On the Stability of a Mixed Type Functional Equation

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ABSTRACT. In this paper, we investigate the stability of the functional equation

$$\begin{aligned} & f(-x+y+z+w) + f(x-y+z+w) + f(x+y-z+w) + f(x+y+z-w) \\ & = 3f(x) + f(-x) + 3f(y) + f(-y) + 3f(z) + f(-z) + 3f(w) + f(-w) \end{aligned}$$

by using the direct method in the sense of Hyers.

1. Introduction

In 1940, Ulam [19] gave a wide ranging talk before the mathematics club of the University of Wisconsin in which he discussed a number of important unsolved problems. Among those was the question concerning the stability of group homomorphisms: Let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot, \cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G_1 \rightarrow G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \rightarrow G_2$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?

The Ulam's problem for the case of approximately additive functions was solved by Hyers [9] under the assumption that G_1 and G_2 are Banach spaces. Indeed, Hyers proved that each solution of the inequality $\|f(x+y) - f(x) - f(y)\| \leq \varepsilon$, for all x and y , can be approximated by an exact solution, say an additive function. In this case, the Cauchy additive functional equation, $f(x+y) = f(x) + f(y)$, is said to satisfy the Hyers-Ulam stability.

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Rassias [18] attempted to weaken the condition for the bound of the norm of the Cauchy difference as follows

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p)$$

and derived Hyers' theorem for the stability of the additive mapping as a special case. Thus in [18], a proof of the generalized Hyers-Ulam stability for the linear mapping between Banach spaces was obtained. In 1950, a special case of Rassias' theorem for the stability of the additive mapping was obtained by Aoki [1] (see also [8], [16], [18]).

The stability concept that was introduced by Rassias' theorem provided large influence to a number of mathematicians to develop the notion of what is known today with the term generalized Hyers-Ulam stability of the linear mappings. Since then, the stability of several functional equations has been extensively investigated by several mathematicians (see, for example, [7], [10], [11], [12], [14], [15] and the references therein).

Almost all subsequent proofs, in this very active area, have used the Hyers' idea from [9]. Namely, starting from the given mapping f , the solution F of a functional equation is explicitly constructed by

$$F(x) = \lim_{n \rightarrow \infty} \frac{1}{2^n} f(2^n x) \quad \text{or} \quad F(x) = \lim_{n \rightarrow \infty} 2^n f\left(\frac{x}{2^n}\right).$$

This method of Hyers is called the direct method. There are also other methods for proving the Hyers-Ulam stability of functional equations and differential equations. One of these methods is called the fixed point method that was applied for the first time by Baker (see [2] and also [3], [4], [5], [6], [17]).

Now, we consider the following functional equation

$$\begin{aligned} & f(-x+y+z+w) + f(x-y+z+w) + f(x+y-z+w) \\ (1.1) \quad & + f(x+y+z-w) \\ & = 3f(x) + f(-x) + 3f(y) + f(-y) + 3f(z) + f(-z) + 3f(w) + f(-w), \end{aligned}$$

which is called the mixed type functional equation. The mapping $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = ax^2 + bx$ is a solution of this functional equation, where a, b are real constants. Every solution of (1.1) is called a quadratic-additive mapping.

In 1998, Jung [13] proved the stability of Eq. (1.1) by decomposing f into the odd and even parts. In his proof, it was necessary to let an additive mapping A and a quadratic mapping Q approximate the odd and even parts of f , respectively, and combine A and Q to prove the existence of a quadratic-additive mapping F which is close to the mapping f .

In this paper, we will prove the stability of the quadratic-additive functional equation (1.1) by making use of the direct method. Indeed, we will approximate the given mapping f by a solution F of Eq. (1.1) without decomposing f into its odd and even parts, while in the previous papers [13] the mapping f was decomposed into

the odd and even parts and they were separately approximated by the corresponding parts of a solution F of Eq. (1.1), respectively.

2. Main Results

Throughout this paper, let X be a (real or complex) linear space and Y a Banach space. For an arbitrarily fixed $p \in \mathbb{R}$, put $s = \text{sign}(2 - p)$ and $t = \text{sign}(1 - p)$.

