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η-Ricci Solitons in δ-Lorentzian Trans Sasakian Manifolds with a Semi-symmetric Metric Connection

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Abstract. The aim of the present paper is to study the δ-Lorentzian trans-Sasakian manifold endowed with semi-symmetric metric connections admitting η-Ricci Solitons and Ricci Solitons. We find expressions for the curvature tensor, the Ricci curvature tensor and the scalar curvature tensor of δ-Lorentzian trans-Sasakian manifolds with a semi-symmetric-metric connection. Also, we discuses some results on quasi-projectively flat and ϕ-projectively flat manifolds endowed with a semi-symmetric-metric connection. It is shown that the manifold satisfying $\bar{R}.\bar{S} = 0$, $\bar{P}.\bar{S} = 0$ is an η-Einstein manifold. Moreover, we obtain the conditions for the δ-Lorentzian trans-Sasakian manifolds with a semi-symmetric-metric connection to be conformally flat and ξ-conformally flat.

1. Introduction

In 1924, the idea of a semi-symmetric linear connection on a differentiable manifold was introduced by A. Friedmann and J. A. Schouten [13]. In 1930, Bartolotti [5] gave a geometrical meaning of such a connection. In 1932, H. A. Hayden [16] defined and studied semi-symmetric metric connections. In 1970, K. Yano [42], started a systematic study of semi-symmetric metric connections in a Riemannian manifold and this was further studied by various authors such as Sharfuddin Ahmad and S. I. Hussain [31], M. M. Tripathi [34], I. E. Hirică and L. Nicolescu [17, 18], G. Pathak and U.C. De [27].

Let ∇ be a linear connection in an n-dimensional differentiable manifold M. The torsion tensor T and the curvature tensor R of ∇ are given respectively by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y],$$

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

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The connection ∇ is said to be symmetric if its torsion tensor T vanishes, otherwise it is non-symmetric. The connection ∇ is said to be a metric connection if there is a Riemannian metric g in M such that $\nabla g = 0$, otherwise it is non-metric. It is well known that a linear connection is symmetric and metric if it is the Levi-Civita connection.

A linear connection ∇ is said to be a semi-symmetric connection if its torsion tensor T is of the form

$$T(X, Y) = \eta(Y)X - \eta(X)Y,$$

where η is a 1-form. Semi-symmetric connections play an important role in the study of Riemannian manifolds. There are various physical problems involving the semi-symmetric metric connection. For example, if a man is moving on the surface of the earth always facing one definite point, say Jaruselam or Mekka or the North pole, then this displacement is semi-symmetric and metric [13].

The study of differentiable manifolds with Lorentizain metric is a natural and interesting topic in differential geometry. In 1996, Ikawa and Erdogan studied Lorentzian Sasakian manifold [20]. Also Lorentzian para contact manifolds were introduced by Matsumoto [24]. Trans Lorentzian para Sasakian manifolds have been used by Gill and Dube [15]. In [41], Yildiz et al. studied Lorentzian α-Sasakian manifold and Lorentzian β-Kenmotsu manifold studied by Funda et al. in [40]. S. S. Pujar and V. J. Khairnar [28] have initiated the study of Lorentzian trans-Sasakian manifolds and studied the some basic results with some of its properties. Earlier to this , S. S. Pujar [29] studied the δ-Lorentzian α-Sasakian manifolds and δ-Lorentzian β-Kenmotsu manifolds.

The study of manifolds with indefinite metrics is of interest from the standpoint of physics and relativity. In 1969, Takahashi [36] has introduced the notion of almost contact metric manifolds equipped with pseudo Riemannian metric. These indefinite almost contact metric manifolds and indefinite Sasakian manifolds are known as (ε)-almost contact metric manifolds. The concept of (ε)-Sasakian manifolds was initiated by Bejancu and Duggal [6] and further investigation was taken up by X. Xufeng and C. Xiaoli [39]. U. C. De and A. Sarkar [11] studied the notion of (ε)-Kenmotsu manifolds with indefinite metric. S. S. Shukla and D. D. Singh [32] extended with indefinite metric which are natural generalization of both (ε)-Sasakian and (ε)-Kenmotsu manifolds called (ε)-trans-Sasakian manifolds. Siddiqi et al. [33] also studied some properties of Indefinite trans-Sasakian manifolds which is closely related to this topic.

The semi Riemannian manifolds has the index 1 and the structure vector field ξ is always a time like. This motivated Thripathi and others [34] to introduced (ε)-almost paracontact structure where the vector filed ξ is space like or time like according as $(\varepsilon) = 1$ or $(\varepsilon) = -1$.

When M has a Lorentzian metric g, that is a symmetric non-degenerate $(0, 2)$ tensor field of index 1, then M is called a Lorentzian manifold. Since the Lorentzian metric is of index 1, Lorentzian manifold M has not only spacelike vector fields but also timelike and lightlike vector fields. This difference with the Riemannian case gives interesting properties on the Lorentzian manifold. A differentiable manifold M has a Lorentzian metric if and only if M has a 1-dimensional distribution. Hence odd dimensional manifold is able to have a Lorentzian metric. Inspired by the above results in 2014, S. M Bhati [8] introduced the notion of δ-Lorentzian trans Sasakian manifolds.

In 1982, R. S. Hamilton [19] said that the Rici solitons move under the Ricci flow simply by diffeomorphisms of the initial metric that is they are sationary points of the Ricci flow is given by

$$\frac{\partial g}{\partial t} = -2Ric(g). \tag{1.1}$$

Definition 1.1. A *Ricci soliton* (g, V, λ) on a Riemannian manifold is defined by

$$L_V g + 2S + 2\lambda = 0, \tag{1.2}$$

where S is the Ricci tensor, L_V is the Lie derivative along the vector field V on M and λ is a real scalar. Ricci soliton is said to be shrinking, steady or expanding according as $\lambda < 0, \lambda = 0$ and $\lambda > 0$, respectively.

In 1925, Levy [22] obtained the necessary and sufficient conditions for the existence of such tensors. later, R. Sharma [30] initiated the study of Ricci solitons in contact Riemannian geometry . After that, Tripathi [35], Nagaraja et al. [25] and others like C. S. Bagewadi et al. [4] extensively studied Ricci soliton. In 2009, J. T. Cho and M. Kimura [9] introduced the notion of η-Ricci solitons and gave a classification of real hypersurfaces in non-flat complex space forms admitting η-Ricci solitons. Later η-Ricci solitons in (ε)-almost paracontact metric manifolds have been studied by A. M. Blaga et al. [3]. A. M. Blaga and various others authors also have been studied η-Ricci solitons in different structures (see [1, 2, 10]). Recently in 2017, K. Venu and G. Nagaraja [38] study the η-Ricci solitons in trans-Sasakian manifold. It is natural and interesting to study η-Ricci soliton in δ-Lorentzian trans-Sasakian manifolds with a semi-symmetric metric connection not as real hypersurfaces of complex space forms but a special contact structures. In this paper we derive the condition for a 3 dimensional δ-Lorentzian Trans-Sasakian manifold with a semi-symmetric metric connection as an η-Ricci soliton and derive expression for the scalar curvature.

Moreover, in this paper we introduced the relation between metric connection and semi-symmetric metric connection in an n-dimensional δ-Lorentzian trans-Sasakian manifolds. Also, we have proved some results on curvature tensor, scalar curvature, quasi projective flat, ϕ-projectively flat, $\bar{R}.\bar{S} = 0$, $\bar{P}.\bar{S} = 0$, Weyl conformally flat, Weyl ξ-conformally flat receptively in n-dimensional δ-Lorentzian trans-Sasakian manifolds with a semi-symmetric metric connection.

2. Preliminaries

Let M be a δ-almost contact metric manifold equipped with δ-almost contact metric structure $(\phi, \xi, \eta, g, \delta)$ [7] consisting of a $(1,1)$ tensor field ϕ, a vector field ξ, a 1-form η and an indefinite metric g such that

$$\phi^2 = X + \eta(X)\xi, \quad \eta \circ \phi = 0, \quad \phi\xi = 0, \tag{2.1}$$

$$\eta(\xi) = -1, \tag{2.2}$$

$$g(\xi, \xi) = -\delta, \tag{2.3}$$

$$\eta(X) = \delta g(X, \xi), \tag{2.4}$$

$$g(\phi X, \phi Y) = g(X, Y) + \delta\eta(X)\eta(Y) \tag{2.5}$$

for all $X, Y \in M$, where δ is such that $\delta^2 = 1$ so that $\delta = \pm 1$. The above structure $(\phi, \xi, \eta, g, \delta)$ on M is called the δ Lorentzian structure on M. If $\delta = 1$ and this is usual Lorentzian structure [8] on M, the vector field ξ is the time like [42], that is M contains a time like vector field.

In [37], Tanno classified the connected almost contact metric manifold. For such a manifold the sectional curvature of the plane section containing ξ is constant, say c. He showed that they can be divided into three classes. (1) homogeneous normal contact Riemannian manifolds with $c > 0$. Other two classes can be seen in Tanno [37].

In Grey and Harvella [14] classification of almost Hermitian manifolds, there appears a class W_4 of Hermitian manifolds which are closely related to the conformal Kaehler manifolds. The class $C_6 \oplus C_5$ [26] coincides with the class of trans-Sasakian structure of type (α, β). In fact, the local nature of the two sub classes, namely C_6 and C_5 of trans-Sasakian structures are characterized completely. An almost conatct metric structure [43] on M is called a trans-Sasakian (see [12, 23, 26]) if $(M \times R, J, G)$ belongs to the class W_4, where J is the almost complex structure on $M \times R$ defined by

$$J\left(X, f\frac{d}{dt}\right) = \left(\phi(X) - f\xi, \eta(X)\frac{d}{dt}\right)$$

for all vector fields X on M and smooth functions f on $M \times R$ and G is the product metric on $M \times R$. This may be expressed by the condition

$$(\nabla_X \phi)Y = \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X) \tag{2.6}$$

for any vector fields X and Y on M, ∇ denotes the Levi-Civita connection with respect to g, α and β are smooth functions on M. The existence of condition (2.3) is ensure by the above discussion.

