On Approximation by Matrix Means of the Multiple Fourier Series in the Hölder Metric

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ABSTRACT. In this work, we shall give the degree of approximation for functions belonging to Hölder class by matrix summability method of multiple Fourier series in the Hölder metric.

1. Introduction and some Notations

Suppose that $f(x, y)$ is integrable in the sense of Lebesgue over the square $S^2 := S(-\pi, \pi; -\pi, \pi)$ and of period 2π in x and in y . If $f(x, y)$ is defined only on the square S^2 , we extend it periodically onto the whole xy -plane. The double Fourier series of $f(x, y)$ can be written in the form

$$f(x, y) \sim \sum_{m, n \in \mathbb{N}} \lambda_{mn} [\eta_{mn} \cos mx \cos ny + \mu_{mn} \sin mx \cos ny + \rho_{mn} \cos mx \sin ny + \zeta_{mn} \sin mx \sin ny]$$

where

$$\lambda_{mn} = \begin{cases} 1/4, & m = n = 0; \\ 1/2, & m > 0, n = 0 \vee m = 0, n > 0; \\ 1, & m > 0, n > 0. \end{cases}$$

and the coefficients η_{mn} , μ_{mn} , ρ_{mn} and ζ_{mn} are calculated by the formulas

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$$\begin{aligned}
\eta_{mn} &= \frac{1}{\pi^2} \iint_{S^2} f(x, y) \cos mx \cos ny dx dy, \\
\mu_{mn} &= \frac{1}{\pi^2} \iint_{S^2} f(x, y) \sin mx \cos ny dx dy, \\
\rho_{mn} &= \frac{1}{\pi^2} \iint_{S^2} f(x, y) \cos mx \sin ny dx dy, \\
\zeta_{mn} &= \frac{1}{\pi^2} \iint_{S^2} f(x, y) \sin mx \sin ny dx dy,
\end{aligned}
\tag{1.1}$$

for $m = 0, 1, 2, \dots$ and $n = 0, 1, 2, \dots$. Now let

$$\begin{aligned}
s_{mn}(x, y) &= \sum_{i=0}^m \sum_{j=0}^n [\eta_{ij} \cos ix \cos jy + \mu_{ij} \sin ix \cos jy \\
&\quad + \rho_{ij} \cos ix \sin jy + \zeta_{ij} \sin ix \sin jy].
\end{aligned}$$

The quantity $s_{mn}(x, y)$ ($m = 0, 1, 2, \dots$; $n = 0, 1, 2, \dots$) are called the partial sums of double Fourier series. According to (1.1), we know that

$$s_{mn}(x, y) = \frac{1}{\pi^2} \iint_{S^2} f(x+u, y+v) \frac{[\sin(m+1/2)u][\sin(n+1/2)v]}{4\sin(u/2)\sin(v/2)} du dv.$$

Moreover, let

$$\tau_{mn}(x, y) = \tau_{mn}(f; A, U; x, y) := \sum_{i=0}^m \sum_{j=0}^n a_{mi} b_{nj} s_{ij}(x, y), \quad \forall m, n \geq 0$$

where $A \equiv (a_{m,i})$ and $U \equiv (b_{n,j})$ are lower triangular infinite matrices such that:

$$(1.2) \quad a_{m,i} = \begin{cases} \geq 0, & i \leq m; \\ 0, & i > m \end{cases} \quad (i, m = 0, 1, 2, \dots) \quad \wedge \quad \sum_{i=0}^m a_{m,i} = 1$$

and

$$(1.3) \quad b_{n,j} = \begin{cases} \geq 0, & j \leq n; \\ 0, & j > n \end{cases} \quad (j, n = 0, 1, 2, \dots) \quad \wedge \quad \sum_{j=0}^n a_{n,j} = 1.$$