For a given mapping $f : X \rightarrow Y$, we use the following abbreviations

$$\begin{aligned} f_o(x) &:= \frac{f(x) - f(-x)}{2}, \\ f_e(x) &:= \frac{f(x) + f(-x)}{2}, \\ J_n f(x) &:= \frac{16^{-sn}}{2} (f(4^{sn}x) + f(-4^{sn}x)) + \frac{4^{-tn}}{2} (f(4^{tn}x) - f(-4^{tn}x)), \\ J'_n f(x) &:= \frac{4^{-sn}}{2} (f(2^{sn}x) + f(-2^{sn}x)) + \frac{2^{-tn}}{2} (f(2^{tn}x) - f(-2^{tn}x)), \\ Af(x, y) &:= f(x + y) - f(x) - f(y), \\ Qf(x, y) &:= f(x + y) + f(x - y) - 2f(x) - 2f(y), \\ Df(x, y, z, w) &:= f(-x + y + z + w) + f(x - y + z + w) + f(x + y - z + w) \\ &\quad + f(x + y + z - w) - 3f(x) - f(-x) - 3f(y) - f(-y) \\ &\quad - 3f(z) - f(-z) - 3f(w) - f(-w) \end{aligned}$$

for all $x, y, z, w \in X$. Then we have

$$\begin{aligned} Df_e(x, y, z, w) &= \frac{1}{2}Df(x, y, z, w) + \frac{1}{2}Df(-x, -y, -z, -w), \\ Df_o(x, y, z, w) &= \frac{1}{2}Df(x, y, z, w) - \frac{1}{2}Df(-x, -y, -z, -w) \end{aligned}$$

for all $x, y, z, w \in X$.

If f is a solution of the functional equation $Df(x, y, z, w) = 0$ for all $x, y, z, w \in X$, then f is called a quadratic-additive mapping.

Theorem 2.1. *A mapping $f : X \rightarrow Y$ is a solution of (1.1) if and only if f_e is a quadratic mapping and f_o is an additive mapping.*

Proof. Let $f : X \rightarrow Y$ satisfy $Df(x, y, z, w) = 0$ for all $x, y, z, w \in X$. Then we get

$$\begin{aligned} Qf_e(x, y) &= \frac{3Df_e(x, y, 0, 0) - Df_e(x, y, 0, 0)}{4} = 0, \\ Af_o(x, y) &= \frac{Df_o(x, y, 0, 0)}{2} = 0 \end{aligned}$$

for all $x, y, z, w \in X$, i.e., f_e is a quadratic mapping and f_o is an additive mapping.

Conversely, assume that f_e is a quadratic mapping and f_o is an additive mapping. Then we get

$$\begin{aligned}
 Df(x, y, z, w) &= Df_e(x, y, z, w) + Df_o(x, y, z, w) \\
 &= Qf_e(x + y, z - w) + Qf_e(x - y, z + w) + 2Qf_e(x, y) \\
 &\quad + 2Qf_e(z, w) + Af_o(x + y, z - w) \\
 &\quad + Af_o(z + w, x - y) + 2Af_o(x, y) + 2Af_o(z, w) \\
 &= 0
 \end{aligned}$$

for all $x, y, z, w \in X$, i.e., f is a solution of (1.1). $\square$

In the following lemma, we will prove that a mapping f is also quadratic-additive provided $Df(x, y, z, w) = 0$ holds for x, y, z, w except zero.

Lemma 2.2. *If a mapping $f : X \rightarrow Y$ satisfies $Df(x, y, z, w) = 0$ for all $x, y, z, w \in X \setminus \{0\}$, then f is a quadratic-additive mapping.*

Proof. It is enough to show that $Df(x, y, z, w) = 0$ for all $x, y, z, w \in X$. For an arbitrarily fixed $x \in X \setminus \{0\}$, we get

$$\begin{aligned}
 f(0) &= \frac{1}{3}(Df(x, x, x, -x) - \frac{1}{4}Df(2x, 2x, 2x, 2x) \\
 &\quad - \frac{3}{4}Df(x, x, x, x) - \frac{1}{4}Df(-x, -x, -x, -x)) \\
 &= 0.
 \end{aligned}$$

