Special Right Jacobson Radicals for Right Near-rings

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ABSTRACT. In this paper three more right Jacobson-type radicals, $J_{g\nu}^r$, are introduced for near-rings which generalize the Jacobson radical of rings, $\nu \in \{0, 1, 2\}$. It is proved that $J_{g\nu}^r$ is a special radical in the class of all near-rings. Unlike the known right Jacobson semisimple near-rings, a $J_{g\nu}^r$ -semisimple near-ring R with DCC on right ideals is a direct sum of minimal right ideals which are right R -groups of type- $g\nu$, $\nu \in \{0, 1, 2\}$. Moreover, a finite right g_2 -primitive near-ring R with eRe a non-ring is a near-ring of matrices over a near-field (which is isomorphic to eRe), where e is a right g_2 -primitive idempotent in R .

1. Introduction

Special radicals for near-rings are introduced in [1] by G. L. Booth and N. J. Groenewald using equiprime near-rings. Among the known left Jacobson-type radicals, $J_3, J_{3(0)}$ are the only special radicals in the class of zero-symmetric near-rings and in the class of all near-rings respectively.

Srinivasa Rao and Siva Prasad [6, 7] introduced and studied J_ν^r , the right Jacobson radical type- ν , $\nu \in \{0, 1, 2\}$. In [9, 10] Srinivasa Rao and Siva Prasad along with T. Srinivas showed that J_ν^r is a Kurosh-Amitsur radical in the Fuchs variety $\mathcal{F}$ of all near-rings R in which the constant part R_c of R is an ideal of R , $\nu \in \{0, 1, 2\}$. But J_ν^r is not s-hereditary in the class of all zero-symmetric near-rings and hence it is not an ideal-hereditary radical in that class, $\nu \in \{0, 1, 2\}$.

Also in [5]([11]) Srinivasa Rao and Siva Prasad (along with T. Srinivas) intro-

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duced and studied the right Jacobson type of radical $J_{\nu(e)}^r$, $\nu \in \{1, 2\}$ ($J_{0(e)}^r$) and showed that it is a Kurosh-Amitsur radical in the class of all near-rings and is an ideal hereditary Kurosh-Amitsur radical in the class of all zero-symmetric near-rings. Moreover, they are special radicals in the class of all near-rings.

In this paper we introduce three more right Jacobson radicals, $J_{g\nu}^r$, $\nu \in \{0, 1, 2\}$. We show that they are special radicals in the class of all near-rings. So, in the class of all near-rings, they are Kurosh-Amitsur radicals, their semisimple classes are hereditary and radicals classes are c -hereditary. Unlike the known right Jacobson semisimple near-rings, a $J_{g\nu}^r$ -semisimple near-ring R with DCC on right ideals is a direct sum of right ideals which are right R -groups of type- $g\nu$, $\nu \in \{0, 1, 2\}$. A finite right g_2 -primitive near-ring R with eRe a non-ring is a near-ring of matrices over a near-field (which is isomorphic to eRe), where e is a right g_2 -primitive idempotent in R .

Near-rings considered are right near-rings (not necessarily zero-symmetric) and R is a near-ring. Now we present some definitions and results of [6] and [7].

A group $(G, +)$ is called a *right R -group* if there is a mapping $((g, r) \rightarrow gr)$ of $G \times R$ into G such that (1) $(g + h)r = gr + hr$, (2) $g(rs) = (gr)s$ for all $g, h \in G$ and $r, s \in R$. A subgroup (normal subgroup) H of a right R -group G is called an *R -subgroup (ideal)* of G if $hr \in H$ for all $h \in H$ and $r \in R$.

Let G be a right R -group. An element $g \in G$ is called a *generator* of G if $gR = G$ and $g(r + s) = gr + gs$ for all $r, s \in R$. G is said to be *monogenic* if G has a generator. G is said to be *simple* if $G \neq \{0\}$ and G , and $\{0\}$ are the only ideals of G .

A monogenic right R -group G is said to be a *right R -group of type-0* if G is simple.

A right R -group G of type-0 is said to be of *type-1* if G has exactly two R -subgroups, namely $\{0\}$ and G .

A right R -group G of type-0 is said to be of *type-2* if $gR = G$ for all $0 \neq g \in G$.

Note that a right R -group of type-2 is of type-1 and a right R -group of type-1 is of type-0.

Let $\nu \in \{0, 1, 2\}$. A right modular right ideal K of R is called *right ν -modular* if R/K is a right R -group of type- ν .

An ideal P of R is called *right ν -primitive* if P is the largest ideal of R contained in a right ν -modular right ideal of R . R is called a *right ν -primitive near-ring* if $\{0\}$ is a right ν -primitive ideal of R .

Now we present some definitions of [11] and [5].

Let G be a right R -group of type- ν , $\nu \in \{0, 1, 2\}$. Suppose that $G0 = \{0\}$ for $\nu = 0$ and P is the largest ideal of R contained in $(0 : G) = \{r \in R \mid Gr = \{0\}\}$. Then G is said to be a *right R -group of type- $\nu(e)$* if $0 \neq g \in G, r_1, r_2 \in R$ and $gxr_1 = gxr_2$ for all $x \in R$ implies $r_1 - r_2 \in P$.

A right modular right ideal K of R is called *right $\nu(e)$ -modular* if R/K is a right R -group of type- $\nu(e)$.

Let G be a right R -group of type- $\nu(e)$. Then $(0 : G)$ is an ideal of R and is called a *right $\nu(e)$ -primitive ideal* of R .

