Polynomial Numerical Index of l_p ($1 < p < \infty$)

SUNG GUEN KIM

Department of Mathematics, Kyungpook National University, Daegu 702-701, Korea
e-mail : sgk317@knu.ac.kr

ABSTRACT. We present some estimates for the polynomial numerical index of l_p ($1 < p < \infty$).

1. Introduction

Given a Banach space E we write B_E for its unit ball and S_E for its unit sphere. The dual space of E is denoted by E^* and let

$$\Pi(E) = \{(x, x^*) : x \in S_E, x^* \in S_{E^*}, x^*(x) = 1\}.$$

A mapping $P : E \rightarrow E$ is called a (continuous) k -homogeneous polynomial if there is a (continuous) k -linear mapping $A : E \times \cdots \times E \rightarrow E$ such that $P(x) = A(x, \dots, x)$ for every $x \in E$. Let $\mathcal{P}(^k E : E)$ denote the Banach space of all k -homogeneous polynomials from E to itself, endowed with the polynomial norm $\|P\| = \sup_{x \in B_E} \|P(x)\|$. We refer to a book [6] by Dineen for background on polynomials. It is natural to generalize the concepts of numerical range and numerical radius of linear operators to homogeneous polynomials. The *numerical range* of $P \in \mathcal{P}(^k E : E)$ is defined to be the set of scalars

$$V(P) := \{x^*(Px) : (x, x^*) \in \Pi(E)\}$$

and the *numerical radius* of P is defined by

$$v(P) := \sup \{|\lambda| : \lambda \in V(P)\}.$$

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Clearly, $v(\cdot)$ is a semi-norm on $\mathcal{P}(^k E : E)$, and $v(P) \leq \|P\|$ for every $P \in \mathcal{P}(^k E : E)$. As in the linear case, the author *et al.* [3] introduced the concept of the *polynomial numerical index of order k* of E to be the constant

$$\begin{aligned} n^{(k)}(E) &:= \inf\{v(P) : P \in \mathcal{P}(^k E : E), \|P\| = 1\} \\ &= \sup\{M \geq 0 : \|P\| \leq \frac{1}{M}v(P) \text{ for all } P \in \mathcal{P}(^k E : E)\}. \end{aligned}$$

Of course, $n^{(1)}(E)$ coincides with the usual numerical index of the space E . Note that $0 \leq n^{(k)}(E) \leq 1$, and $n^{(k)}(E) > 0$ if and only if $v(\cdot)$ is a norm on $\mathcal{P}(^k E : E)$ equivalent to the usual norm. It is obvious that if E_1, E_2 are isometrically isomorphic Banach spaces, then $n^{(k)}(E_1) = n^{(k)}(E_2)$.

The concept of the numerical index was first suggested by G. Lumer in 1968 (see [14]). The author *et al.* [3] introduced and studied the concept of the polynomial numerical index of order k of a Banach space, generalizing to k -homogeneous polynomials the classical numerical index. Recently, the author *et al.* [11] computed that $\frac{1}{2} = n^{(2)}(c_0) = n^{(2)}(l_1) = n^{(2)}(l_\infty)$ for the real spaces c_0, l_1 and l_∞ . Very recently, Garcia *et al.* [9] gave estimates for the polynomial numerical indices of $C(K)$ and $L_1(\mu)$. For general information and background on polynomial numerical index, we refer to ([1]–[4], [7]–[15]).

In this paper, we present some inequality for the norm of 2-homogeneous polynomials from the real space l_p ($1 < p < \infty$) to itself in terms of their coefficients. Using this we give a lower bound for $n^{(2)}(l_p)$. We also present some upper bounds for $n^{(k)}(l_p)$.