The double Fourier series of the function $f(x, y)$ is called to be (A, U) -summable to a finite number s , if $\tau_{mn}(x, y) \rightarrow s$ as $m, n \rightarrow \infty$. The condition of regularity for

double matrix summability means are given by

$$(1.4) \quad \begin{aligned} & \sum_{i=0}^m \sum_{j=0}^n a_{mi} b_{nj} \rightarrow 1, \text{ as } m, n \rightarrow \infty, \\ & \lim_{m,n} \sum_{j=0}^n |a_{mi} b_{nj}| = 0, \text{ for each } i = 1, 2, \dots, \\ & \lim_{m,n} \sum_{i=0}^m |a_{mi} b_{nj}| = 0, \text{ for each } j = 1, 2, \dots \end{aligned}$$

Let

$$H_\alpha = \{f \in C_{2\pi} : |f(x) - f(y)| \leq K|x - y|^\alpha\}$$

where K is a positive constant, not necessarily the same at each occurrence. It is known that H_α is a Banach space (see Prösdorff, [7]) with the norm $\|\cdot\|_\alpha$ defined by

$$(1.5) \quad \|f\|_\alpha = \|f\|_C + \sup_{x \neq y} \Delta^\alpha f(x, y)$$

where

$$\Delta^\alpha f(x, y) = \frac{|f(x) - f(y)|}{|x - y|^\alpha} \quad (x \neq y),$$

by convention $\Delta^0 f(x, y) = 0$ and

$$\|f\|_C = \sup_{x \in [-\pi, \pi]} |f(x)|.$$

The metric induced by the norm (1.5) on H_α is called the Hölder metric. Prösdorff has been studied the degree of approximation in the Hölder metric and proved the following theorem:

Theorem A. ([7]) *Let $f \in H_\alpha$ ($0 < \alpha \leq 1$) and $0 \leq \beta < \alpha \leq 1$. Then*

$$(1.6) \quad \|\sigma_n(f) - f\|_\beta = O(1) \begin{cases} n^{\beta-\alpha} & , 0 < \alpha < 1; \\ n^{\beta-1} \ln n & , \alpha = 1 \end{cases}$$

where $\sigma_n(f)$ is Fejér means of the Fourier series of f .

The case $\beta = 0$ in Theorem A is owing to Alexits [1]. Chandra obtained a generalization of Theorem A in the Voronoi-Nörlund transform [2]. In [6], Mohapatra and Chandra considered the problem by matrix means of the Fourier series of $f \in H_\alpha$. In the one-dimensional case, these problems have been studied in detail. Naturally, similar problems are considered for the periodic functions with two variables. Stepanets investigated the problem of the approximation of functions $f(x, y)$, 2π -periodic with respect to each of the variables by the partial sums of their Fourier sums and under the some conditions in [9, 10]. In [5], Lal studied the approximation

of functions belonging to Lipschitz class by matrix summability method for double Fourier series under the uniform norm.

The Hölder class for $f(x, y)$ continuous functions periodic in both variables with period 2π is defined as

$$H_{(\alpha, \beta)} = \{f : |f(x, y; z, w)| := |f(x, y) - f(z, w)| \leq C_1(|x - z|^\alpha + |y - w|^\beta)\}$$

for some $\alpha, \beta > 0$ and for all x, y, z, w where C_1 is a positive constant may depend on f , but not on x, y, z, w . This class of functions is also called Lipschitz class and denoted by $Lip(\alpha, \beta)$. It can be easily verified that $H_{\alpha, \beta}$ is a Banach space with the norm $\|\cdot\|_{\alpha, \beta}$ defined by

$$(1.7) \quad \|f\|_{\alpha, \beta} = \|f\|_C + \sup_{x \neq z, y \neq w} \Delta^{\alpha, \beta} f(x, y; z, w)$$

where

$$\Delta^{\alpha, \beta} f(x, y; z, w) = \frac{|f(x, y) - f(z, w)|}{|x - z|^\alpha + |y - w|^\beta} \quad (x \neq z, y \neq w),$$

by convention $\Delta^{0,0} f(x, y; z, w) = 0$ and

$$\|f\|_C = \sup_{(x, y) \in S^2} |f(x, y)|.$$

Moreover, a function f in $Lip(\alpha, \beta)$ is said to belong to the little Lipschitz class $lip(\alpha, \beta)$ if

$$\lim_{z \rightarrow x, w \rightarrow y} (|x - z|^\alpha + |y - w|^\beta)^{-1} |f(x, y; z, w)| = 0$$

uniformly in (x, y) . The aim of this paper is as follows. First, the approximation to functions $f(x, y)$ belonging to these Lipschitz classes is given by matrix summability method of double Fourier series in accordance with the norm in (1.7). Later the approximation is generalized to the N -multiple Fourier series.