A near-ring R is called *right $\nu(e)$ -primitive* if $\{0\}$ is a right $\nu(e)$ -primitive ideal of R .

A near-ring R is called an *equiprime near-ring* [2] if $0 \neq a \in R$, $x, y \in R$ and $arx = ary$ for all $r \in R$, implies $x = y$. An ideal I of R is called *equiprime* if R/I is an equiprime near-ring. Moreover, an equiprime near-ring is zero-symmetric.

It is known that a near-ring R is equiprime if and only if ([2])

1. $x, y \in R$ and $xRy = \{0\}$ implies $x = 0$ or $y = 0$.
2. If $\{0\} \neq I$ is an invariant subnear-ring of R , $x, y \in R$ and $ax = ay$ for all $a \in I$ implies $x = y$.

In [1], G. L. Booth and N. J. Groenewald defined special radicals for near-rings. A class $\mathcal{E}$ consisting of equiprime near-rings is called a *special class* if it is hereditary and closed under left invariant essential extensions. If $\mathcal{R}$ is the upper radical in the class of all near-rings determined by a special class of near-rings, then $\mathcal{R}$ is called a special radical. A class of near-rings $\mathcal{E}$ is said to satisfy condition F_l if $J \triangleleft I \triangleleft R$ and I is left invariant in R and $I/J \in \mathcal{E}$ implies $J \triangleleft R$. We need the following theorem:

Theorem 1.1. ([12]) Let $\mathcal{E}$ be a class of zero-symmetric near-rings. If $\mathcal{E}$ is regular, closed under essential left invariant extensions and satisfies condition (F_l) , then $\mathcal{R} := \mathcal{U}\mathcal{E}$ is a c -hereditary radical class in the variety of all near-rings, $\mathcal{SR} = \mathcal{E}$ and $\mathcal{SR}$ is hereditary. So, $\mathcal{R}(R) = \cap \{I \triangleleft R \mid R/I \in \mathcal{E}\}$ for any near-ring R .

2. Right Jacobson Radicals of Type- g_ν

Let G be a right R -group and T be a subset of G . Then $(0 : T) := \{r \in R \mid tr = 0 \text{ for all } t \in T\}$. By Proposition 3.7 of [11], if G is a right R -group of type-0 and $G0 = \{0\}$, then there is a largest ideal of R contained in $(0 : G)$. Moreover, by Proposition 3.1 of [5], if G is a right R -group of type- ν , $\nu \in \{1, 2\}$, then $G0 = \{0\}$, $\nu \in \{1, 2\}$. **Definition 2.1.** Let $\nu \in \{0, 1, 2\}$. Let G be right R -group of type- ν and $G0 = \{0\}$ for $\nu = 0$, and T be the set of all generators of the right R -group G . Then G is said to be a right R -group of type- g_ν if $(0 : T) = P$, where P is the largest ideal of R contained in $(0 : G)$.

We present an example of a right R -group of type- g_0 which is not of type- g_1 .

Example 2.2. Let $(G, +)$ be a finite non-abelian simple group. Since $\{0\}$ is the maximal normal subgroup of $(G, +)$, $\{0\}$ is the maximal right ideal of $M_0(G)$ and hence $M_0(G)$ is a right $M_0(G)$ -group of type-0. This example was considered in [7] and it was shown that $M_0(G)$ is not a right $M_0(G)$ -group of type-1. Each $0 \neq h \in G$ give rise to the inner automorphism t_h of G defined by $t_h(x) = h + x - h$ for all $x \in G$. Clearly, a generator of the right $M_0(G)$ -group $M_0(G)$ is an automorphism of $(G, +)$. Let T be the set of all automorphisms of G . Suppose that for some $t \in M_0(G)$ and $0 \neq h \in G$, $t_h t = 0$. Now $0 = (t_{-h})t_h t = (t_h)^{-1}t_h t = t$. Therefore $\{0\} = (0 : t_h) = (0 : T)$. Since the largest ideal of $M_0(G)$ contained in $(0 : M_0(G))$ is $\{0\}$, $M_0(G)$ is a right $M_0(G)$ -group of type- g_0 but not of type- g_1 .

Now we present an example of a right R -group of type- g_1 which is not of type- g_2 .

Example 2.3. Let $(G, +)$ be a finite cyclic group of prime order p , where $p \neq 2$. Since $\{0\}$ is the only proper subgroup of G , $\{0\}$ is the only proper right $M_0(G)$ -subgroup of $M_0(G)$. Therefore, $M_0(G)$ is a right $M_0(G)$ -group of type-1. Clearly, $M_0(G)$ is not a right $M_0(G)$ -group of type-2, as $M_0(G)$ is not a near-field. This example was considered in [7]. A generator of the right $M_0(G)$ -group $M_0(G)$ is an automorphism $(G, +)$. We know that G has $p - 1$ automorphisms. Let T be the set of all these automorphisms. Suppose that for some $s \in M_0(G)$ and $t \in T$, $ts = 0$. Now $0 = (t^{-1})ts = s$. So $\{0\} = (0 : t) = (0 : T)$. Since the largest ideal of $M_0(G)$ contained in $(0 : M_0(G))$ is $\{0\}$, $M_0(G)$ is a right $M_0(G)$ -group of type- g_1 but not of type- g_2 .

The following are examples of right R -groups of type- g_2 .

Example 2.4. Let R be a near-field. Then R is a right R -group of type-2. Clearly, R is also a right R -group of type- g_2 .