With the above literature, we define the δ-Lorentzian trans-Sasakian manifolds [8] as follows:

Definition 2.1. A δ-Lorentzian manifold with structure $(\phi, \xi, \eta, g, \delta)$ is said to be *δ-Lorentzian trans-Sasakian manifold* of type (α, β) if it satisfies the condition

$$(\nabla_X \phi)Y = \alpha(g(X,Y)\xi - \delta\eta(Y)X) + \beta(g(\phi X, Y)\xi - \delta\eta(Y)\phi X) \tag{2.7}$$

for any vector fields X and Y on M.

If $\delta = 1$, then the δ-Lorentzian trans Sasakian manifold is the usual Lorentzian trans Sasakian manifold of type (α, β) [26]. δ-Lorentzian trans Sasakian manifold of type $(0,0)$, $(0,\beta)$ $(\alpha, 0)$ are the Lorentzian cosymplectic, Lorentzian β-Kenmotsu and Lorentzian α-Sasakian manifolds respectively. In particular if $\alpha = 1$, $\beta = 0$ and $\alpha = 0$, $\beta = 1$, the δ-Lorentzian trans Sasakian manifolds reduces to δ-Lorentzian Sasakian and δ-Lorentzian Kenmotsu manifolds respectively [21].

Form (2.4), we have

$$\nabla_X \xi = \delta\left\{-\alpha\phi(X) - \beta(X + \eta(X)\xi\right\}, \tag{2.8}$$

and

$$(\nabla_X \eta)Y = \alpha g(\phi X, Y) + \beta[g(X,Y) + \delta\eta(X)\eta(Y)]. \tag{2.9}$$

In a δ-Lorentzian trans Sasakian manifold M, we have the following relations:

$$\begin{aligned} R(X,Y)\xi &= (\alpha^2+\beta^2)[\eta(Y)X - \eta(X)Y] + 2\alpha\beta[\eta(Y)\phi X - \eta(X)\phi Y] \\ &\quad + \delta[(Y\alpha)\phi X - (X\alpha)\phi Y + (Y\beta)\phi^2 X - (X\beta)\phi^2 Y], \end{aligned} \tag{2.10}$$

$$\begin{aligned} R(\xi,Y)X &= (\alpha^2+\beta^2)[\delta g(X,Y)\xi - \eta(X)Y] \\ &\quad +\delta(X\alpha)\phi Y + \delta g(\phi X, Y)(grad\alpha) \\ &\quad +\delta(X\beta)(Y + \eta(Y)\xi) - \delta g(\phi Y, \phi X))(grad\beta) \\ &\quad +2\alpha\beta[\delta g(\phi X, Y)\xi + \eta(X)\phi Y], \end{aligned} \tag{2.11}$$

$$\begin{aligned} \eta(R(X,Y)Z) &= \delta(\alpha^2+\beta^2)[\eta(X)g(Y,Z) - \eta(Y)g(X,Z) \\ &\quad +2\delta\alpha\beta[-\eta(X)g(\phi Y, Z) + \eta(Y)g(\phi X, Z)] \\ &\quad -[(Y\alpha)g(\phi X, Z) + (X\alpha)g(Y, \phi Z)] \\ &\quad -(Y\beta)g(\phi^2 X, Z) + (X\beta)g(\phi^2 Y, Z)], \end{aligned} \tag{2.12}$$

$$S(X,\xi) = [((n-1)(\alpha^2+\beta^2) - (\xi\beta)]\eta(X) + \delta((\phi X)\alpha) + (n-2)\delta(X\beta), \tag{2.13}$$

$$S(\xi,\xi) = (n-1)(\alpha^2+\beta^2) - \delta(n-1)(\xi\beta), \tag{2.14}$$

$$Q\xi = (\delta(n-1)(\alpha^2+\beta^2) - (\xi\beta))\xi + \delta\phi(grad\alpha) - \delta(n-2)(grad\beta), \tag{2.15}$$

where R is curvature tensor, while Q is the Ricci operator given by $S(X,Y) = g(QX,Y)$.
Further in an δ-Lorentzian trans-Sasakian manifold , we have

$$\delta\phi(grad\alpha) = \delta(n-2)(grad\beta), \tag{2.16}$$

and

$$2\alpha\beta - \delta(\xi\alpha) = 0. \tag{2.17}$$

The ξ-sectional curvature K_ξ of M is the sectional curvature of the plane spanned by ξ and a unit vector field X. From (2.11), we have

$$K_\xi = g(R(\xi,X),\xi,X) = (\alpha^2+\beta^2) - \delta(\xi\beta). \tag{2.18}$$

It follows from (2.17) that ξ-sectional curvature does not depend on X. From (2.11)

$$\begin{aligned} g(R(\xi,Y)Z,\xi) &= [(\alpha^2+\beta^2) - \delta(\xi\beta)]g(Y,Z) \\ &+[(\xi\beta) - \delta(\alpha^2+\beta^2)]\eta(Y)\eta(Z) + [2\alpha\beta + \delta(\delta\alpha)]g(\phi Y,Z), \end{aligned} \tag{2.19}$$

$$\begin{aligned} C(X,Y)Z = R(X,Y)Z - \frac{1}{(n-2)}[S(Y,Z)X - S(X,Z)Y \\ +g(Y,Z)QX - g(X,Z)QY] + \frac{r}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y]. \end{aligned} \tag{2.20}$$

An affine connection $\bar{\nabla}$ in M is called semi-symmetric connection [13], if its torsion tensor satisfies the following relations

$$\bar{T}(X,Y) = \bar{\nabla}_X Y - \bar{\nabla}_Y X - [X,Y], \tag{2.21}$$

and

$$\bar{T}(X,Y) = \eta(X)Y - \eta(Y)X. \tag{2.22}$$

Moreover, a semi-symmetric connection is called semi-symmetric metric connection if

$$\bar{g}(X,Y) = 0. \tag{2.23}$$

If ∇ is metric connection and $\bar{\nabla}$ is the semi-symmetric metric connection with non-vanishing torsion tensor T in M, then we have

$$T(X,Y) = \eta(Y)X - \eta(X)Y, \tag{2.24}$$

$$\bar{\nabla}_X Y - \nabla_X Y = \frac{1}{2}[T(X,Y) + T^{'}(X,Y) + T^{'}(X,Y)], \tag{2.25}$$

where

$$g(T(Z,X),Y) = g(T^{'}(X,Y),Z). \tag{2.26}$$

By using (2.4), (2.23) and (2.25), we get

$$g(T^{'}(X,Y),Z) = g(\eta(X)Z - \eta(Z)X, Y),$$

$$g(T^{'}(X,Y),Z) = \eta(X)g(Z,Y) - \delta g(X,Y)g(\xi,Z),$$

$$T^{'}(X,Y) = \eta(X)Y - \delta g(X,Y)\xi, \tag{2.27}$$

$$T^{'}(Y,X) = \eta(Y)X - \delta g(X,Y)\xi. \tag{2.28}$$

From (2.23), (2.24),(2.26) and (2.27), we get

$$\bar{\nabla}_X Y = \nabla_X Y + \eta(Y)X - \delta g(X,Y)\xi.$$

Let M be an n-dimensional δ-Lorentzian trans-Sasakian manifold and ∇ be the metric connection on M. The relation between the semi-symmetric metric connection $\bar{\nabla}$ and the metric connection ∇ on M is given by

$$\bar{\nabla}_X Y = \nabla_X Y + \eta(Y)X - \delta g(X,Y)\xi. \tag{2.29}$$

3. Curvature Tensor on δ-Lorentzian Trans-Sasakian Manifold with a Semi-symmetric Metric Connection

Let M be an n-dimensional δ-Lorentzian trans-Sasakian manifold. The curvature tensor $\bar{R}$ of M with respect to the semi-symmetric metric connection $\bar{\nabla}$ is defined by

$$\bar{R}(X,Y)Z = \bar{\nabla}_X \bar{\nabla}_Y Z - \bar{\nabla}_Y \bar{\nabla}_X Z - \bar{\nabla}_{[X,Y]} Z. \tag{3.1}$$

By using (2.4), (2.28) and (3.1), we get

$$\begin{aligned} \bar{R}(X,Y)Z = {} & R(X,Y)Z + (\delta)[g(X,Z)Y - g(Y,Z)X] \\ & +(\beta+\delta)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)]\xi \\ & -(\beta\delta - 1)[\eta(Y)X - \eta(X)Y]\eta(Z), \\ & +\alpha[g(\phi X,Z)Y - g(\phi Y,Z)\phi X - g(X,Z)\phi Y + g(Y,Z)\phi X], \end{aligned} \tag{3.2}$$

where

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

is the Riemannian curvature tensor of connection ∇.