2. Norm and Numerical Radius of 2-Homogeneous Polynomial on l_p

Theorem 2.1. *Let $1 < p < \infty, 1 = \frac{1}{p} + \frac{1}{q}$ and $P(x, y) = (P_1(x, y), P_2(x, y)) \in \mathcal{P}(^2 l_p^2 : l_p^2)$, where $P_k(x, y) = a_k x^2 + b_k y^2 + c_k xy$ for $k = 1, 2$ and $(x, y) \in l_p^2$. Then*

$$(1) \quad \|P\|^p \geq \sup_{0 \leq \lambda \leq 1} |P_1(\lambda^{\frac{1}{p}}, (1-\lambda)^{\frac{1}{p}})|^p + |P_2(\lambda^{\frac{1}{p}}, (1-\lambda)^{\frac{1}{p}})|^p;$$

$$(2) \quad v(P) \geq \sup_{0 \leq \lambda \leq 1} |\lambda^{\frac{1}{q}} P_1(\lambda^{\frac{1}{p}}, (1-\lambda)^{\frac{1}{p}}) + (1-\lambda)^{\frac{1}{q}} P_2(\lambda^{\frac{1}{p}}, (1-\lambda)^{\frac{1}{p}})|.$$

Corollary 2.2. *Let $1 < p < \infty, 1 = \frac{1}{p} + \frac{1}{q}$ and $P(x, y) = (P_1(x, y), P_2(x, y)) \in \mathcal{P}(^2 l_p^2 : l_p^2)$, where $P_k(x, y) = a_k x^2 + b_k y^2 + c_k xy$ for $k = 1, 2$ and $(x, y) \in l_p^2$. Then*

$$(1) \quad \|P\|^p \geq \sup_{0 \leq \lambda \leq 1, k=1,2} \left\{ \left| P_1 \left(\left[\lambda + \frac{(1-\lambda)|a_k - b_k|}{(|c_k|^p + |a_k - b_k|^p)^{\frac{1}{p}}} \right]^{\frac{1}{p}}, \right. \right. \right.$$

$$\left. \left. \left[1 - \lambda - \frac{(1-\lambda)|a_k - b_k|}{(|c_k|^p + |a_k - b_k|^p)^{\frac{1}{p}}} \right]^{\frac{1}{p}} \right| \right\}^p + |P_2([\lambda + \frac{(1-\lambda)|a_k - b_k|}{(|c_k|^p + |a_k - b_k|^p)^{\frac{1}{p}}}]^{\frac{1}{p}},$$

$$[1 - \lambda - \frac{(1 - \lambda)|a_k - b_k|}{(|c_k|^p + |a_k - b_k|^p)^{\frac{1}{p}}}]^{\frac{1}{p}})^p\};$$

$$\begin{aligned} (2) \ v(P) &\geq \sup_{0 \leq \lambda \leq 1, k=1,2} \{[\lambda + \frac{(1 - \lambda)|a_k - b_k|}{(|c_k|^p + |a_k - b_k|^p)^{\frac{1}{p}}}]^{\frac{1}{q}} \\ &\quad P_1([\lambda + \frac{(1 - \lambda)|a_k - b_k|}{(|c_k|^p + |a_k - b_k|^p)^{\frac{1}{p}}}]^{\frac{1}{p}}, [1 - \lambda - \frac{(1 - \lambda)|a_k - b_k|}{(|c_k|^p + |a_k - b_k|^p)^{\frac{1}{p}}}]^{\frac{1}{p}}) \\ &\quad + [1 - \lambda - \frac{(1 - \lambda)|a_k - b_k|}{(|c_k|^p + |a_k - b_k|^p)^{\frac{1}{p}}}]^{\frac{1}{q}} \times P_2([\lambda + \frac{(1 - \lambda)|a_k - b_k|}{(|c_k|^p + |a_k - b_k|^p)^{\frac{1}{p}}}]^{\frac{1}{p}}, \\ &\quad [1 - \lambda - \frac{(1 - \lambda)|a_k - b_k|}{(|c_k|^p + |a_k - b_k|^p)^{\frac{1}{p}}}]^{\frac{1}{p}})]\}. \end{aligned}$$

Corollary 2.3. Let $1 < p < \infty$, $1 = \frac{1}{p} + \frac{1}{q}$ and $P(x, y) = (P_1(x, y), P_2(x, y)) \in \mathcal{P}(^2l_p^2 : l_p^2)$, where $P_k(x, y) = a_k x^2 + b_k y^2 + c_k xy$ for $k = 1, 2$ and $(x, y) \in l_p^2$. Then