Example 2.5. Let $(R, +)$ be a group and let K be a subgroup of $(R, +)$ of index 2. The trivial multiplication on $(R, +)$ determined by $R \setminus K$ is given by $a.b = a$ if $b \in R \setminus K$ and 0 if $b \in K$. Now $(R, +, \cdot)$ is a near-ring. It is clear that K is a maximal (right) ideal of R . Let $a \in R \setminus K$. Now $R = K \cup a + K$. It can be easily verified that $a + K$ is the generator of the right R -group R/K . So R/K is a right R -group of type-2 and $(0 : a + K) = (0 : R/K)$ is the largest ideal of R contained in $(0 : R/K)$. Hence R/K is a right R -group of type- g_2 .

Now we introduce some notions related to the right R -groups of type- g_ν .

Definition 2.6. Let $\nu \in \{0, 1, 2\}$ and K be a right modular right ideal of R . Then K is said to be right g_ν -modular right ideal of R if R/K is a right R -group of type- g_ν .

Definition 2.7. Let $\nu \in \{0, 1, 2\}$. An ideal P of R is called a right g_ν -primitive ideal of R if P is the largest ideal of R contained in $(0 : G) := \{r \in R \mid Gr = \{0\}\}$ for some right R -group G of type- g_ν .

Definition 2.8. Let $\nu \in \{0, 1, 2\}$. A near-ring R is called a *right g_ν -primitive near-ring* if $\{0\}$ is a right g_ν -primitive ideal of R .

Definition 2.9. Let $\nu \in \{0, 1, 2\}$. The intersection of all right g_ν -primitive ideals of R is called the *right Jacobson radical of R of type- g_ν* and is denoted by $J_{g_\nu}^r(R)$. If R has no right g_ν -primitive ideals, then $J_{g_\nu}^r(R)$ is defined to be R .

Note that if R is a ring then $J_{g_\nu}^r(R) = J(R)$, where J is the Jacobson radical of R .

By Proposition 3.1 of [11], for a right R -group G , $G0 = \{0\}$ if and only if $GR_c = \{0\}$. Since for a right R -group G of type- g_ν , $G0 = \{0\}$, R_c is contained in $(0 : g)$ for every generator g of G . So $R_c \subseteq P$ for every right g_ν -primitive ideal P of R . Hence a right g_ν -primitive ideal P of R is invariant. This shows that a right g_ν -primitive near-ring is zero-symmetric.

Proposition 2.10. Let $\nu \in \{0, 1, 2\}$. An ideal P of R is a right g_ν -primitive ideal of R if and only if P is the largest ideal of R contained in a right g_ν -modular right ideal of R .

Proof. Let P be a right g_ν -primitive ideal of R . There is a right R -group G of type- g_ν such that P is the largest ideal of R contained in $(0 : G)$. Let g_0 be a generator of the right R -group G . The mapping $r \rightarrow g_0 r$ is a right R -homomorphism of R on to G with kernel $K := (0 : g_0)$. So R/K is right R -isomorphic to G (as right R -groups). Now K is a right g_ν -modular right ideal of R and P is contained in K . Let Q be the largest ideal of R contained in K . Now $GQ = \{0\}$, that is, $Q \subseteq (0 : G)$ as $RQ \subseteq Q$, Q being invariant ideal of R . Since P is the largest ideal of R contained in $(0 : G)$, $Q \subseteq P$. Now $P \subseteq Q$ as Q is the largest ideal of R contained in K . Therefore $P = Q$, that is, P is the largest ideal of R contained in K . On the other hand suppose that P is the largest ideal of R contained in a right g_ν -modular right ideal K of R . Now $G := R/K$ is a right R -group of type- g_ν . We have $(0 : G) = (0 : R/K) = (K : R)$ and $RP \subseteq P$ as P is an invariant ideal of R . So $P \subseteq (K : R)$. Let T be the largest ideal of R contained in $(K : R) = \{r \in R \mid Rr \subseteq K\}$. Since P is an invariant ideal of R , and $P \subseteq T$, T is an invariant ideal of R . So $RT \subseteq T$. Let K be right modular by e . Now $r - er \in K$ for all $r \in R$. We have $t - et \in K$ for all $t \in T$. Since $RT \subseteq T, T \subseteq K$. Since P is the largest ideal of R contained in K , $T \subseteq P$. So $T = P$. Now P is the largest ideal of R contained in $(K : R)$ and hence P is a right g_ν -primitive ideal of R . $\square$

Proposition 2.11. Let $\nu \in \{0, 1, 2\}$. An ideal P of R is a right g_ν -primitive ideal of R if and only if R/P is a right g_ν -primitive near-ring.

Proof. Let $\nu \in \{0, 1, 2\}$ and P be an ideal of R . Suppose that P is a right g_ν -primitive ideal of R . So, we get a right g_ν -modular right ideal M of R such that P is the largest ideal of R contained in M . Now M/P is a right g_ν -modular right ideal of R/P . Since P is the largest ideal of R contained in M , the zero ideal of R/P is the largest ideal of R/P contained in M/P . Therefore, R/P is a right g_ν -primitive near-ring. Suppose now that R/P is a right g_ν -primitive near-ring. So, we get a right g_ν -modular right ideal M/P of R/P such that the zero ideal of R/P is the largest ideal of R/P contained in M/P . Clearly, M is a right g_ν -modular right ideal of R . Since the zero ideal of R/P is the largest ideal of R/P contained in M/P , P is the largest ideal of R contained in M . Therefore, P is a right g_ν -primitive ideal of R . $\square$

Proposition 2.12. $J_{g_\nu}^r$ is the Hoehnke radical determined by the class of all right g_ν -primitive near-rings, $\nu \in \{0, 1, 2\}$.