Moreover, we have

$$\begin{aligned}
 &Df(x, y, z, 0) \\
 &= Df\left(\frac{x}{2}, y, z, -\frac{x}{2}\right) + Df\left(\frac{x}{2}, y, -x, \frac{x}{2}\right) - Df\left(\frac{y}{2}, \frac{y}{2}, \frac{z}{2}, -\frac{z}{2}\right) \\
 &\quad - Df\left(-\frac{y}{2}, \frac{y}{2}, \frac{z}{2}, \frac{z}{2}\right) - \frac{3}{4}Df\left(\frac{x}{2}, \frac{x}{2}, \frac{x}{2}, \frac{x}{2}\right) - \frac{1}{4}Df\left(-\frac{x}{2}, -\frac{x}{2}, -\frac{x}{2}, -\frac{x}{2}\right) \\
 &\quad + \frac{3}{4}Df\left(\frac{y}{2}, \frac{y}{2}, \frac{y}{2}, \frac{y}{2}\right) + \frac{1}{4}Df\left(-\frac{y}{2}, -\frac{y}{2}, -\frac{y}{2}, -\frac{y}{2}\right) \\
 &\quad + \frac{3}{4}Df\left(\frac{z}{2}, \frac{z}{2}, \frac{z}{2}, \frac{z}{2}\right) + \frac{1}{4}Df\left(-\frac{z}{2}, -\frac{z}{2}, -\frac{z}{2}, -\frac{z}{2}\right) \\
 &= 0
 \end{aligned}$$

for all $x, y, z \in X \setminus \{0\}$.

Furthermore, it is easy to prove that

$$\begin{aligned} Df(x, y, 0, 0) &= Df\left(\frac{y}{2}, \frac{y}{2}, \frac{x}{2}, -\frac{x}{2}\right) + Df\left(-\frac{y}{2}, \frac{y}{2}, \frac{x}{2}, \frac{x}{2}\right) \\ &\quad - \frac{3}{4}Df\left(\frac{x}{2}, \frac{x}{2}, \frac{x}{2}, \frac{x}{2}\right) - \frac{1}{4}Df\left(-\frac{x}{2}, -\frac{x}{2}, -\frac{x}{2}, -\frac{x}{2}\right) \\ &\quad - \frac{3}{4}Df\left(\frac{y}{2}, \frac{y}{2}, \frac{y}{2}, \frac{y}{2}\right) - \frac{1}{4}Df\left(-\frac{y}{2}, -\frac{y}{2}, -\frac{y}{2}, -\frac{y}{2}\right) \\ &= 0 \end{aligned}$$

for all $x, y \in X \setminus \{0\}$. By considering the symmetry in the variables x, y, z, w , it completes the proof. $\square$

In the following theorem, we can prove the generalized Hyers-Ulam stability of the functional equation (1.1) by making use of the direct method.

Theorem 2.3. *Let θ be a nonnegative constant. If a mapping $f : X \rightarrow Y$ satisfies the inequality*

$$(2.1) \quad \|Df(x, y, z, w)\| \leq \theta(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p)$$

for all $x, y, z, w \in X \setminus \{0\}$ with a real constant $p \notin \{1, 2\}$, then there exists a unique quadratic-additive mapping $F : X \rightarrow Y$ such that

$$(2.2) \quad \|f(x) - F(x)\| \leq \begin{cases} \frac{4\theta\|x\|^p}{4^p-16} & (\text{for } p > 2), \\ \left(\frac{4\theta}{16-4^p} + \frac{4\theta}{4^p-4}\right)\|x\|^p & (\text{for } 1 < p < 2), \\ \frac{4\theta\|x\|^p}{4-4^p} & (\text{for } 0 < p < 1), \\ \frac{29}{15}\theta & (\text{for } p = 0) \end{cases}$$

for all $x \in X \setminus \{0\}$ and f is itself a quadratic-additive mapping provided $p < 0$.