Throughout this paper, we shall also use the following notations:

$$\begin{aligned} \Psi(u, v) &:= \Psi(x, y; u, v) := \frac{1}{4} \{f(x + u, y + v) + f(x + u, y - v) \\ &\quad + f(x - u, y + v) + f(x - u, y - v) - 4f(x, y)\} \end{aligned}$$

and

$$F(u, v) = \Phi(u, v) - \Psi(u, v)$$

where $\Phi(u, v) := \Psi(z, w; u, v)$. Since $f(x, y) \in H_{(\alpha, \beta)}$, it is clear that

$$(1.8) \quad |F(u, v)| = O(|x - z|^\alpha + |y - w|^\beta).$$

2. In Case of Double Fourier Series

The approximation by matrix means for double Fourier series is as follows with respect to Hölder metric.

Theorem 2.1. Assume $A \equiv (a_{m,i})$ and $U \equiv (b_{n,j})$ are lower triangular matrices where $(a_{m,i})$ and $(b_{n,j})$ are nondecreasing sequences with respect to $i \leq m$ and $j \leq n$ satisfying the conditions (1.2) and (1.3), respectively such that double matrix method (A, U) is regular. If $f(x, y)$ is a function of period 2π in x and y Lebesgue integrable in S^2 belonging to the class $H_{(\alpha, \beta)}$ for $0 < \alpha, \beta \leq 1$, then

$$\|\tau_{mn} - f\|_{\alpha, \beta} = O(1) \begin{cases} (m+1)^{-\alpha} + (n+1)^{-\beta} & , 0 < \alpha < 1, 0 < \beta < 1; \\ \frac{\log((m+1)\pi)}{(m+1)} + \frac{\log((n+1)\pi)}{(n+1)} & , \alpha = \beta = 1 \end{cases}$$

for $m, n = 0, 1, 2, \dots$

For small Lipschitz class, the analogy of the Theorem can be written if "O" is replaced by "o" as $m, n \rightarrow \infty$ independently one another, and $f \in Lip(\alpha, \beta)$ is replaced by $f \in lip(\alpha, \beta)$ for $0 < \alpha, \beta < 1$. We don't enter in details.

Furthermore, double matrix summability method gives us the following means for some important cases:

- $(C, 1, 1)$ means, when $a_{m,i} = \frac{1}{m+1}$ and $b_{n,j} = \frac{1}{n+1}$ for all i and j , respectively [3];
- (N, p_m, q_n) means, when $a_{m,i} = \frac{p_{m-i}}{P_m}$ and $b_{n,j} = \frac{q_{n-j}}{Q_n}$; where $P_m = \sum_{k=0}^m p_k \neq 0$ and $Q_n = \sum_{k=0}^n q_k \neq 0$ [4];
- $(H, 1, 1)$ means, when $a_{m,i} = \frac{1}{(m-i+1) \log m}$ and $b_{n,j} = \frac{1}{(n-j+1) \log n}$ [8].

Taking into account the first two case above, we write the following results.

Corollary 2.2. If $f(x, y)$ is a function of period 2π in x and y Lebesgue integrable in S^2 belonging to the class $H_{(\alpha, \beta)}$ for $0 < \alpha, \beta \leq 1$, then

$$\|\sigma_{mn} - f\|_{\alpha, \beta} = O(1) \begin{cases} (m+1)^{-\alpha} + (n+1)^{-\beta} & , 0 < \alpha < 1, 0 < \beta < 1; \\ \frac{\log((m+1)\pi)}{(m+1)} + \frac{\log((n+1)\pi)}{(n+1)} & , \alpha = \beta = 1 \end{cases}$$

for $m, n = 0, 1, 2, \dots$, where

$$\sigma_{mn}(x, y) = \frac{1}{(m+1)(n+1)} \sum_{i=0}^m \sum_{j=0}^n s_{ij}(x, y), \quad \forall m, n \geq 0.$$