Theorem 2.13. Let G be a right R -group of type- g_ν and S be an invariant subnear-ring (and right ideal for $\nu = 0$) of R with $GS \neq \{0\}$. Then G is a right S -group of type- g_ν , $\nu \in \{0, 1, 2\}$.

Proof. If G is a right R -group of type-0 and S is an invariant subnear-ring and right ideal of R with $GS \neq \{0\}$, then under the restriction of G to S , by Theorem 3.2 of [9], G is a right S -group type-0. Also if G be a right R -group of type- ν and S is an invariant subnear-ring of R with $GS \neq \{0\}$, then under the restriction of G to S , by Theorems 3.1 and 3.2 of [10], G is a right S -group type- ν , where $\nu \in \{1, 2\}$. Therefore G is a right S -group of type- ν , $\nu \in \{0, 1, 2\}$. Let A be the set of generators of the right R -group G and P be the largest ideal of R contained in $(0 : G)_R := \{r \in R \mid Gr = \{0\}\}$. A generator of the right R -group G is also a generator of the right S -group G . From the proof of Theorem 3.10 of [9] (and Theorems 3.9 and 3.10 of [10] for $\nu \in \{1, 2\}$) as the extension of G from S to R coincides with the action of G on R , it follows that a generator of the right S -group G is also a generator of the right R -group G . So A is the set of generators of the right S -group G . We have $P = (0 : A) = \{r \in R \mid ar = 0 \text{ for all } a \in A\}$. Now $P \cap S = (0 : A) \cap S = \{s \in S \mid As = \{0\}\}$. Let Q be the largest ideal of S contained in $(0 : G)_S := \{s \in S \mid Gs = \{0\}\} = (0 : G) \cap S$. Clearly $P \cap S \subseteq (0 : G)_S$. By the definition of Q , $P \cap S \subseteq Q$. Since $AQ = \{0\}$, $Q \subseteq P$. So $Q \subseteq P \cap S$. Therefore $Q = P \cap S$. Hence G is a right S -group of type- g_ν . $\square$

Proposition 2.14. A right R -group of type- g_ν is an R -group of type- $\nu(e)$, $\nu \in \{0, 1, 2\}$.

Proof. Let G be a right R -group of type- g_ν , $\nu \in \{0, 1, 2\}$. So G is a right R -group of type- ν . In view of Remark 3.9 of [11] G is a right R -group of type- $\nu(e)$ if $r, s \in R$ and $gr = gs$ for all $g \in G$, then $r - s \in P$ where P is the largest ideal of R contained in $(0 : G) := \{r \in R \mid Gr = \{0\}\}$. Let $gr = gs$ for all $g \in G$, $r, s \in R$ and P be the largest ideal of R contained in $(0 : G)$. Let A be the set of all generators of the right R -group G . Now $ar = as$ for all $a \in A$. Since each $a \in A$ is distributive, $a(r - s) = 0$ for all $a \in A$. Therefore $r - s \in P$ as $P = (0 : A)$. Hence G is a right R -group of type- $\nu(e)$. $\square$

Remark 2.15. If G is a right R -group of type- $\nu(e)$, then by Proposition 3.12 of [11], $(0 : G) := \{r \in R \mid Gr = \{0\}\}$ is an ideal of R . Also, by Theorem 3.24 of [11], a right g_ν -primitive near-ring is an equiprime near-ring.

Definition 2.16. Let G be a right R -group of type- g_ν , $\nu \in \{0, 1, 2\}$. Then G is called faithful if $(0 : G) = \{0\}$.

Theorem 2.17. Let G be a faithful right S -group of type- g_ν and S be an essential left invariant ideal of R . Then G is a faithful right R -group of type- g_ν , $\nu \in \{0, 1, 2\}$.

Proof. Let h_0 be a generator of the right S -group G . From the proof of Theorem 3.10 of [9], for $h \in H, r \in R$ the operation defined by $hr := h_0(sr)$ if $h = h_0s, s \in S$, makes G a right R -group and is an extension the action of G on S to R . Moreover, Theorem 3.10 of [9] and Theorems 3.9 and 3.10 of [10], G is a right R -group of type- ν , for $\nu \in \{1, 2\}$. Since G is a right R -group of type- $\nu(e)$, by Theorem 3.33 of [11] and Theorem of [5], G is a faithful R -group of type- $\nu(e)$. Let A be the set of

all generators of the right S -group G . Now $(0 : G)_S := \{s \in S \mid Gs = \{0\}\} = \{0\}$. We have $\{0\} = (0 : A)_S := \{s \in S \mid As = \{0\}\}$. Since G is a faithful right R -group, $(0 : G)_R := \{r \in R \mid Gr = \{0\}\} = \{0\}$. From the proof of Theorem 3.10 of [9], it can be easily seen that a generator of the right S -group G is also a generator of the right R -group G . So A is the set of generators of the right R -group G . Suppose that $r \in (0 : A)$. Now $Ar = \{0\}$. So $\{0\} = (Ar)S = A(rS)$ and hence $rS = \{0\}$ as $rS \subseteq S$. Since S is an ideal, $KS = \{0\}$ and S is a prime near-ring, we have $K = \{0\}$, where K is the ideal of R generated by r . Therefore $r = 0$ and hence $(0 : A)_R = \{0\}$. So G is a faithful right R -group of type- g_ν . $\square$