Proof. By a tedious calculation, we have

$$\begin{aligned} \|f(0)\| &= \frac{1}{3} \left\| Df(2^n x, 2^n x, 2^n x, -2^n x) - \frac{1}{4} Df(2^{n+1} x, 2^{n+1} x, 2^{n+1} x, 2^{n+1} x) \right. \\ &\quad \left. - \frac{3}{4} Df(2^n x, 2^n x, 2^n x, 2^n x) - \frac{1}{4} Df(-2^n x, -2^n x, -2^n x, -2^n x) \right\| \\ &\leq \frac{(8 + 2^p)2^{np}\theta\|x\|^p}{3} \end{aligned}$$

holds for any fixed $x \in X \setminus \{0\}$ and all integers n . Hence, we get $f(0) = 0$ for $p \notin \{0, 1, 2\}$ and $\|f(0)\| \leq 3\theta$ for $p = 0$.

From the definitions of $J_n f(x)$ and $Df(x, y, z, w)$, applying a long calculation,

we get

$$\begin{aligned}
 J_n f(x) - J_{n+1} f(x) = & -\frac{1}{2} \left(16^{\tau-s,n} (Df(-4^{\tau_{s,n}} x, 4^{\tau_{s,n}} x, 4^{\tau_{s,n}} x, 4^{\tau_{s,n}} x) \right. \\
 & + Df(4^{\tau_{s,n}} x, -4^{\tau_{s,n}} x, -4^{\tau_{s,n}} x, -4^{\tau_{s,n}} x)) s \\
 & + 4^{\tau-t,n} (Df(-4^{\tau_{t,n}} x, 4^{\tau_{t,n}} x, 4^{\tau_{t,n}} x, 4^{\tau_{t,n}} x) \\
 & - Df(4^{\tau_{t,n}} x, -4^{\tau_{t,n}} x, -4^{\tau_{t,n}} x, -4^{\tau_{t,n}} x)) t \Big) \\
 & + 3 \cdot 16^{\tau-s,n} f(0)
 \end{aligned} \tag{2.3}$$

for all $x \in X \setminus \{0\}$ and all nonnegative integers n , where $s = \text{sign}(2-p)$, $t = \text{sign}(1-p)$, and $\tau_{k,n}$ are the integers defined by $\tau_{k,n} = k(n+1/2) - 1/2$ for $k \in \{-1, 1\}$.

It follows from (2.1) and (2.3) that

$$\begin{aligned}
 & \|J_n f(x) - J_{n+m} f(x)\| \\
 &= \sum_{j=n}^{n+m-1} \|J_j f(x) - J_{j+1} f(x)\| \\
 &\leq \frac{1}{2} \sum_{j=n}^{n+m-1} \left(\|16^{\tau-s,j} Df(-4^{\tau_{s,j}} x, 4^{\tau_{s,j}} x, 4^{\tau_{s,j}} x, 4^{\tau_{s,j}} x) s \right. \\
 &\quad + 4^{\tau-t,j} Df(-4^{\tau_{t,j}} x, 4^{\tau_{t,j}} x, 4^{\tau_{t,j}} x, 4^{\tau_{t,j}} x) t\| \\
 &\quad + \|16^{\tau-s,j} Df(4^{\tau_{s,j}} x, -4^{\tau_{s,j}} x, -4^{\tau_{s,j}} x, -4^{\tau_{s,j}} x) s \\
 &\quad - 4^{\tau-t,j} Df(4^{\tau_{t,j}} x, -4^{\tau_{t,j}} x, -4^{\tau_{t,j}} x, -4^{\tau_{t,j}} x) t\| \\
 &\quad \left. + \|6 \cdot 16^{\tau-s,j} f(0)\| \right) \\
 &\leq \begin{cases} \sum_{j=n}^{n+m-1} \left(\frac{\theta}{4^j} + \frac{9\theta}{16^{j+1}} \right) & (\text{for } p = 0), \\ \sum_{j=n}^{n+m-1} 4^{-j} \theta \|4^j x\|^p & (\text{for } p < 0 \text{ or } 0 < p < 1), \\ \sum_{j=n}^{n+m-1} \left(\frac{\theta \|4^j x\|^p}{4^{2j+1}} + \frac{4^{j+1} \theta \|x\|^p}{4^{(j+1)p}} \right) & (\text{for } 1 < p < 2), \\ \sum_{j=n}^{n+m-1} 4^{2j+1} \theta \|4^{-j-1} x\|^p & (\text{for } p > 2) \end{cases} \\
 &\leq \begin{cases} \frac{4\theta}{3 \cdot 4^n} + \frac{3\theta}{5 \cdot 16^n} & (\text{for } p = 0), \\ \frac{4^{np} \theta \|x\|^p}{4^{n-1}(4-4^p)} & (\text{for } p < 0 \text{ or } 0 < p < 1), \\ \frac{4^{np} \theta \|x\|^p}{4^{2n-1}(16-4^p)} + \frac{4^{n+1} \theta \|x\|^p}{4^{np}(4^p-4)} & (\text{for } 1 < p < 2), \\ \frac{4^{2n+1} \theta \|x\|^p}{4^{np}(4^p-16)} & (\text{for } p > 2) \end{cases}
 \end{aligned} \tag{2.4}$$