$$\begin{aligned} (1) \ \|P\|^p &\geq \sup_{k=1,2} \{ |P_1([\frac{1}{p} + \frac{|a_k - b_k|}{q(|c_k|^p + |a_k - b_k|^p)^{\frac{1}{p}}}]^{\frac{1}{p}}, [\frac{1}{q} - \frac{|a_k - b_k|}{q(|c_k|^p + |a_k - b_k|^p)^{\frac{1}{p}}}]^{\frac{1}{p}})^p \\ &\quad + |P_2([\frac{1}{p} + \frac{|a_k - b_k|}{q(|c_k|^p + |a_k - b_k|^p)^{\frac{1}{p}}}]^{\frac{1}{p}}, [\frac{1}{q} - \frac{|a_k - b_k|}{q(|c_k|^p + |a_k - b_k|^p)^{\frac{1}{p}}}]^{\frac{1}{p}})^p, \\ &\quad |P_1([\frac{1}{q} - \frac{|a_k - b_k|}{q(|c_k|^p + |a_k - b_k|^p)^{\frac{1}{p}}}]^{\frac{1}{p}}, [\frac{1}{p} + \frac{|a_k - b_k|}{q(|c_k|^p + |a_k - b_k|^p)^{\frac{1}{p}}}]^{\frac{1}{p}})^p \\ &\quad + |P_2([\frac{1}{q} - \frac{|a_k - b_k|}{q(|c_k|^p + |a_k - b_k|^p)^{\frac{1}{p}}}]^{\frac{1}{p}}, [\frac{1}{p} + \frac{|a_k - b_k|}{q(|c_k|^p + |a_k - b_k|^p)^{\frac{1}{p}}}]^{\frac{1}{p}})^p \}; \end{aligned}$$

$$\begin{aligned} (2) \ v(P) &\geq \sup_{k=1,2} \{ [|\frac{1}{p} + \frac{|a_k - b_k|}{q(|c_k|^p + |a_k - b_k|^p)^{\frac{1}{p}}}]^{\frac{1}{q}} P_1([\frac{1}{p} + \frac{|a_k - b_k|}{q(|c_k|^p + |a_k - b_k|^p)^{\frac{1}{p}}}]^{\frac{1}{p}}, \\ &\quad [\frac{1}{q} - \frac{|a_k - b_k|}{q(|c_k|^p + |a_k - b_k|^p)^{\frac{1}{p}}}]^{\frac{1}{p}}) + [\frac{1}{q} - \frac{|a_k - b_k|}{q(|c_k|^p + |a_k - b_k|^p)^{\frac{1}{p}}}]^{\frac{1}{q}} \\ &\quad \times P_2([\frac{1}{p} + \frac{|a_k - b_k|}{q(|c_k|^p + |a_k - b_k|^p)^{\frac{1}{p}}}]^{\frac{1}{p}}, [\frac{1}{q} - \frac{|a_k - b_k|}{q(|c_k|^p + |a_k - b_k|^p)^{\frac{1}{p}}}]^{\frac{1}{p}}) |, \\ &\quad |[\frac{1}{q} - \frac{|a_k - b_k|}{q(|c_k|^p + |a_k - b_k|^p)^{\frac{1}{p}}}]^{\frac{1}{q}} P_1([\frac{1}{q} - \frac{|a_k - b_k|}{q(|c_k|^p + |a_k - b_k|^p)^{\frac{1}{p}}}]^{\frac{1}{p}}, \\ &\quad [\frac{1}{p} + \frac{|a_k - b_k|}{q(|c_k|^p + |a_k - b_k|^p)^{\frac{1}{p}}}]^{\frac{1}{p}}) + [\frac{1}{p} + \frac{|a_k - b_k|}{q(|c_k|^p + |a_k - b_k|^p)^{\frac{1}{p}}}]^{\frac{1}{q}} \\ &\quad \times P_2([\frac{1}{q} - \frac{|a_k - b_k|}{q(|c_k|^p + |a_k - b_k|^p)^{\frac{1}{p}}}]^{\frac{1}{p}}, [\frac{1}{p} + \frac{|a_k - b_k|}{q(|c_k|^p + |a_k - b_k|^p)^{\frac{1}{p}}}]^{\frac{1}{p}}) | \} \end{aligned}$$