Lemma 3.1. *Let M be an n-dimensional δ-Lorentzian trans-Sasakian manifold with a semi-symmetric metric connection, then*

$$(\bar{\nabla}_X\phi)(Y) = \alpha(g(\phi X, Y)\xi - \delta\eta(Y)X) + \beta(g(\phi X, Y)\xi - (\delta\beta + \delta)\eta(Y)\phi X), \tag{3.3}$$

$$\bar{\nabla}_X\xi = -(1+\delta\beta)X - (1+\delta\beta)\eta(X)\xi - \delta\alpha\phi X, \tag{3.4}$$

$$(\bar{\nabla}_X\eta)Y = \alpha g(\phi X, Y) + (\beta+\delta)g(X,Y) - (1+\beta\delta)\eta(X)\eta(Y). \tag{3.5}$$

Proof. By the covariant differentiation of ϕY with respect to X, we have

$$\bar{\nabla}_X\phi Y = (\bar{\nabla}_X\phi) + \phi(\bar{\nabla}_X Y).$$

By using (2.1) and (2.28), we have

$$(\bar{\nabla}_X\phi)Y = (\nabla_X\phi)Y - \eta(Y)\phi X.$$

In view of (2.7), the last equation gives

$$(\bar{\nabla}_X\phi)(Y) = \alpha(g(\phi X, Y)\xi - \delta\eta(Y)X) + \beta(g(\phi X, Y)\xi - (\delta\beta + \delta)\eta(Y)\phi X).$$

To prove (3.4), we replace $Y = \xi$ in (2.28) and we have

$$\bar{\nabla}_X\xi = \nabla_X\xi + \eta(\xi)X - \delta g(X,\xi)\xi.$$

By using (2.2), (2.4) and (2.8), the above equation gives

$$\bar{\nabla}_X\xi = -(1+\delta\beta)X - (1+\delta\beta)\eta(X)\xi - \delta\alpha\phi X.$$

In order to prove (3.5), we differentiate $\eta(Y)$ covariantly with respect to X and using (2.28), we have

$$\bar{\nabla}_X\eta(Y) = (\nabla_X\eta)Y + g(X,Y) - \eta(X)\eta(Y).$$

Using (2.9) in above equation, we get

$$(\bar{\nabla}_X\eta)Y = \alpha g(\phi X, Y) + (\beta+\delta)g(X,Y) - (1+\beta\delta)\eta(X)\eta(Y). \qquad \square$$

Lemma 3.2. *Let M be an n-dimensional δ-Lorentzian trans-Sasakian manifold with a semi-symmetric metric connection, then*

$$\bar{R}(X,Y)\xi = (\alpha^2 + \beta^2 - \delta\beta)[\eta(X)Y - \eta(Y)X]. \\ + (2\alpha\beta + \delta\alpha)[\eta(Y)\phi X - \eta(X)\phi Y] \tag{3.6}$$

$$+\delta[(Y\alpha)\phi X - (-X\alpha)\phi Y - (X\beta)\phi^2 Y + (Y\beta)\phi^2 X].$$

Proof. By replacing $Z = \xi$ in (3.2), we have

$$\begin{aligned}\bar{R}(X,Y)\xi = R(X,Y)\xi + (\delta)[g(X,\xi)Y - g(Y,\xi)X] \\ +(\beta+\delta)[g(Y,\xi)\eta(X) - g(X,\xi)\eta(Y)]\xi \\ -(\beta\delta-1)[\eta(Y)X - \eta(X)Y]\eta(\xi) \\ +\alpha[g(\phi X,\xi)Y - g(\phi Y,\xi)\phi X - g(X,\xi)\phi Y + g(Y,\xi)\phi X]\end{aligned}$$

In view of (2.2), (2.4) and (2.10), the above equation reduces to

$$\begin{aligned}\bar{R}(X,Y)\xi = (\alpha^2+\beta^2-\delta\beta)[\eta(X)Y - \eta(Y)X] \\ +(2\alpha\beta+\delta\alpha)[\eta(Y)\phi X - \eta(X)\phi Y] \\ +\delta[(Y\alpha)\phi X - (X\alpha)\phi Y - (X\beta)\phi^2 Y + (Y\beta)\phi^2 X].\end{aligned}$$ □

Remark 3.1. Replace $Y = \xi$ and using (3.2), (2.11), (2.2) and (2.4), we obtain

$$\begin{aligned}\bar{R}(X,\xi)\xi = (\alpha^2+\beta^2-\delta\beta)[-X - \eta(X)Y] \\ +(2\alpha\beta+\delta\alpha+\delta(\xi\alpha))[\phi X + \delta(\xi\beta)\phi^2 X].\end{aligned} \tag{3.7}$$

Remark 3.2. Now, again replace $X = \xi$ in (3.6), using (2.1), (2.2) and (2.4), we obtain

$$\begin{aligned}\bar{R}(\xi,Y)\xi = (\alpha^2+\beta^2-\delta\beta)[-\eta(Y)\xi - Y] \\ -(2\alpha\beta+\delta\alpha+\delta(\xi\alpha))[\phi Y - \delta(\xi\beta)\phi^2 Y].\end{aligned} \tag{3.8}$$

Remark 3.3. Replace $Y = X$ in (3.8), we get

$$\begin{aligned}\bar{R}(\xi,X)\xi = -(\alpha^2+\beta^2-\delta\beta)[-X - \eta(X)\xi] \\ -(2\alpha\beta+\delta\alpha+\delta(\xi\alpha))[\phi X - \delta(\xi\beta)\phi^2 X].\end{aligned} \tag{3.9}$$

From (3.7) and (3.9), we obtain

$$\bar{R}(X,\xi)\xi = -\bar{R}(\xi,X)\xi. \tag{3.10}$$

Now, contracting X in (3.2), we get

$$\begin{aligned}\bar{S}(Y,Z) = S(Y,Z) - [(\delta)(n-2)+\beta]g(Y,Z) \\ -(\beta\delta-1)(n-2)\eta(Z)\eta(Y) - \alpha(n-2)g(\phi Y,Z),\end{aligned} \tag{3.11}$$

where $\bar{S}$ and S are the Ricci tensors of the connections $\bar{\nabla}$ and ∇, respectively on M.
This gives

$$\bar{Q}Y = QY - [(\delta)(n-2) + \beta]Y \\ -(\beta\delta - 1)(n-2)\eta(Y)\xi - \alpha(n-2)\phi Y, \tag{3.12}$$

where $\bar{Q}$ and Q are Ricci operator with respect to the semi-symmetric metric connection and metric connection respectively and define as $g(\bar{Q}Y, Z) = \bar{S}(Y, Z)$ and $g(QY, Z) = S(Y, Z)$ respectively.

Replace $Y = \xi$ in (3.12) and using (2.15), we get

$$\bar{Q}\xi = \delta(n-1)(\alpha^2 + \beta^2)\xi - (\xi\beta)\xi - 2\delta(n-2)\xi \\ +\delta\phi(grad\alpha) - \delta(n-2)(grad\beta) - \beta(n-1)\xi. \tag{3.13}$$

Putting $Y = Z = e_i$ and taking summation over i, $1 \leq i \leq n-1$ in (3.11), using (2.14) and also the relations $r = S(e_i, e_i) = \sum_{i=1}^{n} \delta_i R(e_i, e_i, e_i, e_i)$, we get

$$\bar{r} = r - (n-1)[(\delta)(n-2) + 2\beta], \tag{3.14}$$

where $\bar{r}$ and r are the scalar curvatures of the connections $\bar{\nabla}$ and ∇, respectively on M.

Now, we have the following lemmas.

Lemma 3.3. *Let M be an n-dimensional δ-Lorentzian trans-Sasakian manifold with a semi-symmetric metric connection, then*

$$\bar{S}(\phi Y, Z) = -\delta(\phi^2 Y)\alpha - \delta(n-2)(\phi Y)\beta - \alpha(n-2)g(\phi Y, \phi Z), \tag{3.15}$$

$$\bar{S}(Y, \xi) = [(n-1)(\alpha^2 + \beta^2 - \delta(\xi\beta) - \delta\beta(n-1)]\eta(Y) \\ +\delta(n-2)(Y\beta) + \delta(\phi Y)\beta, \tag{3.16}$$

$$\bar{S}(\xi, \xi) = [(n-1)(\alpha^2 + \beta^2 - \delta(\xi\beta) - \delta\beta(n-1)]\eta(Y). \tag{3.17}$$

Proof. By replacing $Y = \phi Y$ in equation (3.11) and using (2.13) and (2.5), we have (3.15). Taking $Y = \xi$ in (3.11) and using (2.13) we get (3.16). (3.17) follows from considering $Y = \xi$ in (3.16) we get (3.17). □

Lemma 3.4. *Let M be an n-dimensional δ-Lorentzian trans-Sasakian manifold with a semi-symmetric metric connection, then*

$$\bar{S}(grad\alpha, \xi) = \delta(n-1)(\alpha^2 + \beta^2(\xi\beta) - \beta(n-1)(\xi\alpha) - (\xi\alpha)(\xi\beta) \\ +\delta(\phi grad\alpha)\alpha + \delta(n-2)g(grad\alpha, grad\beta), \tag{3.18}$$

$$\bar{S}(grad\beta, \xi) = \delta(n-1)(\alpha^2 + \beta^2(\xi\beta) - \beta(n-1)(\xi\beta) - (\xi\beta)^2 + \delta(\phi grad\beta)\alpha + \delta(n-2)g(grad\beta)^2. \tag{3.19}$$

Proof. From equation (3.11) and (3.16) and using $Y = grad\alpha$ we have (3.18) . Similarly taking $\xi = grad\beta$ in (3.11) and using (3.16), we get (3.19). Using (3.6), (3.13) and (3.16), for constant α and β, we have

$$\bar{R}(X,Y)\xi = (\alpha^2 + \beta^2 - \delta(\xi\beta)[\eta(Y)X - \eta(X)Y], \tag{3.20}$$

$$\bar{S}(X,Y) = [(n-1)(\alpha^2 + \beta^2 - \delta(\xi\beta) - \delta\beta(n-1)]\eta(Y), \tag{3.21}$$

$$\bar{Q}\xi = \delta(n-1)(\alpha^2 + \beta^2\xi - \delta(\xi\beta)\xi - 2\delta(n-2) - \beta(n-1)\xi. \tag{3.22}$$

□

4. Quasi-projectively flat δ-Lorentzian trans-Sasakian Manifold with a Semi-symmetric Metric Connection

Let M be an n-dimensional δ-Lorentzian trans-Sasakian manifold. If there exists a one to one correspondence between each co-ordinate neighborhood of M and a domain in Euclidean space such that any geodesic of δ-Lorentzian trans-Sasakian manifold corresponds to a straight line in the Euclidean space, then M is said to be locally projectively flat. The projective curvature tensor $\bar{P}$ with respect to semi-symmetric metric connection is defined by

$$\bar{P}(X,Y)Z = \bar{R}(X,Y)Z - \frac{1}{(n-1)}[\bar{S}(Y,Z)X - \bar{S}(X,Z)Y]. \tag{4.1}$$

Definition 4.1. A δ-Lorentzian trans-Sasakian manifold M is said to be *quasi-projectively flat* with respect to semi-symmetric metric connection, if

$$g(\bar{P}(\phi X, Y)Z, \phi U) = 0, \tag{4.2}$$

where $\bar{P}$ is the projective curvature tensor with respect to semi-symmetric metric connection.