Corollary 2.3. *If $f(x, y)$ is a function of period 2π in x and y Lebesgue integrable in S^2 belonging to the class $H_{(\alpha, \beta)}$ for $0 < \alpha, \beta < 1$, then*

$$\|\mathbf{N}_{mn} - f\|_{\alpha, \beta} = O(1) \{ (m+1)^{-\alpha} + (n+1)^{-\beta} \}$$

for $m, n = 0, 1, 2, \dots$, where

$$\mathbf{N}_{mn}(x, y) = \frac{1}{P_m Q_n} \sum_{i=0}^m \sum_{j=0}^n p_{m-i} q_{n-j} s_{ij}(x, y), \quad \forall m, n \geq 0.$$

Before giving the proof of Theorem 2.1, we need the following auxiliary results.

Lemma 2.4. *Let $(a_{m,i})$ and $(b_{n,j})$ be real nonnegative and nondecreasing sequence with (1.2) and (1.3), respectively.*

(i) *For $0 < u \leq 1/(m+1)$, we have $K_m(u) = O(m+1)$ where*

$$K_m(u) := \frac{1}{\pi} \sum_{i=0}^m a_{m,i} \frac{\sin(i + \frac{1}{2})u}{\sin(\frac{u}{2})}.$$

(ii) *For $0 < v \leq 1/(n+1)$, we have $K_n(v) = O(n+1)$ where*

$$K_n(v) := \frac{1}{\pi} \sum_{j=0}^n b_{n,j} \frac{\sin(j + \frac{1}{2})v}{\sin(\frac{v}{2})}.$$

This is easily proved by an elementary calculation.

Lemma 2.5. ([5]) *Assume that $(a_{m,i})$ and $(b_{n,j})$ be real nonnegative and nondecreasing sequence with $i \leq m$ and $j \leq n$, respectively.*

(i) *For $1/(n+1) < v \leq \pi$ and any $n \in \mathbb{N}$, we have*

$$K_n(v) = O\left(\frac{B_{n,\sigma}}{v}\right),$$

where $B_{n,\sigma} = \sum_{j=n-\sigma}^n b_{n,j}$ and σ denote integer part of $\frac{1}{v}$.

(ii) *For $1/(m+1) < u \leq \pi$ and any $m \in \mathbb{N}$, we have*

$$K_m(u) = O\left(\frac{A_{m,\kappa}}{u}\right),$$

where $A_{m,\kappa} = \sum_{i=m-\kappa}^m a_{m,i}$ and κ denote integer part of $\frac{1}{u}$.

3. Proof of the Theorem 2.1

Proof. We know that

$$(3.1) \quad s_{ij}(x, y) - f(x, y) = \frac{1}{\pi^2} \int_0^\pi \int_0^\pi \Psi(u, v) \frac{[\sin(i + 1/2)u][\sin(j + 1/2)v]}{\sin(u/2)\sin(v/2)} dudv.$$

Taking into account (3.1) and $\tau_{mn}(x, y)$ that double matrix means of $s_{mn}(x, y)$, we write

$$\begin{aligned} \tau_{mn}(x, y) - f(x, y) &= \sum_{i=0}^m \sum_{j=0}^n a_{mi} b_{nj} \{s_{ij}(x, y) - f(x, y)\} \\ &= \frac{1}{\pi^2} \int_0^\pi \int_0^\pi \Psi(u, v) \sum_{i=0}^m \sum_{j=0}^n a_{mi} b_{nj} \frac{[\sin(i + 1/2)u][\sin(j + 1/2)v]}{\sin(u/2)\sin(v/2)} dudv \\ &= \int_0^\pi \int_0^\pi \Psi(u, v) K_m(u) K_n(v) dudv \end{aligned}$$