for all $x \in X \setminus \{0\}$. So, it is easy to show that the sequence $\{J_n f(x)\}$ is a Cauchy sequence for all $x \in X \setminus \{0\}$. Since Y is complete and $\lim_{n \rightarrow \infty} f(0) = 0$, the sequence $\{J_n f(x)\}$ converges for all $x \in X$.

Hence, we can define a mapping $F : X \rightarrow Y$ by

$$F(x) := \lim_{n \rightarrow \infty} J_n f(x)$$

for all $x \in X$. Moreover, putting $n = 0$ and letting $m \rightarrow \infty$ in (2.4), we obtain the inequality (2.2).

From the definition of F , we get

$$\begin{aligned}
 & DF(x, y, z, w) \\
 = & \lim_{n \rightarrow \infty} \left(\frac{16^{-sn}}{2} (Df(4^{sn}x, 4^{sn}y, 4^{sn}z, 4^{sn}w) + Df(-4^{sn}x, -4^{sn}y, -4^{sn}z, -4^{sn}w)) \right. \\
 & \quad \left. + \frac{4^{-tn}}{2} (Df(4^{tn}x, 4^{tn}y, 4^{tn}z, 4^{tn}w) - Df(-4^{tn}x, -4^{tn}y, -4^{tn}z, -4^{tn}w)) \right) \\
 \leq & \lim_{n \rightarrow \infty} (4^{-sn(2-p)} + 4^{-tn(1-p)}) \theta (\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p) \\
 = & 0
 \end{aligned}$$

for all $x, y, z, w \in X \setminus \{0\}$. By Lemma 2.2, F is a quadratic-additive mapping.

Now, we show that F is unique. Let $F' : X \rightarrow Y$ be another quadratic-additive mapping satisfying (2.2). It is easy to show that $F'(0) = 0$ for all quadratic-additive mapping F' . It follows from (2.3) that

$$\begin{aligned}
 F'(x) - J_n F'(x) &= \sum_{j=0}^{n-1} (J_j F'(x) - J_{j+1} F'(x)) \\
 &= -\frac{1}{2} \sum_{j=0}^{n-1} \left(16^{\tau-s,j} (DF'(4^{-\tau_{s,j}}x, 4^{\tau_{s,j}}x, 4^{\tau_{s,j}}x, 4^{\tau_{s,j}}x) \right. \\
 & \quad \left. + DF'(4^{\tau_{s,j}}x, -4^{\tau_{s,j}}x, -4^{\tau_{s,j}}x, -4^{\tau_{s,j}}x))s \right. \\
 & \quad \left. + 4^{\tau-t,j} (DF'(-4^{\tau_{t,j}}x, 4^{\tau_{t,j}}x, 4^{\tau_{t,j}}x, 4^{\tau_{t,j}}x) \right. \\
 & \quad \left. - DF'(4^{\tau_{t,j}}x, -4^{\tau_{t,j}}x, -4^{\tau_{t,j}}x, -4^{\tau_{t,j}}x))t \right) \\
 &= 0
 \end{aligned}$$

for all $n \in \mathbb{N}$ and for all $x \in X$.

