On a Type of Semi-Symmetric Non-Metric Connection on Riemannian Manifolds

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ABSTRACT. The object of the present paper is to characterize a Riemannian manifold admitting a type of semi-symmetric non-metric connection.

1. Introduction

In 1924, Friedmann and Schouten [1] introduced the idea of semi-symmetric connection on a differentiable manifold. A linear connection $\bar{\nabla}$ on a differentiable manifold (M^n, g) with Riemannian connection ∇ is said to be a semi-symmetric connection if the torsion tensor T of the connection $\bar{\nabla}$ satisfies

$$(1.1) \quad T(X, Y) = \eta(Y)X - \eta(X)Y,$$

where η is a 1-form and ξ is a vector field defined by $\eta(X) = g(X, \xi)$, for all vector fields $X \in \chi(M^n)$, $\chi(M^n)$ is the set of all differentiable vector fields on M^n .

In 1932, Hayden [4] introduced the idea of semi-symmetric connections on a Riemannian manifold (M^n, g) . A semi-symmetric connection $\bar{\nabla}$ is said to be a semi-symmetric metric connection if

$$\bar{\nabla}g = 0.$$

The study of semi-symmetric metric connection was further developed by Yano [6], Amur and Pujar [7], Chaki and Konar [12], De [17] and many others.

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After long gap the study of a semi-symmetric connection $\bar{\nabla}$ satisfying

$$\bar{\nabla}g \neq 0.$$

was initiated by Prvanović [9] with the name pseudo-metric semi-symmetric connection and was just followed by Andonie [14].

In 1992, Agashe and Chafle [13] introduced and studied a semi-symmetric non-metric connection. The semi-symmetric non-metric connections was further developed by several authors such as De and Biswas [18], Biswas, De and Barua [16], De and Kamilya ([20], [21]) and many others.

In 1967, R.N.Sen and M.C.Chaki [15] studied certain curvature restrictions on a certain kind of conformally flat Riemannian space of class one and they obtained the following expression of the covariant derivative of the curvature tensor :

$$(1.2) \quad K_{ijk,l}^h = 2\lambda_l K_{ijk}^h + \lambda_i K_{ljk}^h + \lambda_j K_{ilk}^h + \lambda_k K_{ijl}^h + \lambda^h K_{lijk},$$

where K_{ijk}^h are the components of the curvature tensor K with respect to the Levi-Civita connection,

$$K_{ijkl} = g_{hl} K_{ijk}^h,$$

λ_i is a non-zero covariant vector and ∇ denotes covariant differentiation with respect to the metric tensor g_{ij} .

Later in 1987, M.C.Chaki [10] called a manifold a pseudo symmetric manifold whose curvature tensor satisfies (1.2). In index free notation this can be stated as follows: A non- flat Riemannian manifold (M^n, g) , $n \geq 2$ is said to be a pseudo symmetric manifold [10] if its curvature tensor K satisfies the condition

$$\begin{aligned} (D_X K)(Y, Z)W &= 2A(X)K(Y, Z)W + A(Y)K(X, Z)W + A(Z)K(Y, X)W \\ &+ A(W)K(Y, Z)X + g(K(Y, Z)W, X)U, \end{aligned}$$

where D denotes the operator of covariant differentiation with respect to the metric tensor g . The 1- form A is called the associated 1-form of the manifold . If $A = 0$, then the manifold reduces to a symmetric manifold in the sense of Cartan [3]. An n -dimensional pseudo symmetric manifold is denoted by $(PS)_n$. In this connection we can mention the notion of weakly symmetric manifold introduced by *Tamássy* and Binh [8]. Such a manifold was denoted by $(WS)_n$.

In a recent paper De and Gazi [19] introduced a type of non-flat Riemannian manifold (M^n, g) , $n \geq 2$ whose curvature tensor K of type (1,3) satisfies the condition

(1.3)

$$(D_X K)(Y, Z)W = [A(X) + B(X)]K(Y, Z)W + A(Y)K(X, Z)W + A(Z)K(Y, X)W \\ + A(W)K(Y, Z)X + g(K(Y, Z)W, X)U,$$

where A, U and D have the meaning already mentioned and B is a non-zero 1-form, V is a vector field defined by $B(X) = g(X, V)$, $\forall X$.

