$ by normalizing $\langle L\rangle$ by the formula

$$f_L(A, Y_1, \ldots) = (-A^3)^{-w(L)}\langle L\rangle.$$

Following their argument we can see that $f_L(A, Y_1, \ldots)$ is a polar link invariant. The two polynomials $\langle L\rangle$ and $f_L(A, Y_1, \ldots)$ take values in the polynomial ring $\mathbb{Z}[A, A^{-1}, Y_1, Y_2, \ldots]$. In particular if we put $Y_1 = Y_2 = \ldots = 1$ in $f_L(A, Y_1, Y_2, \ldots)$ we get the Kauffman polynomial $X_L(A)$ of a virtual link L.

Let $L_+,\ L_-,\ L_0$ and L_∞ be polar link diagrams which are identical except for the shown parts in Figure 14. Since $f_L = (-A^3)^{-w(L)}\langle L\rangle$ for any polar link diagram L, and

$$\begin{cases}\langle L_+\rangle = A\langle L_0\rangle + A^{-1}\langle L_\infty\rangle,\\ \langle L_-\rangle = A^{-1}\langle L_0\rangle + A\langle L_\infty\rangle,\end{cases}$$

we get the following

Lemma 2.1.([2, 13]) *For the quadruple $(L_+, L_-, L_0, L_\infty)$ of polar link diagrams as shown in Figure 14, we get identities*

$$\begin{cases}f_{L_+}(A) = -A^{-2}f_{L_0}(A) - A^{-4}f_{L_\infty} \ \textit{and}\\ f_{L_-}(A) = -A^{2}f_{L_0}(A) - A^{4}f_{L_\infty}.\end{cases}$$

Figure 16: The sign of a crossing.

Similarly to the case of virtual links, we define singular polar links and Vassiliev invariants of polar links. Let $n_0, n_1, \ldots$ be integers such that $n_0 \neq 0$. For a polar link diagram L, we define $h_L(A) \in \mathbb{Z}[A, A^{-1}]$ by

$$h_L(A) = f_L(A^{n_0}, A^{n_1}, \ldots)$$

where $f_L(A, Y_1, \ldots) \in \mathbb{Z}[A, A^{-1}, Y_1, \ldots]$ is the arrow polynomial of L.

Let $v_n(L)$ be the coefficient of x^n in the Maclaurin series of $h_L(e^x)$ for a polar link L. L. H. Kauffman showed that the coefficients in the power expansion of $X_K(e^x)$ are Vassiliev invariants [15]. We can also get numerical Vassiliev invariants $v_n(L)$ of type n from the arrow polynomial as following

Lemma 2.2. *For integers $n_0, n_1, \ldots$ with $n_0 \neq 0$, let $h_L(A)$ be obtained from the arrow polynomial by setting $h_L(A) = f_L(A^{n_0}, A^{n_1}, \ldots)$. Then the coefficient $v_n(L)$ of x^n in the Maclaurin series of $h_L(e^x)$ is a Vassiliev invariant of type n for polar links.*

Proof. Let L_+, L_-, L_0, L_∞ be polar link diagrams as in Figure 14. By Lemma 2.1, we see that

$$h_{L_+}(A) - h_{L_-}(A) = (A^2 - A^{-2})h_{L_0}(A) + (A^4 - A^{-4})h_{L_\infty}(A).$$

We substitute e^x for A. Then

$$\begin{aligned} A^2 - A^{-2} &= e^{2x} - e^{-2x} \\ &= 2\left(\frac{2x}{1!} + \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} + \ldots\right) \end{aligned}$$

and

$$\begin{aligned} A^4 - A^{-4} &= e^{4x} - e^{-4x} \\ &= 2\left(\frac{4x}{1!} + \frac{(4x)^3}{3!} + \frac{(4x)^5}{5!} + \ldots\right). \end{aligned}$$

Then $h_{L_+}(e^x) - h_{L_-}(e^x)$ is a multiple of x. If L has a singular point then the expansion of $h_L(e^x)$ is a multiple of x. Inductively we see that $h_L(e^x)$ is a multiple of x^m if L has m singular points. Therefore we see that $v_n(L)$ is a Vassiliev invariant of type n. $\square$