Now, from (4.1) taking inner product with U, we get

$$g(\bar{P}(X,Y)Z, U) = g(\bar{R}(X,Y)Z, U) - \frac{1}{(n-1)} [\bar{S}(Y,Z)g(X,U) - \bar{S}(X,Z)g(Y,U)]. \tag{4.3}$$

Replace $X = \phi X$ and $U = \phi U$ in (4.3), we get

$$g(\bar{P}(\phi X, Y)Z, \phi U) = g(\bar{R}(\phi X, Y)Z, \phi U) - \frac{1}{(n-1)} \tag{4.4}$$

$$[\bar{S}(Y,Z)g(\phi X,\phi U) - \bar{S}(\phi X,Z)g(Y,\phi U)].$$

From (4.2) and (4.4), we have

$$(4.5)\quad g(\bar{R}(\phi X,Y)Z,\phi U) = \frac{1}{(n-1)}[\bar{S}(Y,Z)g(\phi X,\phi U) - \bar{S}(\phi X,Z)g(Y,\phi U)].$$

Now, using equations (2.1), (2.4), (3.11) and (3.15) in equation (4.5), we have

$$\begin{aligned}(4.6)\quad g(\bar{R}(\phi X,Y)Z,\phi U) &= \frac{1}{(n-1)}[\bar{S}(Y,Z)g(\phi X,\phi U) - \bar{S}(\phi X,Z)g(Y,\phi U)]\\ &-\frac{(\delta+\beta)}{(n-1)}g(\phi X,Z)g(Y,\phi U) + \frac{(\delta+\beta)}{(n-1)}g(Y,Z)g(\phi X,\phi U)\\ &-\frac{(\delta\beta-1)}{(n-1)}\eta(Y)\eta(Z)g(\phi X,\phi U) + \frac{(\delta\alpha)}{(n-1)}\eta(X)\eta(Z)g(\phi X,\phi U)\\ &-\frac{\alpha}{(n-1)}g(X,Z)g(Y,\phi U) - \frac{\alpha}{(n-1)}g(\phi Y,Z)g(\phi X,\phi U)\\ &+\alpha g(Y,Z)g(X,\phi U) + \alpha g(\phi X,Z)g(\phi X,\phi U).\end{aligned}$$

Let $\{e_1, e_2 ....e_{n-1}, \xi\}$ be a local orthonormal basis of vector fileds on δ-Lorentzian trans-Sasakian manifold M, then $\{\phi e_1, \phi e_2 ....\phi e_{n-1}, \xi\}$ is also a local orthonormal basis of vector fields on δ-Lorentzian trans-Sasakian manifold M. Now, putting $X = U = e_i$ in equation (4.6) and using (2.2), (2.3),(2.19), (3.11) and (3.16), we have

$$\begin{aligned}(4.7)\qquad S(Y,Z) &= [(n-2)(\beta+\delta) + \delta(n-1)(\alpha^2+\beta^2) - (n-1)(\xi\beta)]g(Y,Z)\\ &+[\delta(n-2)(\xi\beta) + (n-2)(\beta\delta-1)]\eta(Y)\eta(Z)\\ &-[2\delta(n-1)\alpha\beta + (n-1)(\xi\alpha) - \alpha]g(\phi Y,Z)\\ &-\delta\eta(Y)(\phi Z)\alpha - \delta(n-2)(\xi\beta)\eta(Y).\end{aligned}$$

If $\alpha = 0$ and $\beta =$ constant in (4.7), we get

$$(4.8)\quad S(Y,Z) = [(n-2)(\beta+\delta) + (n-1)\delta\beta^2]g(Y,Z) + (\beta\delta-1)(2-n)\eta(Y)\eta(Z).$$

Therefore, we have

$$S(Y,Z) = ag(Y,Z) + b\eta(Y)\eta(Z),$$

where $a = (n-2)(\beta+\delta) + (n-1)\delta\beta^2$ and $b = (\beta\delta-1)(2-n)$.
These results shows that the manifold under the consideration is an η-Einstein manifold. Thus we can state the following theorem:

Theorem 4.1. *An n-dimensional quasi projectively flat δ-Lorentzian trans-Sasakian manifold M with respect to a semi-symmetric metric connection is an η-Einstein manifold if $\alpha = 0$ and $\beta = constant$.*

5. ϕ-Projectively flat δ-Lorentzian Trans-Sasakian Manifold with a Semi-symmetric Metric Connection

An n-dimensional δ-Lorentzian trans-Sasakian manifold with a semi-symmetric metric connection is said to be ϕ-projectively flat if

$$\phi^2(\bar{P}(\phi X, \phi Y)\phi Z) = 0, \tag{5.1}$$

where $\bar{P}$ is the projective curvature tensor of M n-dimensional δ-Lorentzian trans-Sasakian manifold with respect to a semi-symmetric metric connection. Suppose M be ϕ-projectively flat δ-Lorentzian trans-Sasakian manifold with respect to a semi-symmetric metric connection. It is know that $\phi^2(\bar{P}(\phi, X, \phi Y)\phi Z) = 0$ holds if and only if

$$g(\bar{P}(\phi X, \phi Y)\phi Z, \phi U) = 0, \tag{5.2}$$

for any $X, Y, Z, U \in TM$. Replace $Y = \phi Y$ and $U\phi U$ in (4.4), we have

$$g(\bar{P}(\phi X, \phi Y)\phi Z, \phi U) = g(\bar{R}(\phi X, \phi Y)\phi Z, \phi U) - \frac{1}{(n-1)} [\bar{S}(\phi Y, \phi Z)g(\phi X, \phi U) - \bar{S}(\phi X, \phi Z)g(\phi Y, \phi U)]. \tag{5.3}$$

From (5.2) and (5.3), we have

$$g(\bar{R}(\phi X, \phi Y)\phi Z, \phi U) = \frac{1}{(n-1)}[\bar{S}(\phi Y, \phi Z)g(\phi X, \phi U) - \bar{S}(\phi X, \phi Z)g(\phi Y, \phi U)]. \tag{5.4}$$

Now, using (2.1),(2.2),(2.4),(2.5), (3.2) and (3.11) in equation (5.4), we have

$$\begin{aligned} g(\bar{R}(\phi X, \phi Y)\phi Z, \phi U) &= \frac{1}{(n-1)}[\bar{S}(\phi Y, \phi Z)g(\phi X, \phi U) - \bar{S}(\phi X, \phi Z)g(\phi Y, \phi U)] \\ &- \frac{(\delta+\beta)}{(n-1)}g(\phi Y, \phi Z)g(\phi X, \phi U) + \frac{(\delta+\beta)}{(n-1)}g(\phi X, \phi Z)g(\phi Y, \phi U) \\ &- \frac{\alpha}{(n-1)}g(Y, \phi Z)g(\phi X, \phi U) - \frac{\alpha}{(n-1)}g(X, \phi Y Z)g(\phi X, \phi U) \\ &+ \alpha g(\phi Y, \phi Z)g(X, \phi U) - \alpha g(\phi X, \phi Z)g(Y, \phi U). \end{aligned} \tag{5.5}$$

Let $\{e_1, e_2 ....e_{n-1}, \xi\}$ be a local orthonormal basis of vector fileds on δ-Lorentzian trans-Sasakian manifold M, then $\{\phi e_1, \phi e_2 ....\phi e_{n-1}, \xi\}$ is also a local orthonormal basis of vector fields on δ-Lorentzian trans-Sasakian manifold M. Now putting

$X = U = e_i$ in equation (5.5) and using (2.1)–(2.5), (2.19), (3.11) and (3.16), we have

$$
\begin{aligned}
(5.6) \quad S(Y,Z) = {} & [(n-2)(\beta+\delta) + \delta(n-1)(\alpha^2+\beta^2) - (n-1)(\xi\beta)]g(Y,Z) \\
& + [2\delta(n-2)(\xi\beta) + (n-2)(\beta\delta-1)]\eta(Y)\eta(Z) \\
& + [\alpha - 2\delta\alpha\beta(n-1) - (n-1)(\xi\alpha)]g(\phi Y,Z) \\
& - [\delta(\phi Z)\alpha + \delta(n-2)(Z\beta)]\eta(Y) - [\delta(\phi Y)\alpha + \delta(n-2)(Y\beta)]\eta(Z)
\end{aligned}
$$

If $\alpha = 0$ and $\beta = constant$ in (5.6), we get

$$
(5.7) \quad S(Y,Z) = [(n-2)(\beta+\delta) + (n-1)\delta\beta^2]g(Y,Z) + (\beta\delta-1)(2-n)\eta(Y)\eta(Z).
$$