Such a manifold was called an almost pseudo symmetric manifold and was denoted by $(APS)_n$.

If $B = A$, then from the definitions it follows that $(APS)_n$ deduces to a $(PS)_n$. In the same paper the authors constructed two non-trivial examples of $(APS)_n$. It may be mentioned that almost pseudo symmetric manifolds is not a particular case of weakly symmetric manifolds.

Let (M^n, g) , $(n > 3)$ be a Riemannian manifold admitting a semi-symmetric non-metric connection whose torsion tensor is almost pseudo symmetric, that is,

$$(\nabla_X T)(Y, Z) = [A(X) + B(X)]T(Y, Z) + A(Y)T(X, Z) + A(Z)T(Y, X) + g(T(Y, Z), X)U, \quad (1.4)$$

where A and B are defined earlier.

A non-flat Riemannian manifold (M^n, g) , $n \geq 3$ is said to be a quasi-Einstein manifold [11] if its Ricci tensor $\tilde{S}$ of the Levi-Civita connection is of the form

$$\tilde{S}(X, Y) = ag(X, Y) + bA(X)A(Y),$$

where a and b are smooth functions of the manifold.

In the present paper we consider a Riemannian manifold admitting a semi-symmetric non-metric connection whose torsion tensor is almost pseudo symmetric.

The paper is organized as follows:

After preliminaries in section 3, we first obtain the expressions of the curvature tensor and the Ricci tensor of the semi-symmetric non-metric connection. In this section we prove that if a Riemannian Manifold admits a semi-symmetric non-metric connection whose curvature tensor vanishes and the torsion tensor is almost pseudo symmetric with respect to the semi-symmetric non-metric connection, then the manifold becomes a quasi-Einstein manifold. Finally, we deal with a simply connected $(APS)_n$, $(n > 3)$ admitting such a semi-symmetric non-metric connection.

2. Preliminaries

Let $\tilde{r}$ denotes the scalar curvature of the manifold with respect to the Levi-Civita connection and L denote the symmetric endomorphism of the tangent space at each point corresponding to the Ricci tensor, that is,

$$(2.1) \quad g(LX, Y) = \tilde{S}(X, Y),$$

for any vector field X, Y .

Contracting Y in (1.3), it follows that

$$(2.2) \quad (D_X \tilde{S})(Z, W) = [A(X) + B(X)]\tilde{S}(Z, W) + A(K(X, Z)W) + A(Z)\tilde{S}(X, W) + A(W)\tilde{S}(X, Z) + g(K(X, Z)W, X).$$

Putting $Z = W = e_i$ in (2.2), where $\{e_i\}$, $i = 1, 2, 3, \dots, n$; is an orthonormal basis of the tangent space at each point of the manifold and taking summation over i , $1 \leq i \leq n$, we get

$$(2.3) \quad d\tilde{r}(X) = [A(X) + B(X)]\tilde{r} + 4A(LX).$$

3. Riemannian Manifolds Admitting a Special Type of the Semi-Symmetric Non-Metric Connection

Theorem 3.1. *If a Riemannian manifold admits a semi-symmetric non-metric connection whose curvature tensor R vanishes and torsion tensor T satisfies (1.4), then the manifold is a quasi-Einstein manifold.*

Proof. Let M be an n -dimensional Riemannian manifold with Riemannian metric g . If ∇ is the semi-symmetric non-metric connection of a Riemannian manifold M , then ∇ is given by [13]

$$(3.1) \quad \nabla_X Y = D_X Y + A(Y)X.$$

Let R be the curvature tensor with respect to semi-symmetric non-metric connection. Then R and K are related by [13]

$$(3.2) \quad R(X, Y)Z = K(X, Y)Z + \alpha(X, Z)Y - \alpha(Y, Z)X,$$

for all vector fields X, Y, Z on M , where α is a $(0, 2)$ tensor given by

$$(3.3) \quad \alpha(X, Z) = (D_X A)(Z) - A(X)A(Z).$$

In this section we consider a Riemannian manifold admitting a semi-symmetric non-metric connection whose torsion tensor T satisfies (1.4).