3. Estimates for the Polynomial Numerical Index of l_p

Theorem 3.1. Let $1 < p < \infty$, $1 = \frac{1}{p} + \frac{1}{q}$ and $P(x) = (P_k(x))_{k=1}^\infty \in \mathcal{P}({}^2l_p : l_p)$, where

$$P_k(x) = \sum_{i \leq j} a_{ij}^{(k)} x_i x_j \quad (x = (x_n)_{n=1}^\infty \in l_p).$$

Then

$$\begin{aligned} \|P\| \leq & \left\| \left(\sum_{k=1}^\infty (|a_{11}^{(k)}| + \frac{1}{2} \sum_{1 < j} |a_{1j}^{(k)}|), \sum_{k=1}^\infty (|a_{22}^{(k)}| + \frac{1}{2} |a_{12}^{(k)}| + \frac{1}{2} \sum_{2 < j} |a_{2j}^{(k)}|), \dots, \right. \right. \\ & \left. \left. \sum_{k=1}^\infty (|a_{nn}^{(k)}| + \frac{1}{2} \sum_{m < n} |a_{mn}^{(k)}| + \frac{1}{2} \sum_{n < j} |a_{nj}^{(k)}|), \dots \right) \right\|_q. \end{aligned}$$

Proof. It follows that

$$\begin{aligned} \|P\| &= \sup_{x=(x_n) \in S_{l_p}} \left(\sum_{k=1}^\infty |P_k(x)|^p \right)^{1/p} \\ &\leq \sup_{x=(x_n) \in S_{l_p}} \left(\sum_{k=1}^\infty \left(\sum_{i=1}^\infty |a_{ii}^{(k)}| |x_i|^2 + \sum_{i < j} |a_{ij}^{(k)}| |x_i| |x_j| \right)^p \right)^{1/p} \\ &\leq \sup_{x=(x_n) \in S_{l_p}} \sum_{k=1}^\infty \left(\sum_{i=1}^\infty |a_{ii}^{(k)}| |x_i|^2 + \sum_{i < j} |a_{ij}^{(k)}| |x_i| |x_j| \right) \\ &\leq \sup_{x=(x_n) \in S_{l_p}} \sum_{k=1}^\infty \left(\sum_{i=1}^\infty |a_{ii}^{(k)}| |x_i|^2 + \sum_{i < j} |a_{ij}^{(k)}| \left(\frac{|x_i|^2 + |x_j|^2}{2} \right) \right) \\ &= \sup_{x=(x_n) \in S_{l_p}} \left[\sum_{k=1}^\infty \left(|a_{11}^{(k)}| + \frac{1}{2} \sum_{1 < j} |a_{1j}^{(k)}| \right) |x_1|^2 \right. \\ &\quad + \sum_{k=1}^\infty \left(|a_{22}^{(k)}| + \frac{1}{2} |a_{12}^{(k)}| + \frac{1}{2} \sum_{2 < j} |a_{2j}^{(k)}| \right) |x_2|^2 \\ &\quad + \dots + \sum_{k=1}^\infty \left(|a_{nn}^{(k)}| + \frac{1}{2} \sum_{m < n} |a_{mn}^{(k)}| + \frac{1}{2} \sum_{n < j} |a_{nj}^{(k)}| \right) |x_n|^2 + \dots \Big] \\ &\leq \sup_{x=(x_n) \in S_{l_p}} \left[\sum_{k=1}^\infty \left(|a_{11}^{(k)}| + \frac{1}{2} \sum_{1 < j} |a_{1j}^{(k)}| \right) |x_1| \right. \\ &\quad + \sum_{k=1}^\infty \left(|a_{22}^{(k)}| + \frac{1}{2} |a_{12}^{(k)}| + \frac{1}{2} \sum_{2 < j} |a_{2j}^{(k)}| \right) |x_2| \\ &\quad + \dots + \sum_{k=1}^\infty \left(|a_{nn}^{(k)}| + \frac{1}{2} \sum_{m < n} |a_{mn}^{(k)}| + \frac{1}{2} \sum_{n < j} |a_{nj}^{(k)}| \right) |x_n| + \dots \Big] \end{aligned}$$

$$= \left\| \left(\sum_{k=1}^{\infty} (|a_{11}^{(k)}| + \frac{1}{2} \sum_{1 < j} |a_{1j}^{(k)}|), \sum_{k=1}^{\infty} (|a_{22}^{(k)}| + \frac{1}{2} |a_{12}^{(k)}| + \frac{1}{2} \sum_{2 < j} |a_{2j}^{(k)}|), \dots, \right. \right. \\ \left. \left. \sum_{k=1}^{\infty} (|a_{nn}^{(k)}| + \frac{1}{2} \sum_{m < n} |a_{mn}^{(k)}| + \frac{1}{2} \sum_{n < j} |a_{nj}^{(k)}|), \dots \right) \right\|_q. \quad \square$$