Therefore,

$$
S(Y,Z) = ag(Y,Z) + b\eta(Y)\eta(Z),
$$

where $a = (n-2)(\beta+\delta) + (n-1)\delta\beta^2$ and $b = (\beta\delta-1)(2-n)$.
This result shows that the manifold under the consideration is an η-Einstein manifold. Thus we can state the following theorem:

Theorem 5.1. *An n-dimensional ϕ-projectively flat δ-Lorentzian trans-Sasakian manifold M with respect to a semi-symmetric metric connection is an η-Einstein manifold if $\alpha = 0$ and $\beta = constant$.*

6. δ-Lorentzian trans-Sasakian Manifold with a Semi-symmetric Metric Connection satisfying $\bar{R}.\bar{S} = 0$

Now, suppose that M be an n-dimensional δ-Lorentzian trans-Sasakian manifold with a semi-symmetric metric connection satisfying the condition:

$$
(6.1) \qquad \bar{R}(X,Y).\bar{S} = 0.
$$

Then, we have

$$
(6.2) \qquad \bar{S}(\bar{R}(X,Y)Z,U) + \bar{S}(Z,\bar{R}(X,Y)U) = 0.
$$

Now, we replace $X = \xi$ in equation (6.2), using equations (2.11) and (6.2), we have

(6.3)

$$
\begin{aligned}
& \delta(\alpha^2+\beta^2)g(Y,Z)\bar{S}(\xi,U) - (\alpha^2+\beta^2)\eta(Z)\bar{S}(Y,U) - 2\delta\alpha\beta g(\phi Y,Z)\bar{S}(\xi,U) \\
& + 2\alpha\beta\eta(Z)\bar{S}(\phi Y,U) + \delta(Z\alpha)\bar{S}(\phi Y,U) - \delta g(\phi Y,Z)\bar{S}(grad\alpha,U) \\
& - \delta g(\phi Y,\phi Z)\bar{S}(grad\beta,U) + \delta(Z\beta)\bar{S}(Y,U) - \delta(Z\beta)\eta(Y)\bar{S}(\xi,U) \\
& - \delta g(Y,Z)\bar{S}(\xi,U) + \delta\eta(Z)\bar{S}(Y,U) + \alpha g(\phi Y,Z)\bar{S}(\xi,U) - \delta\alpha\eta(Z)\bar{S}(\phi Y,U) \\
& + \delta(\alpha^2+\beta^2)g(Y,U)\bar{S}(\xi,Z) - (\alpha^2+\beta^2)\eta(U)\bar{S}(Y,Z) - 2\delta\alpha\beta g(\phi Y,U)\bar{S}(\xi,Z)
\end{aligned}
$$

$$
\begin{aligned}
&+ 2\alpha\beta\eta(U)\bar{S}(\phi Y, Z) + \delta(U\alpha)\bar{S}(\phi Y, Z) - \delta g(\phi Y, U)\bar{S}(grad\alpha, Z) \\
&- \delta g(\phi Y, \phi U)\bar{S}(grad\beta, Z) + \delta(U\beta)\bar{S}(Y, Z) - \delta(U\beta)\eta(Y)\bar{S}(\xi, Z) \\
&- \delta g(Y, U)\bar{S}(\xi, Z) + \delta\eta(U)\bar{S}(Y, Z) + \alpha g(\phi Y, U)\bar{S}(\xi, Z) - \delta\alpha\eta(U)\bar{S}(\phi Y, Z) = 0.
\end{aligned}
$$

Using equations (2.1)–(2.5), (2.13), (2.14), (3.11) and (3.15)–(3.19) in equation (6.3)

$$
\begin{aligned}
&[(\alpha^2 + \beta^2) - \delta(\xi\beta) - \delta\beta]S(Y, Z) \\
&= [\delta(n-1)(\alpha^2+\beta^2) - 2\beta(n-1)(\alpha^2+\beta^2) - 2(n-1)(\alpha^2+\beta^2)(\xi\beta) \\
&+ 2\delta\beta(n-1)(\xi\beta) - \delta(\xi\beta)^2 + (\phi grad\beta)\alpha + (n-2)(grad\beta)^2 \\
&+ \delta\beta^2(n-2) + \delta(n-2)(\alpha^2+\beta^2) + \beta(\alpha^2+\beta^2) \\
&- 2\alpha^2\beta(n-2) - \delta\alpha(\xi\alpha) - (n-2)(\xi\beta) - \delta\beta(\xi\beta) \\
&- \beta(n-2) + \delta\alpha^2(n-2)]g(Y, Z) + [-\delta(\phi grad\beta)\alpha \\
&- \delta(n-2)(grad\beta)^2 + (n-2)(\beta\delta - 1)(\alpha^2+\beta^2) \\
&+ 2\delta\alpha^2\beta(n-2) + \alpha(n-2)(\xi\alpha) + (\beta+\delta)(n-2)(\xi\beta) \\
&+ \beta(\beta+\delta)(n-2) - \alpha^2(n-2)]\eta(Y)\eta(Z) + [-2\delta\alpha\beta(n-1)(\alpha^2+\beta^2) \\
&+ 2(n-2)\alpha\beta^2 + 2\alpha\beta(n-2)(\xi\beta) - (n-1)(\alpha^2+\beta^2)(\xi\alpha) \\
&+ \delta\beta(n-2)(\xi\alpha) + \delta(\xi\alpha)(\xi\beta) + (\phi grad\alpha)\alpha + (n-2)(g(grad\alpha, grad\beta) \\
&+ \alpha(\alpha^2+\beta^2) - \delta\alpha(\xi\beta) - 2\alpha\beta(n-2)(\delta) - (n-2)(\delta\alpha) + \alpha(n-2)]g(\phi Y, Z) \\
&+ [\delta(\xi\alpha) + 2\alpha\beta - \delta\alpha]S(\phi Y, Z) + [(n-2)(\xi\beta)(Z\beta) \\
&+ [\delta(\alpha^2+\beta^2)(\phi Z)\alpha - \delta(n-2)(\alpha^2+\beta^2)(Z\beta) + (\xi\beta)(\phi Z)\alpha \\
&\beta(\phi Z)\alpha + \beta(n-2)(Z\beta)]\eta(Y) + [\delta(\alpha^2+\beta^2)(\phi Y)\alpha + \delta(n-2)(\alpha^2+\beta^2)(Y\beta) \\
&- 2\delta\alpha\beta(\phi^2 Y)\alpha - 2\delta\alpha\beta(n-2)(\phi Y\beta) - \beta(\phi Y)\alpha \\
&- \beta(n-2)(Y\beta) + \alpha(\phi^2 Y)\alpha + \alpha(n-2)(\phi Y\beta)]\eta(Z) \\
&- (n-2)(Y\beta)(Z\beta) + (n-2)(Z\beta)(\xi\beta).
\end{aligned}
$$

If $\alpha = 0$ and $\beta =$ constant in (5.6), we get

$$S(Y, Z) = ag(Y, Z) + b\eta(Y)\eta(Z),$$

where $a = -[\frac{(n-1)\delta\beta^4 + (n-2)(grad\beta)^2 + (n-2)\delta\beta^2 + (n-2)\delta\beta^2 - (n-2)\beta + (2n-3)\beta^3}{(\beta+\delta)\beta}]$ and $b = -[\frac{(n-2)(\beta\delta-1)\beta^2 + (n-2)(\beta+\delta)\beta - (n-2)\delta(grad\beta^2}{(\beta+\delta)\beta}]$. This show that M is an η-Einstein manifold. Thus,we can state the following theorem:

Theorem 6.1. *An n-dimensional δ-Lorentzian trans-Sasakian manifold M with respect to a semi-symmetric metric connection $\bar{\nabla}$ satisfying $\bar{R}.\bar{S} = 0$, then δ-Lorentzian trans-Sasakian manifold M is an η-Einstein manifold if $\alpha = 0$ and $\beta = constant$.*

7. δ-Lorentzian Trans-Sasakian Manifold with a Semi-symmetric Metric Connection satisfying $\bar{P}.\bar{S} = 0$

Now, we consider δ-Lorentzian trans-Sasakian manifold with a semi-symmetric metric connection satisfying

$$(\bar{P}(X,Y).\bar{S})(Z,U) = 0, \tag{7.1}$$

where $\bar{P}$ is the projective curvature tensor and $\bar{S}$ is the Ricci tensor with a semi-symmetric metric connection.Then, we have

$$\bar{S}(\bar{P}(X,Y)Z,U) + \bar{S}(Z,\bar{P}(X,Y)U) = 0. \tag{7.2}$$

Replace $X = \xi$ in the equation (7.2), we get

$$\bar{S}(\bar{P}(\xi,Y)Z,U) + \bar{S}(Z,\bar{P}(\xi,Y)U) = 0. \tag{7.3}$$

Putting $X = \xi$ in (4.1), we get

$$\bar{P}(\xi,Y)Z = \bar{R}(\xi,Y)Z - \frac{1}{(n-1)}[\bar{S}(Y,Z)\xi - \bar{S}(\xi,Z)Y]. \tag{7.4}$$

Using (2.1), (2.2), (2.4), (2.11), (3.2), (3.11), (3.17) and (7.4) in (7.3), we get