From (1.1), contracting over X , we get

$$(3.4) \quad (C_1^1 T)(Y) = (n - 1)A(Y).$$

From (3.4), it follows that

$$(3.5) \quad (\nabla_X C_1^1 T)(Y) = (n-1)(\nabla_X A)(Y).$$

Contracting over Z in (1.4) and using (3.4), we obtain

$$(3.6) \quad (\nabla_X C_1^1 T)(Y) = (2n-3)A(X)A(Y) + (n-1)B(X)A(Y) + A(U)g(X, Y).$$

From (3.5) and (3.6) yields

$$(3.7) \quad (n-1)(\nabla_X A)(Y) = (2n-3)A(X)A(Y) + (n-1)B(X)A(Y) + A(U)g(X, Y).$$

Using (3.1) and (3.3), it follows that

$$(3.8) \quad (\nabla_X A)(Y) = (D_X A)(Y) - A(X)A(Y) = \alpha(X, Y).$$

Therefore, from (3.7) and (3.8), we have

$$(3.9) \quad \alpha(X, Y) = \frac{2n-3}{n-1}A(X)A(Y) + B(X)A(Y) + \frac{1}{n-1}A(U)g(X, Y).$$

Using (3.9) in (3.2) yields

$$(3.10) \quad \begin{aligned} R(X, Y)Z &= K(X, Y)Z + \frac{2n-3}{n-1}A(X)A(Z)Y + B(X)A(Z)Y + \frac{1}{n-1}A(U)g(X, Z)Y \\ &\quad - \frac{2n-3}{n-1}A(Y)A(Z)X - B(Y)A(Z)X - \frac{1}{n-1}A(U)g(Y, Z)X. \end{aligned}$$

From (3.10), we get

$$(3.11) \quad \begin{aligned} \tilde{R}(X, Y, Z, W) &= \tilde{K}(X, Y, Z, W) + \frac{2n-3}{n-1}A(X)A(Z)g(Y, W) + B(X)A(Z)g(Y, W) \\ &\quad + \frac{1}{n-1}A(U)g(X, Z)g(Y, W) - \frac{2n-3}{n-1}A(Y)A(Z)g(X, W) \\ &\quad - \frac{1}{n-1}A(U)g(Y, Z)g(X, W) - B(Y)A(Z)g(X, W), \end{aligned}$$

where $\tilde{R}(X, Y, Z, W) = g(R(X, Y)Z, W)$ and $\tilde{K}(X, Y, Z, W) = g(K(X, Y)Z, W)$.

Putting $X = W = e_i$ in (3.11) where $\{e_i\}$, $1 \leq i \leq n$ is an orthonormal basis of the tangent space at any point of the manifold M^n and then summing over i , we

obtain

(3.12)

$$S(Y, Z) = \tilde{S}(Y, Z) - (2n - 3)A(Y)A(Z) - (n - 1)B(Y)A(Z) - A(U)g(Y, Z),$$

where S be the Ricci tensor with respect to the semi-symmetric non-metric connection.

Proposition 3.1. *If a Riemannian manifold admits a semi-symmetric non-metric connection whose torsion tensor is almost pseudo symmetric, then*

- (i) *the curvature tensor of the semi-symmetric non-metric connection is given by (3.10),*
- (ii) *the Ricci tensor of the semi-symmetric non-metric connection is given by (3.12),*
- (iii) *the Ricci tensor S is symmetric if and only if $B(Y)A(Z) = B(Z)A(Y)$.*

Suppose

$$R(X, Y)Z = 0.$$

Then from the above equation, we have

$$S(Y, Z) = 0.$$

Hence the equation (3.12) reduces to

(3.13)

$$\tilde{S}(Y, Z) = (2n - 3)A(Y)A(Z) + (n - 1)B(Y)A(Z) + A(U)g(Y, Z).$$

Since $\tilde{S}$ is symmetric, therefore

Therefore,

(3.14)

$$B(Y)A(Z) = B(Z)A(Y).$$

Putting $Z = U$ in (3.14), it follows that

(3.15)

$$B(Y) = fA(Y).$$

where $f = \frac{B(U)}{A(U)}$

Now using (3.17) in (3.15), we obtain

(3.16)

$$\tilde{S}(Y, Z) = A(U)g(Y, Z) + [(2n - 3) + (n - 1)f]A(Y)A(Z).$$

Therefore, $\tilde{S}(X, Y) = ag(X, Y) + bA(X)A(Y)$,

where $a = A(U)$ and $b = [(2n - 3) + (n - 1)f]$.