Theorem 3.2. Let $1 < p < \infty, 1 = \frac{1}{p} + \frac{1}{q}$. Then

$$n^{(2)}(l_p) \geq \inf \left\{ \frac{\sup_{n \in \mathbb{N}} (\frac{1}{n})^{1+\frac{1}{p}} |\sum_{l=1}^n \sum_{i \leq j} a_{ij}^{(l)}|}{\left\| \left(\sum_{k=1}^n (|a_{11}^{(k)}| + \frac{1}{2} \sum_{j=2}^n |a_{1j}^{(k)}|), \dots, \sum_{k=1}^n (|a_{nn}^{(k)}| + \frac{1}{2} \sum_{m < n} |a_{mn}^{(k)}|), 0, 0, \dots \right) \right\|_q} : \right. \\ \left. \text{nonzero } P(x) = (P_k(x))_{k=1}^{\infty} \in \mathcal{P}^2(l_p : l_p), P_k(x) = \sum_{i \leq j} a_{ij}^{(k)} x_i x_j \right\}.$$

Proof. Suppose that $P(x) = (P_k(x))_{k=1}^{\infty} \in \mathcal{P}^2(l_p : l_p)$ with

$$P_k(x) = \sum_{i \leq j} a_{ij}^{(k)} x_i x_j \quad (x = (x_n)_{n=1}^{\infty} \in l_p).$$

Let $n \in \mathbb{N}$, $x_0 = \sum_{k=1}^n (\frac{1}{n})^{1/p} e_k$, and $x_0^* = \sum_{k=1}^n (\frac{1}{n})^{1/q} e_k$. Then

$$(*) \quad v(P) \geq |x_0^*(P(x_0))| = \left(\frac{1}{n}\right)^{1+\frac{1}{p}} \left| \sum_{l=1}^n \sum_{i \leq j} a_{ij}^{(l)} \right|.$$

It follows that by $(*)$ and Theorem 3.1,

$$n^{(2)}(l_p) = \inf \left\{ \frac{v(P)}{\|P\|} : P(x) = (P_k(x))_{k=1}^{\infty} \in \mathcal{P}^2(l_p : l_p), P \neq 0 \right\} \\ \geq \inf \left\{ \frac{\sup_{n \in \mathbb{N}} (\frac{1}{n})^{1+\frac{1}{p}} |\sum_{l=1}^n \sum_{i \leq j} a_{ij}^{(l)}|}{\left\| \left(\sum_{k=1}^n (|a_{11}^{(k)}| + \frac{1}{2} \sum_{j=2}^n |a_{1j}^{(k)}|), \dots, \sum_{k=1}^n (|a_{nn}^{(k)}| + \frac{1}{2} \sum_{m < n} |a_{mn}^{(k)}|), 0, 0, \dots \right) \right\|_q} : \right.$$

$$\left. \text{nonzero } P(x) = (P_k(x))_{k=1}^{\infty} \in \mathcal{P}^2(l_p : l_p) \right\}. \quad \square$$

Corollary 3.3. We have

$$\inf \left\{ \frac{\sup_{n \in \mathbb{N}} (\frac{1}{n})^{\frac{3}{2}} |\sum_{l=1}^n \sum_{i \leq j} a_{ij}^{(l)}|}{\left\| \left(\sum_{k=1}^n (|a_{11}^{(k)}| + \frac{1}{2} \sum_{j=2}^n |a_{1j}^{(k)}|), \dots, \sum_{k=1}^n (|a_{nn}^{(k)}| + \frac{1}{2} \sum_{m < n} |a_{mn}^{(k)}|), 0, 0, \dots \right) \right\|_2} : \right.$$

$$\left. \text{nonzero } P(x) = (P_k(x))_{k=1}^{\infty} \in \mathcal{P}^2(l_2 : l_2), P_k(x) = \sum_{i \leq j} a_{ij}^{(k)} x_i x_j \right\} = 0.$$