Let us estimate that

$$(3.2) \quad \sup_{x \neq z, y \neq w} \frac{|\tau_{mn}(x, y) - f(x, y) - (\tau_{mn}(z, w) - f(z, w))|}{|x - z|^\alpha + |y - w|^\beta} = O(1).$$

$$\begin{aligned} |\tau_{mn}(x, y) - f(x, y) - (\tau_{mn}(z, w) - f(z, w))| &= \left| \int_0^\pi \int_0^\pi F(u, v) K_m(u) K_n(v) dudv \right| \\ &\leq \left(\int_0^{\frac{1}{(m+1)}} \int_0^{\frac{1}{(n+1)}} + \int_0^{\frac{1}{(m+1)}} \int_{\frac{1}{(n+1)}}^\pi + \int_{\frac{1}{(m+1)}}^\pi \int_0^{\frac{1}{(n+1)}} + \int_{\frac{1}{(m+1)}}^\pi \int_{\frac{1}{(n+1)}}^\pi \right) |F(u, v) K_m(u) K_n(v)| dudv \\ (3.3) \quad &=: J_1 + J_2 + J_3 + J_4. \end{aligned}$$

Therefore, from (1.8) and Lemma 2.4, we obtain

$$\begin{aligned} J_1 &= \int_0^{1/(m+1)} \int_0^{1/(n+1)} |F(u, v) K_m(u) K_n(v)| dudv \\ (3.4) \quad &= (m+1)(n+1) \int_0^{1/(m+1)} \int_0^{1/(n+1)} |F(u, v)| dudv = O(|x - z|^\alpha + |y - w|^\beta) \end{aligned}$$

for $0 < \alpha, \beta \leq 1$. By using Lemma 2.4, Lemma 2.5 and again (1.8), then we have

$$\begin{aligned}
 J_2 &= \int_0^{1/(m+1)} \int_{1/(n+1)}^{\pi} |F(u, v) K_m(u) K_n(v)| du dv \\
 &= (m+1) \int_0^{1/(m+1)} \int_{1/(n+1)}^{\pi} |F(u, v)| \frac{B_{n,\sigma}}{v} du dv = O(|x-z|^\alpha + |y-w|^\beta) \int_{1/(n+1)}^{\pi} \frac{B_{n,\sigma}}{v} dv \\
 &\leq O(|x-z|^\alpha + |y-w|^\beta) \int_{1/(n+1)}^{\pi} \frac{B_{n,1/v}}{v} dv \\
 (3.5) &= O(|x-z|^\alpha + |y-w|^\beta) \int_{1/\pi}^{(n+1)} \frac{B_{n,t}}{t} dt = O(|x-z|^\alpha + |y-w|^\beta)
 \end{aligned}$$

since $\frac{B_{n,t}}{t}$ is monotonic increasing. Similarly, we can prove that

$$(3.6) \quad J_3 = \int_{1/(m+1)}^{\pi} \int_0^{1/(n+1)} |F(u, v) K_m(u) K_n(v)| du dv = O(|x-z|^\alpha + |y-w|^\beta)$$

and

$$(3.7) \quad J_4 = \int_{1/(m+1)}^{\pi} \int_{1/(n+1)}^{\pi} |F(u, v) K_m(u) K_n(v)| du dv = O(|x-z|^\alpha + |y-w|^\beta).$$

By combining (3.3)-(3.7), we obtain (3.2). On the other hand, we know that from [5]

$$(3.8) \quad \|\tau_{mn} - f\|_C = O(1) \begin{cases} (m+1)^{-\alpha} + (n+1)^{-\beta} & , 0 < \alpha, \beta < 1; \\ \frac{\log((m+1)\pi e)}{(m+1)} + \frac{\log((n+1)\pi e)}{(n+1)} & , \alpha = \beta = 1 \end{cases}$$

for $m, n = 0, 1, 2, \dots$. Since $\log e < \log(m+1)\pi$ and $\log e < \log(n+1)\pi$, we omit the number "e" in the formula (3.8). Therefore, according to (3.2) and (3.8), the proof of Theorem 2.1 is completed. $\square$

4. In Case of N -Multiple Fourier Series, $N \geq 3$.

Let $f(x_1, \dots, x_N)$ is integrable over the N dimensional cube S^N and of period