From the above theorem we have:

Theorem 2.18. *The class of all right g_ν -primitive near-rings is closed under essential left invariant extensions, $\nu \in \{0, 1, 2\}$.*

In view of Theorem 1.1, we have the following:

Theorem 2.19. *Let $\nu \in \{0, 1, 2\}$. Let $\mathcal{E}$ be the class of all right g_ν -primitive near-rings and $\mathcal{UE}$ be the upper radical class determined by $\mathcal{E}$. Then $\mathcal{UE}$ is a c-hereditary Kurosh-Amitsur radical class in the variety of all near-rings with hereditary semisimple class $S\mathcal{UE} = \mathcal{E}$. So, $J_{g_\nu}^r$ is a Kurosh-Amitsur radical in the class of all near-rings and for any ideal I of R , $J_{g_\nu}^r(I) \subseteq J_{g_\nu}^r(R) \cap I$ with equality, if I is left invariant.*

Theorem 2.20. *$J_{g_\nu}^r$ is an ideal-hereditary Kurosh-Amitsur radical in the class of all zero-symmetric near-rings.*

Theorem 2.21. *$J_{g_\nu}^r$ is a special radical in the class of all near-rings.*

3. Examples

In this section we present some examples of near-rings R and their right R -groups to show that the present right Jacobson radicals are distinct from the known right Jacobson radicals of near-rings. Now we present an example of a right R -group of type- $\nu(e)$ which is not of type- g_ν , $\nu \in \{0, 1, 2\}$.

Proposition 3.1. *If G be a finite group and G has a subgroup of index two, then $M_0(G)$ is a right 2(e)-primitive near-ring.*

Proof. Let G be a finite group and H be a subgroup of G of index 2. So H is a normal subgroup of G . Let $R = M_0(G)$. Then R/K is a right R -group of type-2(e), where $K = (H : G) = \{r \in R \mid r(g) \in H, \text{ for all } g \in G\}$. To show this we consider the two distinct cosets H and $H + a$ of H in G . Now $G = H \cup H + a$, H and $H + a$ are disjoint sets. K is a right ideal of R which is right modular by the identity element of R . So R/K is a monogenic right R -group. Now we show that R/K is a right R -group of type-2. Let $0 \neq r + K \in R/K$. $(r + K)R = R/K$ if and only if

there is an $s \in R$ such that $(r + K)s = 1 + K$, that is, $1 - rs \in K$. Let $P_1 = \{x \in G \mid r(x) \in H\}$ and $P_2 = \{x \in G \mid r(x) \in H + a\}$. Let $b \in P_2$ and $r(b) = h' + a$, $h' \in H$. Define $s : G \rightarrow G$ by $s(g) = b$, if $g \in H + a$, and 0, if $g \in H$. We have $s \in R$. For $y \in H$, $(1 - rs)(y) = y - r(s(y)) = y - r(0) = y \in H$ and for $z = h + a \in H + a$, $(1 - rs)(z) = z - r(s(z)) = z - r(b) = (h + a) - (h' + a) = h - h' \in H$. Therefore, $1 - rs \in (H : G) = K$ and hence R/K is a right R -group of type-2. Since R is simple, $\{0\}$ is the largest ideal of R contained in $(0 : R/K) = (K : R) = \{t \in R \mid Rt \subseteq K\}$. Let $u, v \in R$ and $(t + K)u = (t + K)v$ for all $t + K \in R/K$. Now $tu - tv \in K$, for all $t \in R$. Suppose that $g \in G$ and $u(g) \neq v(g)$. We can choose a $t \in R$ such that $(tu)(g) - (tv)(g) \in H + a$, a contradiction to the fact that $tu - tv \in K$. Therefore, $u = v$ and hence R/K is a right R -group of type-2(e). Since R is simple, it is a right 2(e)-primitive near-ring. $\square$

Example 3.2. Let G be the non-abelian group of order 6. Let N be the subgroup of G of order 3. By Proposition 3.1, $M_0(G)/(N : G)$ is a right $M_0(G)$ -group of type-2(e) and $M_0(G)$ is a right 2(e)-primitive near-ring. Since N is the maximal (normal) subgroup of G , $(N : G)$ is the only proper (maximal) right ideal of $M_0(G)$. So a right $M_0(G)$ -group of type-0 is $M_0(G)$ -isomorphic to $M_0(G)/(N : G)$. Therefore, if $f + (N : G)$ is a generator of the right $M_0(G)$ -group $M_0(G)/(N : G)$, then $(0 : f + (N : G)) = (N : G) \neq \{0\}$. Note that as $M_0(G)$ is a simple near-ring, $\{0\}$ is the largest ideal of $M_0(G)$ contained in $(0 : M_0(G)/(N : G))$. Hence $M_0(G)/(N : G)$ is not a right $M_0(G)$ -group of type- g_ν , $\nu \in \{0, 1, 2\}$.

Now we present another example to show that there are right R -groups of type- $\nu(e)$ which are not of type- g_ν . The following example was considered in [3] and [11].