Theorem 3.4. Let $k \in \mathbb{N}$ with $k \geq 2$. Then

$$n^{(2)}(l_k^2) = \inf \{ \|Q_P\|_{l_k^2} : P \in \mathcal{P}^2(l_k^2 : l_k^2),$$

$$\|P\| = 1, P(x, y) = (ax^2 + by^2 + cxy, a'x^2 + b'y^2 + c'xy),$$

where $Q_P \in \mathcal{P}^{(k+1)}(l_k^2)$ with $Q_P(x, y) = ax^{k+1} + bx^{k-1}y^2 + cx^ky + a'x^2y^{k-1} + b'y^{k+1} + c'xy^k$.

Theorem 3.5. Let $1 < p < \infty$ and $k \in \mathbb{N}$ with $k \geq 2$. Then

$$n^{(k)}(l_p) \leq n^{(k)}(l_p^2) \leq \min\left\{\left(\frac{p-1}{k+p-1}\right)^{1-\frac{1}{p}} \left(\frac{k}{k+p-1}\right)^{\frac{k}{p}}, \left(\frac{p+k-1}{2p+k-2}\right)^{\frac{1}{p}} \frac{\left(\frac{p+k-2}{2p+k-2}\right)^{\frac{p+k-2}{p}}}{\left(\frac{k-1}{p+k-1}\right)^{\frac{k-1}{p}}}\right\}.$$

Proof. Let $P_1(x, y) = (x^k, 0)$ and $P_2(x, y) = (x^{k-1}y, 0)$ for $(x, y) \in l_p^2$. Then $P_j \in \mathcal{P}^k(l_p^2)$ for $j = 1, 2$ and $\|P_1\| = 1$. Some computation shows that

$$\begin{aligned} \|P_2\| &= \left(\frac{k-1}{p+k-1}\right)^{\frac{k-1}{p}} \left(\frac{p}{p+k-1}\right)^{\frac{1}{p}}, \\ v(P_2) &= \left(\frac{p+k-2}{2p+k-2}\right)^{\frac{p+k-2}{p}} \left(\frac{p}{2p+k-2}\right)^{\frac{1}{p}}, \\ \text{and } v(P_1) &= \left(\frac{p-1}{k+p-1}\right)^{1-\frac{1}{p}} \left(\frac{k}{k+p-1}\right)^{\frac{k}{p}}. \end{aligned}$$

It follows that

$$\begin{aligned} n^{(k)}(l_p) &\leq n^{(k)}(l_p^2) \leq \min\{v(P_1), \frac{v(P_2)}{\|P_2\|}\} \\ &= \min\left\{\left(\frac{p-1}{k+p-1}\right)^{1-\frac{1}{p}} \left(\frac{k}{k+p-1}\right)^{\frac{k}{p}}, \left(\frac{p+k-1}{2p+k-2}\right)^{\frac{1}{p}} \frac{\left(\frac{p+k-2}{2p+k-2}\right)^{\frac{p+k-2}{p}}}{\left(\frac{k-1}{p+k-1}\right)^{\frac{k-1}{p}}}\right\}, \end{aligned}$$

which completes the proof. $\square$

Theorem 3.6. Let $k \in \mathbb{N}$. Then $n^{(2k+1)}(l_{2k}) = 0$ for the real space l_{2k} .

Proof. Let

$$P_0(x) := (-x_1x_2^{2k}, x_1^{2k}x_2, 0, 0, \dots) \quad (x = (x_j) \in l_{2k}).$$

Then $\|P_0\| \neq 0$. Note that if $(x_1, x_2, 0, 0, \dots) \in S_{l_{2k}}$, then $(x_1^{2k-1}, x_2^{2k-1}, 0, 0, \dots) \in S_{(l_{2k})^*} = S_{l_{\frac{2k}{2k-1}}}$. It follows that

$$\begin{aligned} &v(P_0) \\ &= \sup_{x_1^{2k} + x_2^{2k} = 1} | \langle \text{sign}(x_1^{2k})x_1^{2k-1}, \text{sign}(x_2^{2k})x_2^{2k-1}, 0, \dots \rangle, P_0(x_1, x_2, 0, \dots) \rangle | \\ &= \sup\{ | \langle (x_1^{2k-1}, x_2^{2k-1}, 0, 0, \dots), P_0(x_1, x_2, 0, 0, \dots) \rangle | : x_1^{2k} + x_2^{2k} = 1 \} \\ &= \sup\{ | -x_1x_2^{2k} + x_1x_2^{2k} | : x_1^{2k} + x_2^{2k} = 1 \} = 0. \end{aligned}$$