2π in each variable. The N -multiple Fourier series of $f(x_1, \dots, x_N)$ can be written in the form

$$f(x_1, \dots, x_N) \sim \sum_{m_1 \in \mathbb{Z}} \sum_{m_2 \in \mathbb{Z}} \cdots \sum_{m_N \in \mathbb{Z}} c_{m_1, m_2, \dots, m_N} e^{i(m_1 x_1 + m_2 x_2 + \cdots + m_N x_N)},$$

where $c_{m_1, m_2, \dots, m_N}$ is the Fourier coefficients of f (see, [11, p. 300]). The series is denoted by $S[f]$ and the partial sums of it are given by

$$S_{m_1 m_2 \dots m_N}(x_1, \dots, x_N) := \pi^{-N} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} f(x_1 + t_1, \dots, x_N + t_N) \prod_{j=1}^N D_{m_j}(t_j) dt_1 \dots dt_N$$

where $D_{m_j}(t_j)$ are the Dirichlet kernels for each j . Moreover, similar to the two-dimensional, we can write

$$\begin{aligned} \tau_{m_1 m_2 \dots m_N}(x_1, \dots, x_N) &=: \tau_{m_1 m_2 \dots m_N}(f; \{A_k\}_1^N; x_1, \dots, x_N) \\ &:= \sum_{i_1=0}^{m_1} \sum_{i_2=0}^{m_2} \cdots \sum_{i_N=0}^{m_N} a_{m_1 i_1} \cdots a_{m_N i_N} S_{i_1 i_2 \dots i_N}(x_1, \dots, x_N) \end{aligned}$$

for all $m_k \geq 0$. Here $\{A_k\}_{k=1}^N \equiv \{(a_{m_k, i_k})\}_{k=1}^N$ are lower triangular infinite matrices such that:

$$(4.1) \quad a_{m_k, i_k} = \begin{cases} \geq 0, & i_k \leq m_k; \\ 0, & i_k > m_k \end{cases} \quad (m_k, i_k = 0, 1, 2, \dots) \quad \wedge \quad \sum_{i_k=0}^{m_k} a_{m_k, i_k} = 1$$

for each $k = 1, 2, \dots, N$. The N -multiple Fourier series of function $f(x_1, \dots, x_N)$ is called to be $(A_1, \dots, A_N)$ -summable to a finite number ℓ , if $\tau_{m_1 m_2 \dots m_N}(x_1, \dots, x_N) \rightarrow \ell$ as $m_1, m_2, \dots, m_N \rightarrow \infty$. The condition of regularity for N -multiple matrix summability means are given by

$$\begin{aligned} \sum_{i_1=0}^{m_1} \sum_{i_2=0}^{m_2} \cdots \sum_{i_N=0}^{m_N} (a_{m_1 i_1} \cdots a_{m_N i_N}) &\rightarrow 1, \text{ as } m_1, m_2, \dots, m_N \rightarrow \infty, \\ \lim_{m_i} \sum_{i_2=0}^{m_2} \sum_{i_3=0}^{m_3} \cdots \sum_{i_N=0}^{m_N} (a_{m_1 i_1} \cdots a_{m_N i_N}) &= 0, \text{ for each } i_1 = 1, 2, \dots, \\ \lim_{m_i} \sum_{i_1=0}^{m_1} \sum_{i_3=0}^{m_3} \cdots \sum_{i_N=0}^{m_N} (a_{m_1 i_1} \cdots a_{m_N i_N}) &= 0, \text{ for each } i_2 = 1, 2, \dots, \\ &\vdots \\ \lim_{m_i} \sum_{i_1=0}^{m_1} \sum_{i_2=0}^{m_2} \cdots \sum_{i_{N-1}=0}^{m_{N-1}} (a_{m_1 i_1} \cdots a_{m_N i_N}) &= 0, \text{ for each } i_N = 1, 2, \dots \end{aligned}$$