Example 3.3. Consider $G := Z_8$, the group of integers under addition modulo 8. Now $T : G \rightarrow G$ defined by $T(g) = 5g$ for all $g \in G$ is an automorphism of G . T fixes 0, 2, 4, 6 and maps 1 to 5, 5 to 1, 3 to 7 and 7 to 3. Now $A := \{I, T\}$ is an automorphism group of G and $\{0\}, \{2\}, \{4\}, \{6\}, \{1, 5\}$ and $\{3, 7\}$ are the orbits. Let R be the centralizer near-ring $M_A(G)$, the near-ring of all self maps of G which fix 0 and commute with T . An element of R is completely determined by its action on $\{1, 2, 3, 4, 6\}$. Note that for $f \in R$ we have $f(2), f(4), f(6)$ are arbitrary in $2G$ and $f(1), f(3)$ are arbitrary in G . In [3] shown that $I := (0 : 2G) = \{f \in R \mid f(h) = 0, \text{ for all } h \in 2G\}$ is the only non-trivial ideal of R . Let $K := (2G : G) = \{t \in R \mid t(G) \subseteq 2G\} \neq R$. Let t_0 be the identity element in R . Now $t_0 + K$ is a generator of the right R -group R/K . Let $h \in R - K$. We show now that $(h + K)R = R/K$. Since $h \notin K$, there is an $a \in G - 2G$ such that $b := h(a) \notin 2G$. We construct an element $s \in R$ such that $s(1) = s(3) = a$, so that $s(5) = s(7) = a + 4$, and $s = 0$ on $2G$. Since s maps $G - 2G$ to $G - 2G$, we get that $t_0 - hs \in K$ and hence $(h + K)s = t_0 + K$. So $(h + K)R = R/K$. Therefore, R/K is a right R -group of type- ν . Moreover, $(R/K)I \neq \{K\}$. Therefore, $\{0\}$ is the largest ideal of R contained in $(K : R)$ and hence $J_\nu^r(R) = \{0\}$. Consider $s_1, s_1 \in R$, where $s_1(1) = 1$ and 0 on $G - \{1, 5\}$ and

$s_2(1) = 5$ and 0 on $G - \{1, 5\}$. Clearly $(h + K)s_1 = (h + K)s_2$ for all $h \in R$ as $h(1) - h(5) \in 2G$ for all $h \in R$. But $s_1 - s_2 \notin \{0\}$. Therefore, R/K is not a right R -group of type- $\nu(e)$.

Proposition 3.4. Let R be the near-ring considered in the Example 3.3 and let K be a right ideal of R . Then $H_1 := \{f(g) \mid f \in K, g \in G\} \subseteq G$ and $H_2 := \{f(g) \mid f \in K, g \in 2G\} \subseteq 2G$ are (normal) subgroups of G and $2G$ respectively.

Proof. We show that H_1 is a subgroup of G . Since $0 \in H_1$, H_1 is non-empty. Let $h_1, h_2 \in H_1$. We get $f_1, f_2 \in K$ and $g_1, g_2 \in G$ such that $h_1 = f_1(g_1)$ and $h_2 = f_2(g_2)$. Clearly, $-h_1 = (-f_1)(g_1) \in H_1$ as $-f_1 \in K$. Suppose that one of the g_i is in $G - 2G$. With out loss of generality, suppose that $g_1 \in G - 2G$. We get $f_3 \in R$ such that $f_3(g_1) = g_2$. Now $f_1 - f_2f_3 \in K$ and $h_1 - h_2 = (f_1 - f_2f_3)(g_1) \in H_1$. Assume now that $g_1, g_2 \in 2G$. So, $h_1, h_2 \in 2G$. If $g_1 = 0$, then $h_1 - h_2 = -h_2 \in H_1$. Suppose that $g_1 \neq 0$. So, we get $f_4 \in R$ such that $f_4(g_1) = g_2$. Now $f_1 - f_2f_4 \in K$ and $h_1 - h_2 = (f_1 - f_2f_4)(g_1) \in H_1$. Therefore, H_1 is a subgroup of G . Similarly, we get that H_2 is a subgroup of $2G$. $\square$

Proposition 3.5. Let R, K, H_1 and H_2 be as defined in Proposition 3.4. If $H_1 = G$ and $H_2 = 2G$, then $K = R$.

Proof. Suppose that $H_1 = G$ and $H_2 = 2G$. We have $1, 3 \in H_1$. So, for $i \in \{1, 3\}$, we get $f_i \in K$ such that $f_i(g_i) = i$, where $g_i \in \{1, 3, 5, 7\} = G - 2G$. For $i = 1, 3$ we also get $m_i \in R$ such that $m_i(i) = g_i$, so that $m_i(i + 4) = g_i + 4$ and $m_i = 0$ on $G - \{i, i + 4\}$. Now $f_i m_i \in K$, $i = 1, 3$. Clearly, $f_1 m_1 + f_3 m_3$ fixes all the elements of $G - 2G$ and maps all the elements of $2G$ to 0. We have $2, 4, 6 \in H_2 = 2G = \{0, 2, 4, 6\}$. For $i = 2, 4, 6$ we get $f_i \in K$ such that $f_i(g_i) = i$, $g_i \in 2G$. So, for $i = 2, 4, 6$ we get $m_i \in R$ such that $m_i(i) = g_i$ and m_i is 0 on $G - \{i\}$. Now $f_i m_i \in K$, $i = 2, 4, 6$. $f_2 m_2 + f_4 m_4 + f_6 m_6$ fixes all the elements of $2G$ and maps all the elements of $G - 2G$ to 0. Therefore, the identity map I of G can be expressed as $I = f_1 m_1 + f_2 m_2 + f_3 m_3 + f_4 m_4 + f_6 m_6 \in K$. Hence, $K = R$. $\square$