We extended the Okada's theorem for virtual knots by using the normalized bracket polynomial $X_K(A)$. The crossing change, which transforms a positive crossing to a negative crossing or vice versa, is an unknotting operation for classical links but it is not for virtual ones. So the skein relations which define the Conway polynomial are not enough to define an invariant for virtual links. H. Murakami showed that $V_K^{(2)}(1) = -6a_2(K)$ for any knot K, where $V_K(t) = X_K(t^{-1/4})$ [19]. We will see that any Vassiliev invariant of type < 2 is invariant under the Δ-move. Assume that K_1 and K_2 are knots related by a Δ-move. We have proved that $\frac{X''_{K_1}(1)+X'_{K_1}(1)}{2} - \frac{X''_{K_2}(1)+X'_{K_2}(1)}{2} = \pm 48$ in [11]. Therefore

$$X''_{K_1}(1) - X''_{K_2}(1) = \pm 96.$$

2. Main Theorem

For integers $n_0, n_1, \ldots$ with $n_0 \neq 0$, let $v_n(L)$ be the coefficient of $h_L(e^x)$ where $h_L(A) = f_L(A^{n_0}, A^{n_1}, \ldots)$. If K_1 and K_2 are virtual knots related by a Δ-move, then we represent the difference of K_1 and K_2 as a difference of two singular virtual knots in a Vassiliev skein module. Then by calculating $v_2(K_1) - v_2(K_2)$ we will show that $h''_{K_1}(1) - h''_{K_2}(1) = \pm 96$.

Let $\mathcal{K}$ be the set of singular virtual knot diagrams modulo the equivalence relation of virtual knot diagrams and $\mathbb{Z}\mathcal{K}$ be the free $\mathbb{Z}$-module generated by $\mathcal{K}$. We also denote by $\mathcal{V}$ the submodule of $\mathbb{Z}\mathcal{K}$ generated by the relations

$$L_\times = L_+ - L_-,$$

where $L_\times, L_+$ and L_- are singular virtual knot diagrams which are identical except for the shown parts in Figure 13. We will often denote the equivalence class of a singular virtual knot diagram K in $\mathbb{Z}\mathcal{K}/\mathcal{V}$ simply by K too. We extend a Vassiliev invariant on $\mathbb{Z}\mathcal{K}$ linearly and it induces a natural quotient map on $\mathbb{Z}\mathcal{K}/\mathcal{V}$.

Lemma 3.1. *Let K_1 and K_2 be virtual knots related by a Δ-move and $K_{\times_{11}\times_{12}}$ and $K'_{\times_{21}\times_{22}}$ be 2-singular knots as shown in Figure 17. Then, in $\mathbb{Z}\mathcal{K}/\mathcal{V}$ we have*

$$K_1 - K_2 = K_{\times_{11}\times_{12}} - K'_{\times_{21}\times_{22}}.$$

Proof. See Figure 18. $\square$

From now on $n_0, n_1, \ldots$ are assumed to be integers such that $n_0 \neq 0$, unless otherwise stated. Let $h_L(A) = f_L(A^{n_0}, A^{n_1}, \ldots)$ be obtained from the arrow polynomial $f_L(A, Y_1, Y_2, \ldots)$ by substituting variables. Since $v_n(L)$ is defined as the coefficient of x^n in the Maclaurin series of $h_L(e^x)$, $v_n(L)$ can be given as $v_n(L) = \frac{H_L^{(n)}(0)}{n!}$, where $H_L(x) = h_L(e^x)$.

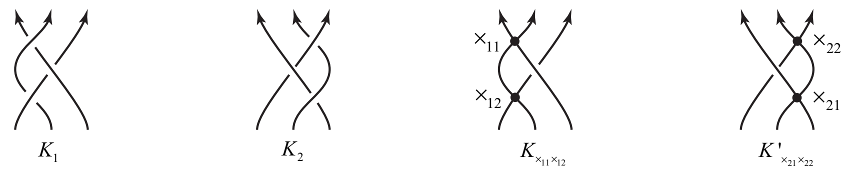

Figure 17:

Then we see that
$$\begin{cases} v_0(L) = H_L(0) = h_L(1), \\ v_1(L) = H'_L(0) = h'_L(1), \\ v_2(L) = \frac{H''_L(0)}{2} = \frac{h''_L(1)+h'_L(1)}{2}. \end{cases}$$
We also see that
$$\begin{cases} h_L(1) = v_0(L), \\ h'_L(1) = v_1(L), \\ h''_L(1) = 2v_2(L) - v_1(L). \end{cases}$$

Lemma 3.2. *For any polar link diagram L, $v_0(L) = (-2)^{\mu(L)-1}$ where $\mu(L)$ is the number of components of L.*

Proof. Since $v_0(L) = h_L(1)$, we will show that $h_L(1) = f_L(1, 1, \ldots) = (-2)^{\mu(L)-1}$. We will apply mathematical induction on the number of crossings of L. If L has no crossing then $h_L(1) = f_L(1, 1, \ldots) = (-2)^{\mu(L)-1}$. Assume that L has a positive crossing. Let $L = L_+$, L_-, L_0 and L_∞ be polar link diagrams as shown in Figure 14. Since
$$\begin{cases} f_{L_+}(A, K_1, \ldots) = -A^{-2} f_{L_0}(A, K_1, \ldots) - A^{-4} f_{L_\infty}(A, K_1, \ldots), \\ f_{L_-}(A, K_1, \ldots) = -A^{2} f_{L_0}(A, K_1, \ldots) - A^{4} f_{L_\infty}(A, K_1, \ldots), \end{cases}$$
we see that
$$h_{L_+}(1) = -f_{L_0}(1, 1, \ldots) - f_{L_\infty}(1, 1, \ldots) = h_{L_-}(1).$$
Assume that the polar link diagram L is as shown in Figure 19. Then $\mu(L_0) = \mu(L) + 1$ and $\mu(L_\infty) = \mu(L)$. By induction hypothesis, we get the equations
$$\begin{cases} h_{L_0}(1) = (-2)^{\mu(L_0)-1} = (-2)^{\mu(L)}, \\ h_{L_\infty}(1) = (-2)^{\mu(L_\infty)-1} = (-2)^{\mu(L)-1}. \end{cases}$$
Therefore
$$h_L(1) = h_{L_+}(1) = -h_{L_0}(1) - h_{L_\infty}(1) = -(-2)^{\mu(L)} - (-2)^{\mu(L)-1} = (-2)^{\mu(L)-1}.$$

Assume that the polar link diagram L is as shown in Figure 20. Then $\mu(L_0) =$

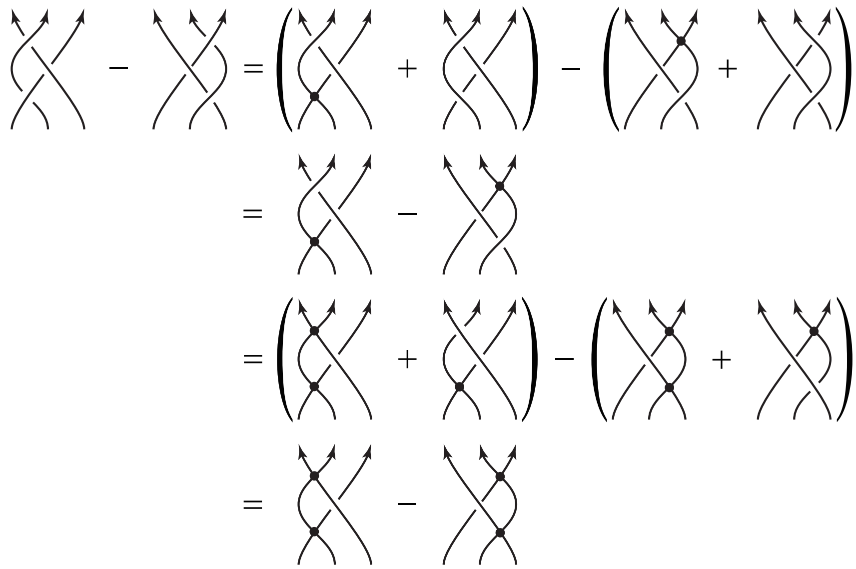

Figure 18:

$\mu(L) - 1$ and $\mu(L_\infty) = \mu(L) - 1$. By induction hypothesis, we get the equations

$$\begin{cases} h_{L_0}(1) = (-2)^{\mu(L_0)-1} = (-2)^{\mu(L)-2}, \\ h_{L_\infty}(1) = (-2)^{\mu(L_\infty)-1} = (-2)^{\mu(L)-2}. \end{cases}$$

Therefore

$$h_L(1) = h_{L_+}(1) = -h_{L_0}(1) - h_{L_\infty}(1) = -(-2)^{\mu(L)-2} - (-2)^{\mu(L)-2} = (-2)^{\mu(L)-1}.$$

Similarly we can show the statement when L has a negative crossing. □

For a singular polar link diagram $L_\times$ we can calculate $v_1(L_\times)$ and $v_2(L_\times)$ as following

Lemma 3.3. *Let $L_\times, L_+, L_-, L_0$ and L_∞ be polar link diagrams as shown in Figure 21. Then $v_i(L_\times) = 4v_{i-1}(L_0) + 8v_{i-1}(L_\infty)$ for $i = 1, 2$.*

Proof. Since $f_{L_+} - f_{L_-} = (A^2 - A^{-2})f_{L_0} + (A^4 - A^{-4})f_{L_\infty}$ by Lemma 2.1 and $h_L(A) = f_L(A^{n_0}, A^{n_1}, \ldots)$, we see that

$$h_{L_\times}(A) = h_{L_+}(A) - h_{L_-}(A) = (A^2 - A^{-2})h_{L_0}(A) + (A^4 - A^{-4})h_{L_\infty}(A).$$

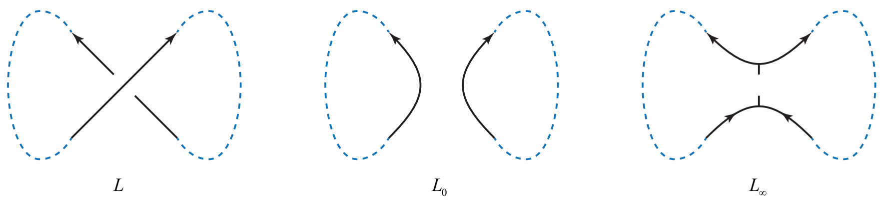

Figure 19:

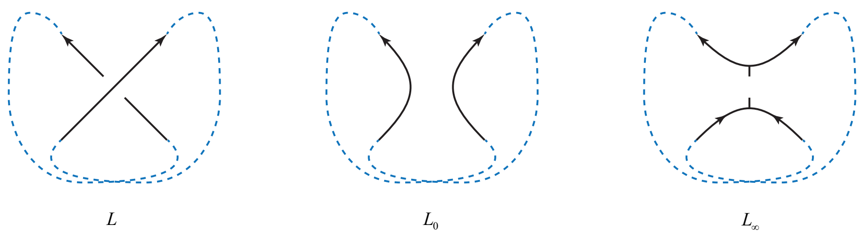

Figure 20:

By substituting the variable A with e^x, we have

$$
\begin{aligned}
h_{L_\times}(e^x) &= h_{L_+}(e^x) - h_{L_-}(e^x) \\
&= (e^{2x} - e^{-2x}) h_{L_0}(e^x) + (e^{4x} - e^{-4x}) h_{L_\infty}(e^x) \\
&= \left(4x + \frac{8}{3}x^3 + \dots\right) h_{L_0}(e^x) + \left(8x + \frac{64}{3}x^3 + \dots\right) h_{L_\infty}(e^x).
\end{aligned}
$$

Since $v_n(L)$ is the coefficient of x^n in the Maclaurin series of $h_L(e^x)$, we see that $v_i(L_\times) = 4v_{i-1}(L_0) + 8v_{i-1}(L_\infty)$ for $i = 1, 2$. $\square$

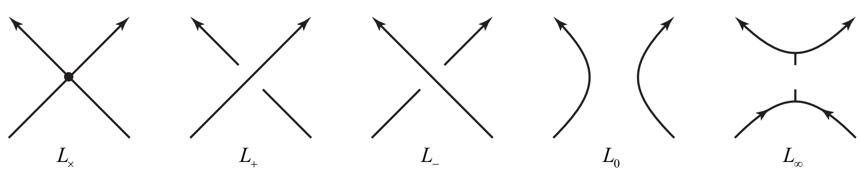

Figure 21:

Now we can evaluate $v_2(L_{\times\times})$ for a polar link diagram $L_{\times\times}$ with two singular points as following

Lemma 3.4. *Let $L_{\times\times}$ be a polar link diagram with two singular points and $v_2(L)$ be the coefficient of x^2 in the Maclaurin series of $h_L(e^x)$. Then*

$$v_2(L_{\times\times}) = 16\left((-2)^{\mu(L_{00})-1} + 2(-2)^{\mu(L_{\infty 0})-1} + 2(-2)^{\mu(L_{0\infty})-1} + 4(-2)^{\mu(L_{\infty\infty})-1}\right).$$

Proof. By applying Lemma 3.3, we see that

$$\begin{aligned} v_2(L_{\times\times}) &= 4v_1(L_0\times) + 8v_1(L_{\infty\times}) \\ &= 4\left(4v_0(L_{00}) + 8v_0(L_{0\infty})\right) + 8\left(4v_0(L_{\infty 0}) + 8v_0(L_{\infty\infty})\right) \\ &= 16\left(v_0(L_{00}) + 2v_0(L_{0\infty}) + 2v_0(L_{\infty 0}) + 4v_0(L_{\infty\infty})\right) \end{aligned}$$

By Lemma 3.2, $v_0(L) = (-2)^{\mu(L)-1}$ for any polar link diagram L.
Then we have

$$v_2(L_{\times\times}) = 16\left((-2)^{\mu(L_{00})-1} + 2(-2)^{\mu(L_{\infty 0})-1} + 2(-2)^{\mu(L_{0\infty})-1} + 4(-2)^{\mu(L_{\infty\infty})-1}\right). \quad \square$$

Now we get a necessary condition for two virtual knots to be Δ-homotopic by using the arrow polynomial as following

Theorem 3.5. *Let $n_0, n_1, \ldots$ be integers such that $n_0 \neq 0$. Let $h_L(A)$ be obtained from the arrow polynomial by setting $h_L(A) = f_L(A^{n_0}, A^{n_1}, \ldots)$. If K_1 and K_2 are virtual knots related by a single Δ-move, then*

$$h''_{K_1}(1) - h''_{K_2}(1) = \pm 96.$$

Proof. Let K_1 and K_2 be virtual knots related by a Δ-move and $K_{\times_{11}\times_{12}}$ and $K'_{\times_{21}\times_{22}}$ be singular knots as shown in Figure 17. By applying Lemma 3.1 and Lemma 3.4, we see that

$$\begin{aligned} &v_2(K_1) - v_2(K_2) \\ &= v_2(K_{\times_{11}\times_{12}}) - v_2(K'_{\times_{21}\times_{22}}) \\ &= 16\left((-2)^{\mu(K_{00})-1} + 2(-2)^{\mu(K_{\infty 0})-1} + 2(-2)^{\mu(K_{0\infty})-1} + 4(-2)^{\mu(K_{\infty\infty})-1}\right) \\ &\quad -16\left((-2)^{\mu(K'_{00})-1} + 2(-2)^{\mu(K'_{\infty 0})-1} + 2(-2)^{\mu(K'_{0\infty})-1} + 4(-2)^{\mu(K'_{\infty\infty})-1}\right). \end{aligned}$$

There are two types of K_1 as shown in Figure 22.

Assume that K_1 is of type 1. Then K_{00}, $K_{0\infty}$, $K_{\infty 0}$, $K_{\infty\infty}$, K'_{00}, $K'_{0\infty}$, $K'_{\infty 0}$, $K'_{\infty\infty}$ are as shown in Figure 23. Since $\mu(K_{00}) = 3$, $\mu(K_{0\infty}) = \mu(K_{\infty 0}) = 2$, $\mu(K_{\infty\infty}) = 1$, $\mu(K'_{00}) = \mu(K'_{0\infty}) = \mu(K'_{\infty 0}) = 1$, and $\mu(K'_{\infty\infty}) = 2$, we have