(7.5)
$$\begin{aligned}
&\frac{\delta(\alpha^2+\beta^2)(n-1)+(\beta+\delta)(n-2)}{(n-1)}g(Y,Z)\bar{S}(\xi,U) - \frac{1}{(n-1)}S(Y,Z)\bar{S}(\xi,U)\\
&-\frac{(n-2)}{(n-1)}(\beta\delta-1)\eta(Y)\eta(Z)\bar{S}(\xi,U) + \frac{\alpha-2\delta\alpha\beta(n-1)}{(n-1)}g(\phi Y,Z)\bar{S}(\xi,U)\\
&-\delta g(\phi Y,Z)\bar{S}(grad\alpha,U) - \delta g(\phi Y,\phi Z)\bar{S}(grad\beta,U) + 2\alpha\beta\eta(Z)\bar{S}(\phi Y,U)\\
&+\delta(Z\alpha)\bar{S}(\phi Y,U) + \delta(Z\beta)\bar{S}(Y,U) - \delta(Z\beta)\eta(Y)\bar{S}(\xi,U) - \delta\alpha\eta(Z)\bar{S}(\phi Y,U)\\
&-\frac{1}{(n-1)}\delta(\xi\beta)\eta(Z)\bar{S}(Y,U)\frac{(n-2)}{(n-1)}\delta(Z\beta)\bar{S}(Y,U) - \frac{1}{(n-1)}\delta(\phi Z)\alpha\bar{S}(Y,U)\\
&\frac{\delta(\alpha^2+\beta^2)(n-1)+(\beta+\delta)(n-2)}{(n-1)}g(Y,U)\bar{S}(\xi,Z) - \frac{1}{(n-1)}S(Y,U)\bar{S}(\xi,Z)\\
&-\frac{(n-2)}{(n-1)}(\beta\delta-1)\eta(Y)\eta(U)\bar{S}(\xi,Z) + \frac{\alpha-2\delta\alpha\beta(n-1)}{(n-1)}g(\phi Y,U)\bar{S}(\xi,Z)\\
&-\delta g(\phi Y,U)\bar{S}(grad\alpha,Z) - \delta g(\phi Y,\phi U)\bar{S}(grad\beta,Z) + 2\alpha\beta\eta(U)\bar{S}(\phi Y,Z)\\
&+\delta(U\alpha)\bar{S}(\phi Y,Z) + \delta(Z\beta)\bar{S}(Y,Z) - \delta(U\beta)\eta(Y)\bar{S}(\xi,Z) - \delta\alpha\eta(U)\bar{S}(\phi Y,Z)\\
&-\frac{1}{(n-1)}\delta(\xi\beta)\eta(Z)\bar{S}(Y,Z)\frac{(n-2)}{(n-1)}\delta(U\beta)\bar{S}(Y,Z) - \frac{1}{(n-1)}\delta(\phi U)\alpha\bar{S}(Y,Z) = 0
\end{aligned}$$

Putting $U = \xi$ and Using (2.1)–(2.5), (3.11)and (3.15)–(3.20) in (7.5), we get

(7.6)
$$\begin{aligned}
&[(\alpha^2+\beta^2)-\delta(\xi\beta)-\delta\beta]S(Y,Z)\\
&=[\delta(n-1)(\alpha^2+\beta^2)+(n-2)(\beta\delta)(\alpha^2+\beta^2)-\beta(n-1)(\alpha^2+\beta^2)\\
&-\delta(n-2)(\beta\delta-1)-2(n-1)(\xi\beta)(\alpha^2+\beta^2)-(n-2)(\beta\delta-1)(\xi\beta)\\
&-2\alpha^2\beta(n-2)\delta\alpha(n-2)(\xi\alpha)+\delta\alpha^2(n-2)+\delta\beta(n-1)+\delta(\xi\beta)^2\\
&+(\phi grad\alpha)\alpha+(n-2)(grad\beta)^2]g(Y,Z)+[(n-2)\beta(\beta+\delta)-(n-2)(\alpha^2+\beta^2)\\
&+2(n-2)\delta\alpha^2\beta+\alpha(n-2)(\xi\alpha)+(n-2)(\beta+\delta)(\xi\beta)-\alpha^2(n-2)\\
&-\delta(n-2)(grad\beta)^2-\delta(\phi grad\beta)\alpha]\eta(Y)\eta(Z)+[\alpha(\alpha^2+\beta^2)\\
&-2\delta\alpha\beta(\alpha^2+\beta^2)(n-1)-2\alpha\beta^2n-\delta(\xi\beta)-\delta\beta(\xi\alpha)+2\alpha\beta(\xi\beta)\\
&-2\delta\alpha\beta(n-2)-(n-1)(\xi\alpha)+\alpha(n-2)-(n-1)(\alpha^2+\beta^2)(\xi\alpha)+(n-1)\delta\beta(\xi\alpha)\\
&+\delta(\xi\alpha)(\xi\beta)+(\phi grad\alpha)\alpha+)n-2)g(grad\alpha,grad\beta)]g(\phi Y,z)+[\delta\alpha+\delta(\xi\alpha)\\
&-\delta\alpha]S(\phi Y,Z)+[\delta(n+3)(\alpha^2+\beta^2)(Z\beta)+\beta(n-2)(Z\beta)-delta(\alpha^2+\beta^2)(\phi Z)\alpha\\
&+(n-1)\beta(\phi Z)\alpha+(\xi\beta)(\phi Z)\alpha)]\eta(Y)+[-2\delta\alpha\beta(\phi^2Y)\alpha-2\delta\alpha\beta(n-2)(\phi Y\beta)\\
&+\alpha(\phi^2Y)\alpha+\alpha(n-2)(\phi Y\beta)+\delta(\alpha^2+\beta^2)(\phi Y)\alpha+\delta(n-2)(\alpha^2+\beta^2)(Y\beta)\\
&-\beta(\phi Y)\alpha-\beta(n-2)(Y\beta)]\eta(Z)-(Z\alpha)(\phi^2Y)\alpha-(n-2)(Z\beta)(\phi Y\beta)\\
&-(Z\beta)(\phi Y)\alpha-\beta(n-2)(Y\beta).
\end{aligned}$$

If $\alpha = 0$ and $\beta = constant$ in (7.6), we get

$$S(Y,Z) = ag(Y,Z) + b\eta(Y)\eta(Z), \tag{7.7}$$

where $a = -[\frac{(n-1)\beta^4+(n-2)\beta^2(\beta\delta)+(n-1)\beta^3-(n-2)\beta(\beta\delta-1)+(n-1)\delta\beta+(n-2)(grad\beta)^2}{\beta(\beta\delta)}]$
and
$b = -[\frac{(n-2)\beta(\beta+\delta)+(n-2)\beta^2-(n-2)\delta(grad\beta)^2}{\beta(\beta+\delta)}]$.

This result show that the manifold under the consideration is an η-Einstein manifold. Thus we have the following theorem:

Theorem 7.1. *An n-dimensional δ-Lorentzian trans-Sasakian manifold M with respect to a semi-symmetric metric connection $\bar{\nabla}$ satisfying $\bar{P}.\bar{S} = 0$, then δ-Lorentzian trans-Sasakian manifold M is an η-Einstein manifold if $\alpha = 0$ and $\beta = constant$.*

8. Weyl Conformal Curvature Tensor on δ-Lorentzian Trans-Sasakian Manifold with a Semi-symmetric Metric Connection

The Weyl conformal curvature tensor $\bar{C}$ of type $(1,3)$ of M an n-dimensional δ-Lorentzian trans-Sasakian manifold a with semi-symmetric metric connection $\bar{\nabla}$ is given by [16]

$$\bar{C}(X,Y)Z = \bar{R}(X,Y)Z - \frac{1}{(n-2)}[\bar{S}(Y,Z)X - \bar{S}(X,Z)Y + g(Y,Z)\bar{Q}X - g(X,Z)\bar{Q}Y] + \frac{\bar{r}}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y], \tag{8.1}$$

where $\bar{Q}$ is the Ricci operator with respect to the semi-symmetric metric connection $\bar{\nabla}$. Let M ba an n-dimensional δ-Lorentzian trans-Sasakian manifold. The Weyl conformal curvature tensor $\bar{C}$ of M with respect to the semi-symmetric metric connection $\bar{\nabla}$ is defined in equation (8.1).

Now, taking inner product with U in (8.1), we get

$$g(\bar{C}(X,Y)Z,U) = g(\bar{R}(X,Y)Z,U) - \frac{1}{(n-2)}[\bar{S}(Y,Z)g(X,U) - \bar{S}(X,Z)g(Y,U) + g(Y,Z)g(\bar{Q}X,U) - g(X,Z)g(\bar{Q}Y,U)] + \frac{\bar{r}}{(n-1)(n-2)}[g(Y,Z)g(X,U) - g(X,Z)g(Y,U)]. \tag{8.2}$$

Using (2.4), (3.2), (3.11), (3.12) and (3.14) in (8.2), we get

$$\bar{C}(X,Y,Z,U) = g(\bar{R}(X,Y)Z,U) - \frac{1}{(n-2)}[S(Y,Z)g(X,U) - S(X,Z)g(Y,U) + g(Y,Z)g(QX,U) - g(X,Z)g(QY,U)] + \frac{r}{(n-1)(n-2)}[g(Y,Z)g(X,U) - g(X,Z)g(Y,U)], \tag{8.3}$$

where $g(\bar{C}(X,Y)Z,U) = \bar{C}(X,Y,Z,U)$ and $R(X,Y)Z,U) = C(X,Y,Z,U)$ are Weyl curvature tensor with respect to the semi-symmetric metric connection respectively, we have

$$\bar{C}(X,Y,Z,U) = C(X,Y,Z,U), \tag{8.4}$$

where

$$C(X,Y,Z,U) = g(\bar{R}(X,Y)Z,U) - \frac{1}{(n-2)}[S(Y,Z)g(X,U) - S(X,Z)g(Y,U) + g(Y,Z)g(QX,U) - g(X,Z)g(QY,U)] + \frac{r}{(n-1)(n-2)}[g(Y,Z)g(X,U) - g(X,Z)g(Y,U)]. \tag{8.5}$$

Theorem 8.1. *The Weyl conformal curvature tensor of a δ-Lorentzian trans-Sasakian manifold M with respect to a metric connection is equal to the Weyl curvature of δ-Lorentzian trans-Sasakian manifold with respect to the semi-symmetric*

metric connection.