Since F and F' are quadratic-additive, replacing x with $4^n x$ in (2.2), we have

$$\begin{aligned}
 \|F(x) - F'(x)\| &= \|J_n F(x) - J_n F'(x)\| \\
 &\leq \frac{16^{-sn}}{2} \left(\|(F - f)(4^{sn}x)\| + \|(f - F')(4^{sn}x)\| \right. \\
 & \quad \left. + \|(F - f)(-4^{sn}x)\| + \|(f - F')(-4^{sn}x)\| \right) \\
 & \quad + \frac{4^{-tn}}{2} \left(\|(F - f)(4^{tn}x)\| + \|(F' - f)(4^{tn}x)\| \right. \\
 & \quad \left. + \|(F - f)(-4^{tn}x)\| + \|(F' - f)(-4^{tn}x)\| \right) \\
 &\leq \left(\frac{8\theta}{|16 - 4^p|} + \frac{8\theta}{|4^p - 4|} \right) (4^{-sn(2-p)} + 4^{-tn(1-p)}) \|x\|^p
 \end{aligned}$$

for all $x \in X \setminus \{0\}$ and all positive integers n provided $p \neq 0$. We also have

$$\|F(x) - F'(x)\| \leq \frac{49}{15}(16^{-n} + 4^{-n})\theta$$

for all $x \in X \setminus \{0\}$ and all positive integers n provided $p = 0$.

Taking the limit in the above inequality as $n \rightarrow \infty$, we can conclude that $F(x) = F'(x)$ for all $x \in X$. This proves the uniqueness of F . Since

$$\begin{aligned} & \|f(x) - F(x)\| \\ & \leq \|Df((3k-1)x, kx, kx, kx) - DF((3k-1)x, kx, kx, kx)\| \\ & \quad + 3\|(F-f)((4k-1)x)\| + 3\|(f-F)((3k-1)x)\| \\ & \quad + \|(f-F)((1-3k)x)\| + 9\|(f-F)(kx)\| + 3\|(f-F)(-kx)\| \\ & \leq \left((3k-1)^p + 3k^p + \frac{4(3(4k-1)^p + 4(3k-1)^p + 12k^p)}{4-4^p} \right) \theta \|x\|^p \end{aligned}$$

holds for all $x \in X \setminus \{0\}$ and all $k \in \mathbb{R}$, we conclude that $f(x) = F(x)$ for all $x \in X \setminus \{0\}$ provided $p < 0$ by letting $k \rightarrow \infty$ in the above inequality. Since $f(0) = 0$, f is itself a quadratic-additive mapping. $\square$

In the case of $p \geq 0$, x, y, z, w can take 0 in the inequality (2.1). For this case, we obtain a little bit different result from Theorem 2.3.

Theorem 2.4. *Let θ be a nonnegative constant. If a mapping $f : X \rightarrow Y$ satisfies the inequality*

$$(2.5) \quad \|Df(x, y, z, w)\| \leq \theta(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p)$$

for all $x, y, z, w \in X$ with a nonnegative real constant $p \notin \{1, 2\}$, then there exists a unique quadratic-additive mapping $F : X \rightarrow Y$ such that

$$(2.6) \quad \|f(x) - F(x)\| \leq \begin{cases} \frac{\theta\|x\|^p}{2^p-4} & (\text{for } p > 2), \\ \left(\frac{\theta}{4-2^p} + \frac{\theta}{2^p-2} \right) \|x\|^p & (\text{for } 1 < p < 2), \\ \frac{\theta\|x\|^p}{2-2^p} & (\text{for } 0 < p < 1), \\ \frac{19}{9}\theta & (\text{for } p = 0) \end{cases}$$

for all $x \in X$.