$$v_2(K_1) - v_2(K_2) = 48.$$

Assume that K_1 is of type 2. Then $\mu(K_{00}) = \mu(K_{0\infty}) = \mu(K_{\infty 0}) = 1$, $\mu(K_{\infty\infty}) = 2$, $\mu(K'_{00}) = 3$, $\mu(K'_{0\infty}) = 2$, $\mu(K'_{\infty 0}) = 2$, $\mu(K'_{\infty\infty}) = 1$. So we see that

$$v_2(K_1) - v_2(K_2) = -48.$$

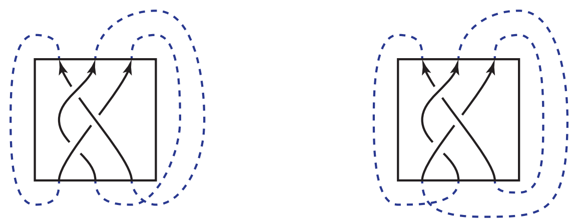

Figure 22:

Since K_1 and K_2 are related by a Δ-move, by Lemma 3.1 $v_1(K_1) = v_1(K_2)$. Since $h''_L(1) = 2v_2(L) - v_1(L)$ for any polar link L, we see that

$$\begin{aligned} h''_{K_1}(1) - h''_{K_2}(1) &= 2v_2(K_1) - v_1(K_1) - 2v_2(K_2) + v_1(K_2) \\ &= \pm 96. \end{aligned}$$ □

Take $h_L(A) = f_L(A, 1, \ldots) = X_L(A)$ in Theorem 3.5. If K_1 and K_2 are related by a Δ-move then by Lemma 3.1, $v_1(K_1) = v_1(K_2)$. Since $v_1(L) = h'_L(1)$, we have the following

Corollary 3.6.([11]) *Let K_2 be a virtual knot obtained from a virtual knot K_1 by applying a single Δ-move. Then*

$$vx_2(K_1) - vx_2(K_2) = \pm 48,$$

where $vx_2(K) = \frac{X''_K(1) + X'_K(1)}{2}$.

We have a lower bound for the Δ-Gordian distance of homotopic virtual knots as following

Corollary 3.7. *Let $n_0, n_1, \ldots$ be fixed integers such that $n_0 \neq 0$. Let $h_K(A) = f_K(A^{n_0}, A^{n_1}, \ldots)$ be obtained from the arrow polynomial $f_L(A, Y_1, \ldots)$ by change of variables. If K_1 and K_2 are Δ-homotopic virtual knots then*

$$d_G^{\Delta}(K_1, K_2) \geq \frac{|h''_{K_1}(1) - h''_{K_2}(1)|}{96}.$$

The following example shows that Theorem 3.5 is useful to determine whether two given virtual knots are Δ-homotopic or not.

Example 3.8. Let K be the virtual knot as shown in Figure 24. Then by calculating the arrow polynomial of K we see that

$$f_K(A, Y_1, \ldots) = A^8 + (1 - A^8)Y_1^2 \text{ and } X_K(A) = 1.$$

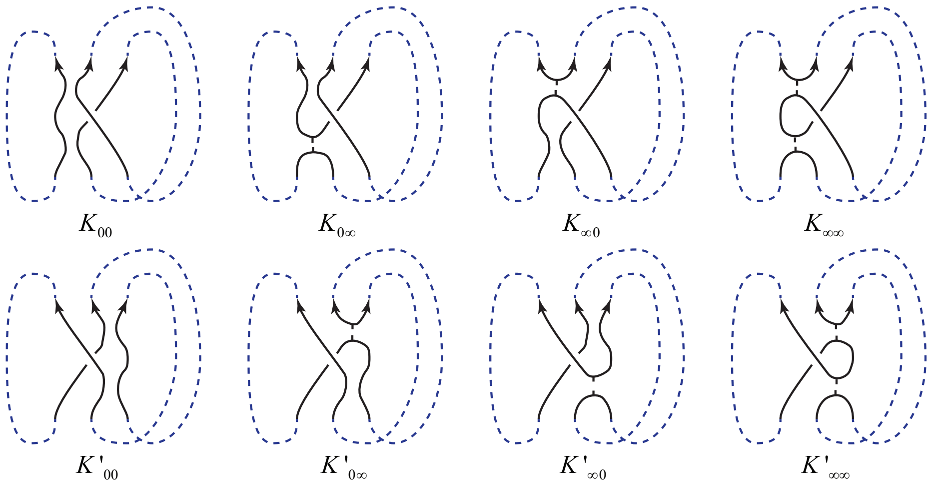

Figure 23:

Take $h_L(A) = f_L(A, A, 1, 1, \ldots)$ for any virtual link L. Then $h''_K(1) = -32$ and $X''_K(1) = 0$. Although we may not determine whether K is Δ-homotopic to the trivial knot or not by $X_K(A)$, we see that K is not Δ-homotopic to the trivial knot by applying Theorem 3.5.

If two virtual knots are Δ-homotopic then they are homotopic. We see that the knot K in Figure 24 is neither Δ-homotopic nor homotopic to the trivial knot. Therefore the Δ-move is not an unknotting operation for virtual knots.

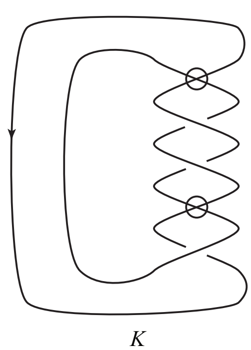

Figure 24:

In the following example we can see that Corollary 3.7 is useful to find the

Δ-Gordian distance between two Δ-homotopic virtual knots.

Example 3.9. Let K_1 and K_2 be the virtual knots as shown in Figure 25. Then

$$f_{K_1}(A, Y_1, \ldots) = 2 - 2A^4 + 2A^8 - A^{12} + (-A^{-6} + 2A^{-2} - 2A^2 + A^6)Y_1$$

and

$$f_{K_2}(A, Y_1, \ldots) = A^{-8} - A^{-4} + 2 - 2A^4 + A^8 + (-A^6 + A^{10})Y_1.$$

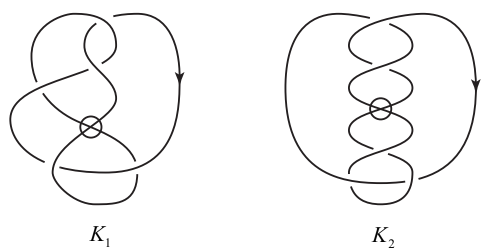

Figure 25:

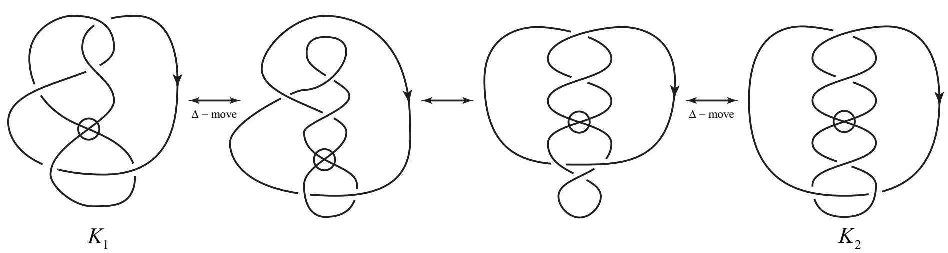

Figure 26:

Take $h_L(A) = f_L(A, A^6, 1, 1, \ldots)$ for any virtual link L. Since

$$h_{K_1}(A) = 1$$

and

$$h_{K_2}(A) = A^{-8} - A^{-4} + 2 - 2A^4 + A^8 - A^{12} + A^{10},$$

we have

$$h''_{K_1}(1) - h''_{K_2}(1) = -192.$$

If we assume that K_1 and K_2 are Δ-homotopic, then by Corollary 3.7, we get a lower bound for the Δ-Gordian distance $d_G^{\Delta}(K_1, K_2)$ between K_1 and K_2 as following

$$d_G^{\Delta}(K_1, K_2) \geq 2.$$

We can see that K_1 and K_2 are Δ-homotopic and $d_G^{\Delta}(K_1, K_2) = 2$ from Figure 26.

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References

[1] J. S. Birman and X.–S. Lin, *Knot polynomials and Vassiliev's invariants*, Invent. Math., **111**(1993), 225–270.