Thus

$$n^{(2k+1)}(l_{2k}) \leq \frac{v(P_0)}{\|P_0\|} = 0.$$

□

Corollary 3.7. Let $k, s \in \mathbb{N}$ with $s \geq 2k + 1$. Then $n^{(s)}(l_{2k}) = 0$ for the real space l_{2k} .

Notice that there exist $1 < p < q < \infty, N \geq 2$ such that

$$n^{(N)}(l_p) = n^{(N)}(l_q).$$

In fact, take $p = 2k, q = 2(k + 1), N = 2k + 3$. By Corollary 3.7, $n^{(N)}(l_p) = 0 = n^{(N)}(l_q)$.

We are in position to prove our main result.

Theorem 3.8. Let $1 < p < \infty$. Then for the complex space l_p ,

$$n^{(k)}(l_p) \leq 2^{\frac{1-k}{p}} \quad (\forall k \in \mathbb{N}).$$

Proof. Simple computation shows that

$$\sup_{|x_1|^p + |x_2|^p = 1} (|x_1|^{p+k-1} + |x_2|^{p+k-1}) = 2^{\frac{1-k}{p}} = \sup_{|x_1|^p + |x_2|^p = 1} (|x_1|^{kp} + |x_2|^{kp})^{\frac{1}{p}}.$$

For every $b \in \mathbb{C}$, define

$$Q_b(x) := (x_1^k + bx_2^k, bx_1^k + x_2^k, 0, 0, \dots) \in \mathcal{P}^k(l_p) \quad (x = (x_j) \in l_p).$$

Then $\|Q_b\| \geq |Q_b(1, 0)| = (1 + |b|^p)^{\frac{1}{p}}$ and

$$\begin{aligned} & v(Q_b) \\ &= \sup_{|x_1|^p + |x_2|^p = 1} | \langle \text{sign}(x_1^p)x_1^{p-1}, \text{sign}(x_2^p)x_2^{p-1}, 0, 0, \dots, Q_b(x_1, x_2, 0, 0, \dots) \rangle | \\ &\leq \sup_{|x_1|^p + |x_2|^p = 1} (|x_1|^{p+k-1} + |x_2|^{p+k-1}) \\ &\quad + |b| \sup_{|x_1|^p + |x_2|^p = 1} (|x_1|^{p-1}|x_2|^k + |x_2|^{p-1}|x_1|^k) \\ &\leq 2^{\frac{1-k}{p}} + |b| \sup_{|x_1|^p + |x_2|^p = 1} (|x_1|^{(p-1)q} + |x_2|^{(p-1)q})^{\frac{1}{q}} \sup_{|x_1|^p + |x_2|^p = 1} (|x_1|^{kp} + |x_2|^{kp})^{\frac{1}{p}} \\ &\quad (\text{by Holder's inequality}) \\ &= 2^{\frac{1-k}{p}} + |b| \sup_{|x_1|^p + |x_2|^p = 1} (|x_1|^p + |x_2|^p)^{\frac{1}{q}} \sup_{|x_1|^p + |x_2|^p = 1} (|x_1|^{kp} + |x_2|^{kp})^{\frac{1}{p}} \\ &= 2^{\frac{1-k}{p}} + |b| 2^{\frac{1-k}{p}} \\ &= 2^{\frac{1-k}{p}} (1 + |b|). \end{aligned}$$

It follows that

$$n^{(k)}(l_p) \leq \inf_{b \in \mathbb{C}} \frac{v(Q_b)}{\|Q_b\|} \leq \inf_{b \in \mathbb{C}} \frac{2^{\frac{1-k}{p}}(1+|b|)}{(1+|b|^p)^{\frac{1}{p}}} = 2^{\frac{1-k}{p}}.$$

□

Corollary 3.9. Let $1 < p < \infty$. Then

- (1) $n^{(k)}(l_p) \leq 2^{-\frac{1}{p}} < 1$ ($\forall k \geq 2$).
- (2) $n^{(k)}(l_p) \leq \frac{1}{2}$ ($\forall k \geq p+1$).
- (3) $\frac{2^k}{2+M_k(2^k-2)} \leq n^{(k)}(l_1) \leq (\frac{1}{2})^{k-1}$, where $k \geq 1$ and $M_k = \sum_{j=1}^k \frac{j^k}{j!(k-j)!}$.
- (4) $\lim_{k \rightarrow \infty} n^{(k)}(l_p) = 0$.