Proof. Since

$$\|f(0)\| = \left\| \frac{1}{12} Df(0, 0, 0, 0) \right\| \leq \frac{\|0\|^p \theta}{3},$$

we get $f(0) = 0$ for $p \notin \{0, 1, 2\}$ and $\|f(0)\| \leq \theta/3$ for $p = 0$. From the definitions of $J_n f(x)$ and $Df(x, y, z, w)$, we get

$$\begin{aligned}
 & J'_n f(x) - J'_{n+1} f(x) \\
 &= -\frac{1}{4} \left(4^{\tau_{-s,n}} (Df(2^{\tau_{s,n}} x, 2^{\tau_{s,n}} x, 0, 0) + Df(-2^{\tau_{s,n}} x, -2^{\tau_{s,n}} x, 0, 0)) s \right. \\
 (2.7) \quad & \quad \left. + 2^{\tau_{-t,n}} (Df(2^{\tau_{t,n}} x, 2^{\tau_{t,n}} x, 0, 0) - Df(-2^{\tau_{t,n}} x, -2^{\tau_{t,n}} x, 0, 0)) t \right) \\
 & \quad + 4^{\tau_{-s,n}+1} f(0)
 \end{aligned}$$

for all $x \in X$ and all nonnegative integers n , where $s = \text{sign}(2-p)$, $t = \text{sign}(1-p)$, and $\tau_{k,n}$ are the integers defined by $\tau_{k,n} = k(n+1/2) - 1/2$ for $k \in \{-1, 1\}$.

It follows from (2.5) and (2.7) that

$$\begin{aligned}
 & \|J'_n f(x) - J'_{n+m} f(x)\| \\
 &= \sum_{j=n}^{n+m-1} \|J'_j f(x) - J'_{j+1} f(x)\| \\
 &\leq \frac{1}{4} \sum_{j=n}^{n+m-1} \left(\|4^{\tau_{-s,j}} Df(2^{\tau_{s,j}} x, 2^{\tau_{s,j}} x, 0, 0) s + 2^{\tau_{-t,j}} Df(2^{\tau_{t,j}} x, 2^{\tau_{t,j}} x, 0, 0) t\| \right. \\
 & \quad + \|4^{\tau_{-s,j}} Df(-2^{\tau_{s,j}} x, -2^{\tau_{s,j}} x, 0, 0) s \\
 & \quad \quad \left. - 2^{\tau_{-t,j}} Df(-2^{\tau_{t,j}} x, -2^{\tau_{t,j}} x, 0, 0) t\| \right. \\
 (2.8) \quad & \quad \left. + \|16 \cdot 4^{\tau_{-s,j}} f(0)\| \right) \\
 &\leq \begin{cases} \sum_{j=n}^{n+m-1} \left(\frac{\theta}{2^j} + \frac{\theta}{3 \cdot 4^j} \right) & (\text{for } p = 0), \\ \sum_{j=n}^{n+m-1} 2^{-j-1} \theta \|2^j x\|^p & (\text{for } 0 < p < 1), \\ \sum_{j=n}^{n+m-1} (2^{-2j-2} \theta \|2^j x\|^p + 2^j \theta \|2^{-j-1} x\|^p) & (\text{for } 1 < p < 2), \\ \sum_{j=n}^{n+m-1} 2^{2j} \theta \|2^{-j-1} x\|^p & (\text{for } p > 2) \end{cases} \\
 &\leq \begin{cases} \frac{2\theta}{2^n} + \frac{\theta}{9 \cdot 4^n} & (\text{for } p = 0), \\ \frac{2^{np} \theta \|x\|^p}{2^n (2-2^p)} & (\text{for } 0 < p < 1), \\ \frac{2^{np} \theta \|x\|^p}{4^n (4-2^p)} + \frac{2^n \theta \|x\|^p}{2^{np} (2^p-2)} & (\text{for } 1 < p < 2), \\ \frac{4^n \theta \|x\|^p}{2^{np} (2^p-4)} & (\text{for } p > 2) \end{cases}
 \end{aligned}$$

for all $x \in X$.

So, it is easy to show that the sequence $\{J'_n f(x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{J'_n f(x)\}$ converges for all $x \in X$. Hence, we can define a mapping $F : X \rightarrow Y$ by

$$F(x) := \lim_{n \rightarrow \infty} J'_n f(x)$$

for all $x \in X$. Moreover, putting $n = 0$ and letting $m \rightarrow \infty$ in (2.8), we get the inequality (2.6). The remaining part of the proof is similar to the proof of Theorem 2.3. $\square$

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