[2] H. A. Dye and L. H. Kauffman, *Virtual crossing number and the arrow polynomial*, J. Knot Theory Ramifications, **18(10)**(2009), 1335–1357.

[3] M. Goussarov, M. Polyak and O. Viro, *Finite type invariants of classical and virtual knots*, Topology, **39**(2000), 1045–1068.

[4] M. N. Goussarov, *On n-equivalence of knots and invariants of finite degree*, in "Topology of Manifolds and Varieties"(O. Viro, Editor), Amer. Math. Soc., Providence 1994, 173–192.

[5] M. N. Goussarov, *Finite type invariants and n-equivalence of* 3*-manifolds*, C. R. Acad. Sci. Paris, Topologie/Topology 1999, t. 329, Série I, 517–522.

[6] M. N. Goussarov, *Variations of knotted graphs. The geometric technique of n-equivalence*, Algebra i Analiz, **12(4)**(2000), 79–125.

[7] K. Habiro, *Claspers and finite type invariants of links*, Geom. and Topol., **4**(2000), 1–83.

[8] A. Ishii, *The pole diagram and the Miyazawa polynomial*, Internat. J. Math., **19(2)**(2008), 193–207.

[9] T. Kanenobu, *Forbidden moves unknot a virtual knot*, J. Knot Theory Ramifications, **10(1)**(2001), 89–96.

[10] M.–J. Jeong, *A lower bound for the number of forbidden moves to unknot a long virtual knot*, J. Knot Theory Ramifications, **22(6)**(2013), 1350024, 13pp.

[11] M.–J. Jeong, *Delta moves and Kauffman polynomials of virtual knots*, J. Knot Theory Ramifications, **23(10)**(2014), 1450053, 17pp.

[12] V. F. R. Jones, *A polynomial invariant for knots via von Neumann algebras*, Bull. Amer. Math. Soc., **12**(1985), 103–111.

[13] M.–J. Jeong and C.–Y. Park, *Similarity indices and the Miyazawa polynomials of virtual links*, J. Knot Theory Ramifications, **23(7)**(2014), 1460003, 17pp.

[14] L. H. Kauffman, *State models and the Jones polynomial*, Topology, **26**(1987), 395–407.

[15] L. H. Kauffman, *Virtual knot theory*, Europ. J. Combinatorics, **20**(1999), 663–690.

[16] T. Kanenobu and R. Nikkuni, *Delta move and polynomial invariants of links*, Topology Appl., **146/147**(2005), 91–104.

[17] Y. Miyazawa, *Magnetic graphs and an invariant for virtual links*, J. Knot Theory Ramifications, **15(10)**(2006), 1319–1334.

[18] Y. Miyazawa, *A multi–variable polynomial invariant for virtual knots and links*, J. Knot Theory Ramifications, **17(11)**(2008), 1311–1326.

[19] H. Murakami, *On derivatives of the Jones polynomial*, Kobe J. Math., **3**(1986), 61–64.

[20] H. Murakami and Y. Nakanishi, *On a certain move generating link-homology*, Math. Ann., **284**(1989), 75–89.

[21] Y. Nakanishi, *Delta link homotopy for two component links*, Topology Appl., **121**(2002), 169–182.

[22] Y. Nakanishi and T. Shibuya, *Relations among self Delta-equivalence and self Sharp-equivalences for links*, Knots in Hellas '98, World Scientific, Singapore, 2000, 353–360.

[23] Y. Nakanishi and Y. Ohyama, *Knots with given finite type invariants and C_k-distances*, J. Knot Theory Ramifications, **10(7)**(2001), 1041–1046.

[24] S. Nelson, *Unknotting virtual knots with Gauss diagram forbidden moves*, J. Knot Theory Ramifications, **10(6)**(2001), 931–935.

[25] K. Nakamura, Y. Nakanishi and Y. Uchida, *Delta-unknotting number for knots*, J. Knot Theory Ramifications, **7(5)**(1998), 639–650.

[26] M. Okada, *Delta-unknotting operation and the second coefficient of the Conway polynomial*, J. Math. Soc. Japan, **42**(1990), 713–717.

[27] M. Sakurai, *2- and 3-variations and finite type invariants of degree 2 and 3*, J. Knot Theory Ramifications, **22(8)**(2013), 11350042, 20pp.