Proposition 3.6. Let R, K, H_1 and H_2 be as defined in Proposition 3.4. If K is a maximal right ideal of R , then $K = (2G : G) = \{f \in R \mid f(G) \subseteq 2G\}$ or $(4G : 2G) = \{f \in R \mid f(2G) \subseteq 4G\}$

Proof. Suppose that K is a maximal right ideal of R . Clearly, if H and T are (normal) subgroups of G and $2G$ respectively, then $(H : G) = \{f \in R \mid f(G) \subseteq H\}$ and $(T : 2G) = \{f \in R \mid f(2G) \subseteq T\}$ are right ideals of R . Now $2G$ and $4G$ are the maximal (normal) subgroups of G and $2G$ respectively. We have $K \subseteq (H_1 : G)$ and $K \subseteq (H_2 : 2G)$. Since K is a maximal right ideal of R , by Proposition 3.5, either $H_1 \neq G$ or $H_2 \neq 2G$.

Case(i) Suppose that $H_2 \neq 2G$. Since K is a maximal right ideal of R and $K \subseteq (H_2 : 2G) \neq R$, we get that $H_2 = 4G$ and $K = (4G : 2G)$.

case(ii) Suppose that $H_1 \neq G$. Since K is a maximal right ideal of R and $K \subseteq (H_1 : G) \neq R$, we get that $H_1 = 2G$ and $K = (2G : G)$.

Therefore, either $K = (2G : G)$ or $(4G : 2G)$. $\square$

Proposition 3.7. Let R be the near-ring considered in the Example 3.3. Let $U = (4G : 2G) = \{f \in R \mid f(2G) \subseteq 4G\}$. Then U is a maximal right ideal of R and R/U is a right R -group of type-2(e).

Proof. Clearly, U is a right ideal of R . Consider the right R -group R/U . We prove that R/U is a right R -group of type-2. Since R has identity I , $I + U$ is a generator of the right R -group R/U and hence R/U is a monogenic right R -group. Let $0 \neq f + U \in R/U$. So, $f \notin U$. We get $0 \neq a \in 2G$ such that $b := f(a) \notin 4G$. So, $2G = \{0, b, 2b, 3b\}$ as 2 and 6 are generators of $2G$. Construct $r \in R$ by $r(b) = a$, $r(2b) = 0$, $r(3b) = a$ and $r = 0$ on $G - \{0, 1, 3, 5, 7\}$. Now $(I - fr)(x) \in 4G$ for all $x \in 2G$. Therefore, $I - fr \in U$ and hence $(f + U)r = I + U$. This shows that $(f + U)R = R/U$. So, R/U is a right R -group of type-2. We know that $P := (0 : 2G)$ is the only non-trivial ideal of R . Therefore, P is the largest ideal of R contained in $U = (4G : 2G)$ and hence P is the largest ideal of R contained in $(0 : R/U) = (U : R) = \{f \in R \mid Rf \subseteq U\}$. Let $0 \neq s + U \in R/U$ and $f, h \in R$. Suppose that $(s + U)rf = (s + U)rh$ for all $r \in R$. So, $srf - srh \in U$ for all $r \in R$. We show that $f - h \in P$. If possible, suppose that $f - h \notin P$. We get $0 \neq a \in 2G$ such that $(f - h)(a) = f(a) - h(a) \neq 0$ with $h(a) \neq 0$. Let $s(c) \notin \{0, 4\}$ for some $c \in 2G$. Choose $r \in R$ such that $r(f(a)) = 0$ and $r(h(a)) = c$. Now $(srf)(a) = 0$ and $(srh)(a) = s(c)$. So, $(srf - srh)(a) = 0 - s(c) \notin \{0, 4\}$, a contradiction to the fact that $srf - srh \in U$. Therefore, $f(a) = h(a)$ for all $a \in 2G$. Hence $f - h \in P$. So, R/U is a right R -group of type-2(e). $\square$

Proposition 3.8. Let R be the near-ring considered in Example 3.3. Then $J_\nu^r(R) = \{0\}$ and $J_{\nu(e)}^r(R) = (0 : 2G) \neq \{0\}$.

Proof. We know that $\{0\}$ and $I := (0 : 2G) = \{f \in R \mid f(2G) = \{0\}\}$ are the only proper ideals of R . Let $K_1 := (2G : G) = \{f \in R \mid f(G) \subseteq 2G\}$ and $K_2 := (4G : 2G) = \{f \in R \mid f(2G) \subseteq 4G\}$. By Proposition 3.6, a maximal right ideal of R is either K_1 or K_2 . So, a right R -group of type-0 is isomorphic to R/K_1 or R/K_2 . By Example 3.3, R/K_1 is a right R -group of type-2 but not of type-2(e). Since $\{0\}$ is the largest ideal of R contained in K_1 , $\{0\}$ is a right 2-primitive ideal of R but not a right 2(e)-primitive ideal of R . By Proposition 3.7, R/K_2 is a right R -group of type-2(e). Since $I = (0 : 2G)$ is the largest ideal of R contained in K_2 , I is a right 2(e)-primitive ideal of R . Therefore, $J_\nu^r(R) = \{0\}$ and $J_{\nu(e)}^r(R) = (0 : 2G)$. $\square$

Proposition 3.9. Let R be the near-ring considered in Example 3.3. Then $J_{g_\nu}^r(R) = R$, $\nu \in \{0, 1, 2\}$.