Next, we give the notion of Lipschitz classes of functions on S^N . Let $f(x_1, \dots, x_N)$ be a continuous periodic function with period 2π in each variable. The function f belongs to the Lipschitz class $Lip(\alpha_1, \alpha_2, \dots, \alpha_N)$ (or $H_{(\alpha_1, \alpha_2, \dots, \alpha_N)}$) for some $\alpha_1, \alpha_2, \dots, \alpha_N \geq 0$ if there exists a constant K_1 such that

$$|f(x_1, \dots, x_N; y_1, \dots, y_N)| := |f(x_1, \dots, x_N) - f(y_1, \dots, y_N)| \leq K_1 \sum_{k=1}^N |x_k - y_k|^{\alpha_k}$$

for all x_k, y_k where $k = 1, \dots, N$. Furthermore, a function f in $Lip(\alpha_1, \alpha_2, \dots, \alpha_N)$ is said to belong to little Lipschitz class $lip(\alpha_1, \alpha_2, \dots, \alpha_N)$ if

$$\lim_{y_1 \rightarrow x_1, \dots, y_N \rightarrow x_N} \frac{|f(x_1, \dots, x_N; y_1, \dots, y_N)|}{\sum_{k=1}^N |x_k - y_k|^{\alpha_k}} = 0$$

uniformly in $(x_1, \dots, x_N)$.

The function space $H_{(\alpha_1, \alpha_2, \dots, \alpha_N)}$ is a Banach space with respect to the norm $\|\cdot\|_{\alpha_1, \alpha_2, \dots, \alpha_N}$ defined by

$$\|f\|_{\alpha_1, \alpha_2, \dots, \alpha_N} = \|f\|_C + \sup_{x_1 \neq y_1, \dots, x_N \neq y_N} \Delta^{\alpha_1, \alpha_2, \dots, \alpha_N} f(x_1, \dots, x_N; y_1, \dots, y_N)$$

where

$$\Delta^{\alpha_1, \alpha_2, \dots, \alpha_N} f(x_1, \dots, x_N; y_1, \dots, y_N) = \frac{|f(x_1, \dots, x_N; y_1, \dots, y_N)|}{\sum_{k=1}^N |x_k - y_k|^{\alpha_k}},$$

for $x_1 \neq y_1, \dots, x_N \neq y_N$ by convention $\Delta^{0, \dots, 0} f(x_1, \dots, x_N; y_1, \dots, y_N) = 0$ and

$$\|f\|_C = \sup_{(x_1, \dots, x_N) \in S^N} |f(x_1, \dots, x_N)|.$$

Now as an extension of Theorem 2.1, we write the following theorem.

Theorem 4.1. *Let $\{A_k\}_{k=1}^N \equiv \{(a_{m_k, i_k})\}_{k=1}^N$, $N \geq 3$, are lower triangular matrices where $\{(a_{m_k, i_k})\}_{k=1}^N$ are nondecreasing sequences with respect to $i_k \leq m_k$, $k = 1, \dots, N$, satisfying the conditions (4.1), respectively such that N -multiple matrix method $(A_1, A_2, \dots, A_N)$ is regular. If $f(x_1, x_2, \dots, x_N)$ is a function of period 2π in each variable Lebesgue integrable in S^N belonging to the class $H_{(\alpha_1, \alpha_2, \dots, \alpha_N)}$ for $0 < \alpha_1, \alpha_2, \dots, \alpha_N \leq 1$, then*

$$\|\tau_{m_1 m_2 \dots m_N} f\|_{\alpha_1, \alpha_2, \dots, \alpha_N} = O(1) \begin{cases} \sum_{k=1}^N (m_k + 1)^{-\alpha_k} & , 0 < \alpha_1, \alpha_2, \dots, \alpha_N < 1; \\ \sum_{k=1}^N \frac{\log((m_k + 1)\pi)}{(m_k + 1)} & , \alpha_1 = \alpha_2 = \dots = \alpha_N = 1, \end{cases}$$

for $m_k = 0, 1, 2, \dots$, where $k = 1, 2, \dots, N$ and $N \geq 3$ is a fixed integer.