Theorem 3.10. Let p be an even number and k an odd number with $k \leq p-1$. Then

$$n^{(k)}(l_p) \leq \left(\frac{k}{p-1}\right)^{\frac{k}{p-1-k}} - \left(\frac{k}{p-1}\right)^{\frac{p-1}{p-1-k}}, \text{ when } l_p \text{ is the real space.}$$

Proof. We define

$$Q_0(x) := (-x_2^k, x_1^k, 0, 0, \dots) \in \mathcal{P}^k(l_p) \quad (x = (x_j) \in l_p).$$

Then $\|Q_0\| = 1$ and

$$\begin{aligned} & v(Q_0) \\ &= \sup_{x_1^p + x_2^p = 1} |< (x_1^{p-1}, x_2^{p-1}, 0, 0, \dots), Q_0(x_1, x_2, 0, 0, \dots) >| \\ &= \sup_{x_1^p + x_2^p = 1} |-x_1^{p-1}x_2^k + x_1^kx_2^{p-1}| \\ &\leq \sup_{x_1^p + x_2^p = 1} |x_1x_2|^k - |x_1x_2|^{p-1} \\ &= \left(\frac{k}{p-1}\right)^{\frac{k}{p-1-k}} - \left(\frac{k}{p-1}\right)^{\frac{p-1}{p-1-k}}, \text{ when } |x_1x_2| = \left(\frac{k}{p-1}\right)^{\frac{1}{p-1-k}}. \end{aligned}$$

Thus

$$n^{(k)}(l_p) \leq \frac{v(Q_0)}{\|Q_0\|} = \left(\frac{k}{p-1}\right)^{\frac{k}{p-1-k}} - \left(\frac{k}{p-1}\right)^{\frac{p-1}{p-1-k}}.$$

□

Theorem 3.11. Let p and s be even numbers with $s < p-1$. Then

$$n^{(s)}(l_p) \leq \min\left\{\left(\frac{s-1}{p-1}\right)^{\frac{s-1}{p-s}} - \left(\frac{s-1}{p-1}\right)^{\frac{p-1}{p-s}}, 2^{-\frac{s}{p}}\right\}, \text{ when } l_p \text{ is the real space.}$$

Proof. By Theorem 3.10,

$$n^{(s)}(l_p) \leq n^{(s-1)}(l_p) \leq \left(\frac{s-1}{p-1}\right)^{\frac{s-1}{p-s}} - \left(\frac{s-1}{p-1}\right)^{\frac{p-1}{p-s}}.$$

We define

$$Q_0(x) := (-x_2^s, x_1^s, 0, 0, \dots) \in \mathcal{P}({}^s l_p) \quad (x = (x_j) \in l_p).$$

Then $\|Q_0\| = 1$ and

$$\begin{aligned} & v(Q_0) \\ &= \sup_{x_1^p + x_2^p = 1} | \langle (x_1^{p-1}, x_2^{p-1}, 0, 0, \dots), Q_0(x_1, x_2, 0, 0, \dots) \rangle | \\ &= \sup_{x_1^p + x_2^p = 1} | -x_1^{p-1}x_2^s + x_1^s x_2^{p-1} | \\ &= \sup_{x_1^p + x_2^p = 1} |x_1|^{p-1}|x_2|^s + |x_1|^s|x_2|^{p-1} \\ &\leq \sup_{x_1^p + x_2^p = 1} 2|x_1 x_2|^s \\ &= 2^{-\frac{s}{p}}. \end{aligned}$$

Thus

$$n^{(k)}(l_p) \leq \frac{v(Q_0)}{\|Q_0\|} = 2^{-\frac{s}{p}}. \quad \square$$

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