Proof. Let R be the near-ring considered in the Example 3.3 and $K = (2G : G)$, $U = (4G : 2G)$. As seen above K, U are the only maximal right ideals of R and R/K is a right R -group of type-2 but not of type-2(e), where as R/U is a right R -group of type-2(e). If $f + K$ is a generator of the right R -group R/K , then the maximal right ideal $(0 : f + K)$ must be either K or U . Since $0(K) = 2^{10} \neq 2^9 = 0(U)$, and $R/(0 : f + K)$ is right R -isomorphic R/K , $(0 : f + K) = K$. Hence R/K is not a right R -group of type- g_ν as $\{0\}$, $(0 : 2G)$ and R are the only ideals of R . By a similar argument we get that R/U is not a right R -group of type- g_ν . So $J_{g_\nu}^r(R) = R$. $\square$

4. $J_{g_\nu}^r$ -semisimple Near-rings, $\nu \in \{0, 1, 2\}$

In this section we present structure theorems for $J_{g_\nu}^r$ -semisimple near-rings.

Proposition 4.1. Let $R (\neq \{0\})$ be a $J_{g_\nu}^r$ -semisimple near-rings satisfying DCC on right ideals of R , $\nu \in \{0, 1, 2\}$. Then R is a finite direct sum of minimal right ideals which are right R -groups of type- g_ν .

Proof. Let $P_i, i \in I$ be the collection of right g_ν -primitive ideals of R . Since R is a $J_{g_\nu}^r$ -semisimple near-ring, $\cap\{P_i \mid i \in I\} = \{0\}$. We get a right R -group G_i of type- g_ν such that $P_i = (0 : G_i) := \{r \in R \mid G_i r = \{0\}\}, i \in I$. Let A_i be the set of generators of $G_i, i \in I$. Now $P_i = (0 : A_i) := \{r \in R \mid A_i r = \{0\}\}$. Note that for each $a \in A_i, (0 : a) := \{r \in R \mid ar = 0\}$ is a right g_ν -modular right ideal of R and the right R -group $R/(0 : a)$ is right R -isomorphic to $G_i, i \in I$. Since each P_i is an intersection of right g_ν -modular right ideal of R and $\cap\{P_i \mid i \in I\} = \{0\}$, the intersection of all right g_ν -modular right ideal of R is zero. We get a finite number of right g_ν -modular right ideals $K_1, K_2, \dots, K_n$ of R such that $\cap\{K_j \mid j = 1, 2, \dots, n\} = \{0\}$. Let $T_i := K_1 \cap K_2 \cap \dots \cap K_{i-1} \cap K_{i+1} \cap \dots \cap K_n, i = 1, 2, \dots, n$. We may assume that $T_i \neq \{0\}$ for all $i = 1, 2, \dots, n$. Now by Proposition 3.12[(2)] of [8], $R = T_1 \oplus T_2 \oplus \dots \oplus T_n$, a direct sum of minimal right ideals T_i of R which are right R -groups of type- g_ν . $\square$

In [8](Definition 3.5), if R is a direct sum of n minimal right ideals of R , then the *dimension* of R is defined as n and is denoted by $\dim R$.

Definition 4.2. A distributive idempotent e of R is called *right g_ν -primitive* if eR is a right R -group of type- $g_\nu, \nu \in \{0, 1, 2\}$.

Theorem 4.3. Let R be a right g_ν -primitive near-rings satisfying DCC on right ideals of $R, \nu \in \{0, 1, 2\}$. Then R is a simple near-ring with identity and R has a subnear-ring which is isomorphic to the matrix near-ring $M_n(S)$, where $S = eRe, e$ is a right g_ν -primitive idempotent and $n = \dim R$. If, in addition, R is distributively generated, then R is isomorphic to $M_n(S)$.

Proof. R satisfies the hypothesis of Theorem 4.3 of [8] and hence the conclusion follows from it. $\square$

Theorem 4.4. Let R be a finite right g_2 -primitive near-ring and eRe be a non-ring. Then R is (isomorphic to) the matrix near-ring $M_n(F)$, where $n = \dim R, F := eRe$ is a near-field and e is a right g_2 -primitive idempotent in R .

Proof. Proof follows from Theorem 4.16 of [8]. $\square$

Theorem 4.5. Let $R (\neq \{0\})$ be a $J_{g_\nu}^r$ -semisimple near-rings satisfying DCC on right ideals of $R, \nu \in \{0, 1, 2\}$. Then R is a direct sum of minimal ideals which are simple right g_ν -primitive near-rings with identity.

Proof. Let $P_i, i \in I$ be the collection of right g_ν -primitive ideals of R , $\nu \in \{0, 1, 2\}$. Now $\cap\{P_i \mid i \in I\} = \{0\}$. Since R has DCC on right ideals of R , we get a finite number of right g_ν -primitive ideals of $P_1, P_2, \dots, P_n$ of R such that $P_1 \cap P_2 \cap \dots \cap P_n = \{0\}$. We may assume that $K_j := P_1 \cap P_2 \cap \dots \cap P_{j-1} \cap P_{j+1} \cap \dots \cap P_n \neq \{0\}, j = 1, 2, \dots, n$. By Theorem 4.3, R/P_i is a simple near-ring with identity as R/P_i is a right g_ν -primitive near-ring with DCC on right ideals. Now by Theorem 2.50 of Pilz [4], $R = K_1 \oplus K_2 \oplus \dots \oplus K_n$, K_i are minimal ideals of R and are simple right g_ν -primitive near-rings with identity. $\square$

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