Proof. One needs the extensions of Lemma 2.4 and Lemma 2.5 with respect to each variable from double to N -multiple. After this, the proof runs along the same lines as that of Theorem 2.1. $\square$

Let $0 < \alpha_1, \alpha_2, \dots, \alpha_N < 1$. The analogy of statement in the Theorem 2.1 can be written if "O" is replaced by "o" as $m_1, m_2, \dots, m_N \rightarrow \infty$, and $f \in H_{(\alpha_1, \alpha_2, \dots, \alpha_N)}$ is replaced by $f \in \text{lip}(\alpha_1, \alpha_2, \dots, \alpha_N)$.

N - multiple matrix summability method gives us the $(C, 1, 1, \dots, 1)$ means, when $a_{m_k, i_k} = \frac{1}{m_k + 1}$ for all $i_k, (k = 1, 2, \dots, N)$ [11]. Then, it will be in the form

$$\sigma_{m_1 m_2 \dots m_N}(x_1, \dots, x_N) = \left(\prod_{k=1}^N \frac{1}{m_k + 1} \right) \sum_{i_1=0}^{m_1} \sum_{i_2=0}^{m_2} \dots \sum_{i_N=0}^{m_N} S_{i_1 i_2 \dots i_N}(x_1, \dots, x_N)$$

Therefore, we observe the next result from the Theorem 4.1.

Corollary 4.2. *If $f(x_1, x_2, \dots, x_N)$ is a function of period 2π in each variable Lebesgue integrable in S^N belonging to the class $H_{(\alpha_1, \alpha_2, \dots, \alpha_N)}$ for $0 < \alpha_1, \alpha_2, \dots, \alpha_N \leq 1$, then*

$$\|\sigma_{m_1 m_2 \dots m_N} f - f\|_{\alpha_1, \alpha_2, \dots, \alpha_N} = O(1) \begin{cases} \sum_{k=1}^N (m_k + 1)^{-\alpha_k} & , 0 < \alpha_1, \alpha_2, \dots, \alpha_N < 1; \\ \sum_{k=1}^N \frac{\log((m_k + 1)\pi)}{(m_k + 1)} & , \alpha_1 = \alpha_2 = \dots = \alpha_N = 1 \end{cases}$$

for $m_k = 0, 1, 2, \dots$, where $k = 1, 2, \dots, N$ and $N \geq 3$ is a fixed integer.

References

- [1] G. Alexits, *Convergence problems of orthogonal series*, New York: Pergamon Press, (1961).
- [2] P. Chandra, *On the generalized Fejer means in the metric of the Hölder space*, Math. Nachr., **109**(1982), 39–45.
- [3] Y. S. Chow, *On the Cesáro summability of double Fourier series*, Tohoku Math. J., **5**(1935), 277–283.
- [4] E. Hille and J. D. Tamarkin, *On the summability of Fourier series*, Trans. Amer. Math. Soc., **34**(1932), 757–783.
- [5] S. Lal, *On the approximation of function $f(x, y)$ belonging to Lipschitz class by matrix summability method of double Fourier series*, Journal of the Indian Math. Soc., **78**(1-4)(2011), 93–101.

- [6] R. N. Mohapatra and P. Chandra, *Degree of approximation of functions in the Hölder metric*, Acta Math. Hung., **41** (1-2)(1983), 67–76.
- [7] S. Prösdorff, *Zur Konvergenz der Fourier reihen Hölder Stetiger Funktionen*, Math. Nachr., **69**(1975), 7–14.
- [8] P. L. Sharma, *On the harmonic summability of double Fourier series*, Proc. Amer. Math. Soc., **91**(1958), 979–986.
- [9] A. I. Stepanets, *The approximation of certain classes of differentiable periodic functions of two variables by Fourier sums*, Ukrainian Mathematical Journal (Translated from Ukrainskii Matematicheskii Zhurnal, **25**(5)(1973), 599-609), **26**(1973), 498–506.
- [10] A. I. Stepanets, *Approximation of certain classes of periodic functions of two variables by linear methods of summation of their Fourier series*, Ukrainian Mathematical Journal (Translated from Ukrainskii Matematicheskii Zhurnal, **26**(2)(1974), 205-215), **26**(1974), 168–179.
- [11] A. Zygmund, *Trigonometric Series, Vol. II*, Cambridge University Press, Cambridge, (1959).