9. δ-Lorentzian Trans-Sasakian Manifold with Weyl Conformal Flat Conditions with a Semi-symmetric Metric Connection

Let us consider that the δ-Lorentzian trans-Sasakian manifold M with respect to the semi-symmetric metric connection is Weyl conformally flat, that is $\bar{C} = 0$. Then from equation (8.1), we get

$$\begin{aligned} \bar{R}(X,Y)Z = \frac{1}{(n-2)}[&\bar{S}(Y,Z)X - \bar{S}(X,Z)Y \\ &+ g(Y,Z)\bar{Q}X - g(X,Z)\bar{Q}Y] \\ &+ \frac{\bar{r}}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{9.1}$$

Now, taking the inner product of equation (9.1) with U. then we get

$$\begin{aligned} g(\bar{R}(X,Y)Z,U) = \frac{1}{(n-2)}[&\bar{S}(Y,Z)g(X,U) - \bar{S}(X,Z)g(Y,U) \\ &+ g(Y,Z)g(\bar{Q}X,U) - g(X,Z)g(\bar{Q}Y,U)] \\ &- \frac{\bar{r}}{(n-1)(n-2)}[g(Y,Z)g(X,U) - g(X,Z)g(Y,U)]. \end{aligned} \tag{9.2}$$

Using equations (2.4), (3.2), (3.11), (3.12) and (3.14) in equation (9.2), we get

$$\begin{aligned} g(R(X,Y)Z,U) = \frac{1}{(n-2)}[&S(Y,Z)g(X,U) - S(X,Z)g(Y,U) \\ &+ g(Y,Z)g(QX,U) - g(X,Z)g(QY,U)] \\ &- \frac{r}{(n-1)(n-2)}[g(Y,Z)g(X,U) - g(X,Z)g(Y,U)]. \end{aligned} \tag{9.3}$$

Putting $X = U = \xi$ in equation (9.3) and using (2.2), (2.3) and (2.4), we get

$$\begin{aligned} g(R(\xi,Y)Z,\xi) = \frac{1}{(n-2)}[&\delta S(Y,Z) - \delta\eta(Y)S(\xi,Z) \\ &+ g(Y,Z)S(\xi,\xi) - \delta\eta(Z)S(Y,\xi)] \\ &- \frac{r}{(n-1)(n-2)}[\delta g(Y,Z) - \eta(Y)\eta(Z)], \end{aligned} \tag{9.4}$$

where $g(QY,Z) = S(Y,Z)$.

Now, using equations (2.13), (2.14) and (2.16), we get

$$(9.5)\quad \begin{aligned} S(Y,Z) &= [(\delta(\alpha^2+\beta^2)-(\xi\beta)]+\frac{r}{(n-1)}]g(Y,Z)\\ &+[\delta(n-4)(\xi\beta)+n(\alpha^2+\beta^2)-\frac{\delta}{r}(n-1)]\eta(Y)\eta(Z)\\ &-[2\delta\alpha\beta(n-2)+(n-2)(\xi\alpha)]g(\phi Y,Z)\\ &-[\delta(\phi Z)\alpha+\delta(Z\beta)(n-2)]\eta(Y)-[\delta(\phi Y)\alpha+\delta(n-2)(Y\beta)]\eta(Z). \end{aligned}$$

If $\alpha = 0$ andd $\beta =$ constant in (7.6), we get

$$(9.6)\quad S(Y,Z)=[\delta\beta^2+\frac{r}{(n-1)}]g(Y,Z)+[n\beta^2-\frac{\delta r}{(n-1)}]\eta(Y)\eta(Z).$$

Therefore

$$S(Y,Z)=ag(Y,Z)+b\eta(Y)\eta(Z),$$

where $a=[\delta\beta^2+\frac{r}{(n-1)}]$ and $b=[n\beta^2-\frac{\delta r}{(n-1)}]$. This shows that M is an η-Einstein manifold. Thus we can state the following theorem:

Let M ba an n-dimensional Weyl conformally flat δ-Lorentzian trans-Sasakian manifold with respect to the semi-symmetric metric connection $\bar{\nabla}$ is an η-Einstein manifold if $\alpha = 0$ and $\beta =$constant. Now, taking equation (8.1)

$$(9.7)\quad \begin{aligned} \bar{C}(X,Y)Z &= \bar{R}(X,Y)Z\\ &-\frac{1}{(n-2)}[\bar{S}(Y,Z)X-\bar{S}(X,Z)Y+g(Y,Z)\bar{Q}X-g(X,Z)\bar{Q}Y]\\ &+\frac{\bar{r}}{(n-1)(n-2)}[g(Y,Z)X-g(X,Z)Y]. \end{aligned}$$

Using (2.20), (3.2), (3.11), (3.12) and (3.14) in equation (9.7), we get

$$(9.8)\quad \begin{aligned} \bar{C}(X,Y)Z &= C(X,Y)Z+\delta[g(X,Z)Y-g(Y,Z)X]\\ &+(\delta+\beta)[\eta(X)g(Y,Z)-\eta(Y)g(X,Z)]\xi\\ &-(\beta\delta-1)\eta(Z)[\eta(Y)X-\eta(X)Y]+\alpha[g(\phi X,Z)Y\\ &-g(\phi,Z)X-g(Y,Z)\phi X+g(X,Z)\phi Y]+\frac{1}{(n-2)}\\ &(\beta\delta-1)(n-2)\eta(Y)\eta(Z)-((\delta)(n-2)+\beta)g(Y,Z)X\\ &+\alpha(n-2)g(\phi Y,Z)X+((\delta)(n-2)+\beta)g(X,Z)Y\\ &+(\beta\delta-1)(n-2)\eta(X)\eta(Z)Y-\alpha(n-2)g(\phi X,Z)Y\\ &-((\delta)(n-2)+\beta)g(Y,Z)X+(\beta+\delta)(n-2)g(Y,Z)\eta(X)\xi\\ &\alpha(n-2)g(Y,Z)\phi X+((\delta)(n-2)+\beta)g(X,Z)Y\\ &-(\beta+\delta)(n-2)g(X,Z)\eta(Y)\xi-\alpha(n-2)g(X,Z)\phi Y]\\ &-\frac{\beta+\delta+(n-2)}{(n-2)}[g(Y,Z)X-g(X,Z)Y]. \end{aligned}$$

Let X and Y are orthogonal basis to ξ. Putting $Z = \xi$ and using (2.1), (2.2) and (2.4) in (9.8), we get

$$\bar{C}(X,Y)\xi = C(X,Y)\xi.$$

Theorem 9.1. *An n-dimensinal δ-Lorentzian trans-Sasakian manifold M is Weyl ξ-conformally flat with respect to the semi-symmetric metric connection if and only if the manifold is also Weyl ξ-conformally flat with respect to the metric connection provided that the vector fields are horizontal vector fields.*

10. η-Ricci Solitons and Ricci Solitons in δ-Lorentzian Trans-Sasakian Manifold with a Semi-symmetric Metric Connection

Let M be 3-dimensional δ-Lorentzian trans-Sasakian manifold with a semi-symmetric metric connection and V be pointwise collinear with ξ *i.e.* $V = b\xi$, where b is a function. Then consider the equation [9]

$$L_V g + 2\bar{S} + 2\lambda g + 2\mu\eta \otimes \eta = 0, \tag{10.1}$$

where L_V is the Lie derivative operator along the vector field V, $\bar{S}$ is the Ricci curvature tensor field of the metric g and λ and μ are real constants. Then equation (10.1) implies,

$$g(\bar{\nabla}_X b\xi, Y) + g(\bar{\nabla}_Y b\xi, X) + 2\bar{S}(X,Y) + 2\lambda g(X,Y) + 2\mu\eta(X)\eta(Y) = 0, \tag{10.2}$$

or

$$\begin{aligned} &bg(\bar{\nabla}_X \xi, Y) + (Xb)\eta(Y) + bg(\bar{\nabla}_Y \xi, X) + (Yb)\eta(X) \\ &+ 2\bar{S}(X,Y) + 2\lambda g(X,Y) + 2\mu\eta(X)\eta(Y) = 0. \end{aligned} \tag{10.3}$$

Using (3.4) in (10.3), we get

$$\begin{aligned} &bg[-(1+\delta\beta)X - (1+\delta\beta)\eta(X)\xi - \delta\alpha\phi X, Y] + (Xb)\eta(Y) \\ &+ bg[-(1+\delta\beta)Y - (1+\delta\beta)\eta(Y)\xi - \delta\alpha\phi Y, X] + (Yb)\eta(X) \\ &+ 2\bar{S}(X,Y) + 2\lambda g(X,Y) + 2\mu\eta(X)\eta(Y) = 0. \end{aligned} \tag{10.4}$$

$$\begin{aligned} &-2b(1+\delta\beta)g(X,Y) - 2b(1+\delta\beta)\eta(Y)\eta(X) + (Xb)\eta(Y) + (Yb)\eta(X) \\ &+ 2\bar{S}(X,Y) + 2\lambda g(X,Y) + 2\mu\eta(X)\eta(Y) = 0. \end{aligned} \tag{10.5}$$