Hence the proof is completed. $\square$

4. Special Conformally Flat $(APS)_n$ Admitting a Special Type of the Semi-Symmetric Non-Metric Connection

Theorem 4.1. *If a $(APS)_n$ ($n > 3$) admits a semi-symmetric non-metric connection whose torsion tensor is almost pseudo symmetric and the curvature tensor of the semi-symmetric non-metric connection vanishes, then the manifold is a particular kind of a special conformally flat manifold, namely a subprojective manifold.*

Proof. Chen and Yano [2] introduced the notion of a special conformally flat manifold which generalizes the notion of a subprojective manifold. A conformally flat manifold is called a special conformally flat manifold if the tensor H of type $(0, 2)$ defined by

$$(4.1) \quad H(X, Y) = -\frac{1}{n-2}\tilde{S}(X, Y) + \frac{\tilde{r}}{2(n-1)(n-2)}g(X, Y),$$

is expressible in the form

$$H(X, Y) = -\frac{\alpha^2}{2}g(X, Y) + \beta(D_X\alpha)(D_Y\alpha),$$

where α and β are two scalars such that α is positive. In particular, if β is a function of α then the special conformally flat manifold is called a subprojective manifold [5].

Let us consider $(APS)_n$ admitting a semi-symmetric non-metric connection whose torsion tensor is almost pseudo symmetric and the curvature tensor of the semi-symmetric non-metric connection vanishes.

Using (3.16) in (4.1), we get

$$(4.2) \quad H(X, Y) = \frac{\tilde{r} - 2(n-1)A(U)}{2(n-1)(n-2)}g(X, Y) - \frac{2n-3+f(n-1)}{n-2}A(X)A(Y).$$

Now, put

$$(4.3) \quad \alpha^2 = -\frac{\tilde{r} - 2(n-1)A(U)}{(n-1)(n-2)}.$$

From (3.16), we get

$$(4.4) \quad \tilde{r} = (n-1)(3+f)A(U), n \geq 3.$$

Since $\tilde{r} \neq 0$, it follows that α^2 will be positive provided that $\tilde{r} < 0$.

From (3.16) and (2.1), it follows that

$$(4.5) \quad LY = [2n-3+(n-1)f]A(Y)U + A(U)Y.$$

From (4.5), we obtain

$$(4.6) \quad A(LY) = (n-1)(2+f)A(U)A(Y).$$

Using (4.6) and (3.15) in (2.3), we have

$$(4.7) \quad d\tilde{r}(X) = [(1+f)\tilde{r} + 4(n-1)(2+f)A(U)]A(X).$$

Let us take the covariant derivative of both side of (4.3) with respect to X and using (4.7), we obtain

$$(4.8) \quad D_X \alpha = -\frac{(1+f)\tilde{r} + 4(n-1)(2+f)A(U)}{2(n-1)(n-2)\alpha} A(X).$$

From (4.8), we have

$$(4.9) \quad A(X) = -\frac{2(n-1)(n-2)\alpha}{(1+f)\tilde{r} + 4(n-1)(2+f)A(U)} D_X \alpha.$$

Thus, due to (4.3), (4.9) and (4.2) can be expressed in the form

$$H(X, Y) = -\frac{\alpha^2}{2} g(X, Y) + \beta(D_X \alpha)(D_Y \alpha),$$

where

$$(4.10) \quad \beta = -\frac{4(2n-3) + (n-1)f(n-1)^2(n-2)}{[(1+f)\tilde{r} + 4(n-1)(2+f)A(U)]^2} \alpha^2.$$

In virtue of (4.10), we deduce that β is a function of α . Thus the theorem is proved. $\square$

Corollary 4.1. ([2]) *Every simply connected subprojective space can be isometrically immersed in a Euclidean space as a hypersurface.*

Moreover, using this Corollary, we can also state the following theorem:

Theorem 4.2. *If a simply connected $(APS)_n$ ($n > 3$) admits a semi-symmetric non-metric connection whose torsion tensor is almost pseudo symmetric and the curvature tensor of the semi-symmetric non-metric connection vanishes, then the manifold can be isometrically immersed in a Euclidean space E^{n+1} as a hypersurface.*

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