With the substitution of Y with ξ in (10.5) and using (3.21) for constants α and β, it follows that

$$\begin{aligned} &(Xb) + (\xi b)\eta(X) - 4b(1+\delta\beta)\eta(X) \\ &+ 2[2(\alpha^2 + \beta^2 - \delta(\xi\beta)) - 2\delta\beta]\eta(X) \\ &+ 2\lambda\eta(X) + 2\mu\eta(X) = 0. \end{aligned} \tag{10.6}$$

or

$$(XB) + (\xi b)\eta(X) + \tag{10.7}$$
$$[-4b(1+\delta\beta) + 2(2(\alpha^2+\beta^2-\delta(\xi\beta)) - 2\delta\beta + 2\lambda + 2\mu]\eta(X) = 0.$$

Again replacing $X = \xi$ in (10.7), we obtain

$$\xi b = -[-2b(1+\delta\beta) + (2(\alpha^2+\beta^2-\delta(\xi\beta)) - \delta\beta + \lambda + \mu] \tag{10.8}$$

Putting (10.8) in (10.7), we obtain

$$db = [2b(1+\delta\beta) - (2(\alpha^2+\beta^2-\delta(\xi\beta)) - \delta\beta - \lambda - \mu]\eta. \tag{10.9}$$

By applying d on (10.9), we get

$$[2b(1+\delta\beta) - (2(\alpha^2+\beta^2-\delta(\xi\beta)) - \delta\beta - \lambda - \mu]d\eta = 0. \tag{10.10}$$

Since $d\eta \neq 0$ from (10.10), we have

$$[2b(1+\delta\beta) - (2(\alpha^2+\beta^2-\delta(\xi\beta)) - \delta\beta - \lambda - \mu] = 0. \tag{10.11}$$

By using (10.9) and (10.11), we obtain that b is a constant. Hence from (10.5) it is verified

$$\bar{S}(X,Y) = [b(1+\delta\beta) - \lambda]g(X,Y) + [b(1+\delta\beta) - \mu]\eta(X)\eta(Y). \tag{10.12}$$

which implies that M is an η-Einstien manifold. This lead to the following:

Theorem 10.1. *In a 3-dimensional δ-Lorentzian trans-Sasakian manifold with a semi-symmetric metric connection, the metric g is an η-Ricci soliton and V is a positive collinear with ξ, then V is a constant multiple of ξ and g is an η-Einstien manifold of the form (10.12) and η-Ricci soliton is expanding or shrinking according as the following relation is positive and negative*

$$\lambda = -[2b(1+\delta\beta) - (2(\alpha^2+\beta^2-\delta(\xi\beta)) - \delta\beta - \mu]. \tag{10.13}$$

For $\mu = 0$, we deduce equation (10.12)

$$\bar{S}(X,Y) = [b(1+\delta\beta) - \lambda]g(X,Y) + [b(1+\delta\beta)]\eta(X)\eta(Y). \tag{10.14}$$

Now, we have the following corollary:

Corollary 10.1. *In a 3-dimensional δ-Lorentzian trans-Sasakian manifold with a semi-symmetric metric connection, the metric g is a Ricci soliton and V is a positive collinear with ξ, then V is a constant multiple of ξ and g is an η-Einstien manifold and Ricci soliton is shrinking according as the following relation is negative. For $\mu = 0$, (10.13) reduce to*

$$\lambda = -[2b(1+\delta\beta) - (2(\alpha^2+\beta^2-\delta(\xi\beta)) - \delta\beta]. \tag{10.15}$$

Here is an example of η-Ricci soliton on δ-Lorentzian trans-Sasakian manifold with a semi-symmetric metric connection.

Example 10.1. *We consider the three dimensional manifold* $M = [(x, y, z) \in \mathbb{R}^3 \mid z \neq 0]$, *where* (x, y, z) *are the Cartesian coordinates in* $\mathbb{R}^3$. *Choosing the vector fields*

$$e_1 = z\frac{\partial}{\partial x}, \quad e_2 = z\frac{\partial}{\partial y}, \quad e_3 = -z\frac{\partial}{\partial z},$$

which are linearly independent at each point of M. Let g be the Riemannian metric define by

$$g(e_1, e_3) = g(e_2, e_3) = g(e_2, e_2) = 0, \qquad g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = \delta,$$

where $\delta = \pm 1$. Let η be the 1-form defined by $\eta(Z) = \epsilon g(Z, e_3)$ for any vector field Z on TM. Let ϕ be the $(1,1)$ tensor field defined by $\phi(e_1) = -e_2, \quad \phi(e_2) = e_1, \quad \phi(e_3) = 0$. Then by the linearity property of ϕ and g, we have

$$\phi^2 Z = Z + \eta(Z)e_3, \quad \eta(e_3) = 1 \quad \text{and} \quad g(\phi Z, \phi W) = g(Z, W) - \delta\eta(Z)\eta(W)$$

for any vector fields Z, W on M.

Let ∇ be the Levi-Civita connection with respect to the metric g. Then we have

$$[e_1, e_2] = 0, \qquad [e_1, e_3] = \delta e_1, \qquad [e_2, e_3] = \delta e_2.$$

The Riemannian connection ∇ with respect to the metric g is given by

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\ &\quad + g([X, Y], Z) - g([Y, Z], X) + g([Z, X], Y). \end{aligned}$$

From above equation which is known as Koszul's formula, we have

$$\begin{aligned} &\nabla_{e_1} e_3 = \delta e_1, && \nabla_{e_2} e_3 = \delta e_2, && \nabla_{e_3} e_3 = 0, \\ &\nabla_{e_1} e_2 = 0, && \nabla_{e_2} e_2 = -\delta e_3, && \nabla_{e_3} e_2 = 0, \\ &\nabla_{e_1} e_1 = -\delta e_3, && \nabla_{e_2} e_1 = 0, && \nabla_{e_3} e_1 = 0. \end{aligned} \tag{10.16}$$

Using the above relations, for any vector field X on M, we have

$$\nabla_X \xi = \delta(X - \eta(X)\xi)$$

for $\xi \in e_3$, $\alpha = 0$ and $\beta = 1$. Hence the manifold M under consideration is an δ-Lorentzian trans Sasakian of type $(0, 1)$ manifold of dimension three.

Now, we consider this example for semi-symmetric metric connection from (2.9) and (10.14), we obtain:

$$\begin{aligned} &\bar{\nabla}_{e_1} e_3 = (1 + \delta)e_1, && \bar{\nabla}_{e_2} e_3 = (1 + \delta)e_2, && \bar{\nabla}_{e_3} e_3 = 0, \\ &\bar{\nabla}_{e_1} e_2 = 0, && \bar{\nabla}_{e_2} e_2 = -(1 + \delta)e_3, && \bar{\nabla}_{e_3} e_2 = 0, \\ &\bar{\nabla}_{e_1} e_1 = -(1 + \delta)e_3, && \bar{\nabla}_{e_2} e_1 = 0, && \bar{\nabla}_{e_3} e_1 = 0. \end{aligned} \tag{10.17}$$

Then the Riemannian and the Ricci curvature tensor fields with respect to the semi-symmetric metric connection are given by:

$$\bar{R}(e_1,e_2)e_2 = -(1+\delta)^2 e_1, \quad \bar{R}(e_1,e_3)e_3 = -\delta(1+\delta)e_2, \quad \bar{R}(e_2,e_1)e_1 = -(1+\delta)^2 e_2,$$
$$\bar{R}(e_2,e_3)e_3 = -\delta(1+\delta)e_2, \quad \bar{R}(e_3,e_1)e_1 = \delta(1+\delta)e_3, \quad \bar{R}(e_3,e_2)e_2 = -\delta(1+\delta)e_3,$$

$$\bar{S}(e_1,e_1) = \quad \bar{S}(e_2,e_2) = -(1+\delta)(1+2\delta), \quad \bar{S}(e_3,e_3,) = 2\delta(1+\delta).$$

From (10.14), for $\lambda = \frac{(1+\delta)^2}{\delta}$ and $\mu = -(1+\delta)(1+3\delta)$, the data (g,ξ,λ,μ) is an η-Ricci soliton on (M,ϕ,ξ,η,g) which is expanding.

References

[1] A. M. Blaga, *η-Ricci solitons on Lorentzian para-Sasakian manifolds*, Filomat, **30(2)**(2016), 489–496.

[2] A. M. Blaga, *η-Ricci solitons on para-Kenmotsu manifolds*, Balkan J. Geom. Appl., **20**(2015), 1–13.

[3] A. M. Blaga, S. Y. Perktas, B. L. Acet and F. E. Erdogan, *η-Ricci solitons in (ε)-almost para contact metric manifolds*, Glas. Mat. Ser. III, **53**(2018), 205-–220.

[4] C. S. Bagewadi and G. Ingalahalli, *Ricci Solitons in Lorentzian α-Sasakian Manifolds*, Acta Math. Acad. Paedagog. Nyhzi.(N.S.), **28(1)**(2012), 59–68.

[5] E. Bartolotti, *Sulla geometria della variata a connection affine*. Ann. di Mat., **4(8)**(1930), 53–101.

[6] A. Bejancu and K. L. Duggal, *Real hypersurfaces of indefinite Kaehler manifolds*, Internet. J. Math. Math. Sci., **16**(1993), 545–556.

[7] D. E. Blair, *Contact manifolds in Riemannian geometry*, Lecture note in Mathematics **509**, Springer-Verlag, Berlin-New York, 1976.

[8] S. M. Bhati, *On weakly Ricci ϕ-symmetric δ-Lorentzian trans Sasakian manifolds*, Bull. Math. Anal. Appl., **5 (1)**(2013), 36–43.

[9] J. T. Cho and M. Kimura, *Ricci solitons and Real hypersurfaces in a complex space form*, Tohoku math.J., **61**(2009), 205–212.

[10] O. Chodosh, F. T. H. Fong, *Rotational symmetry of conical Kahler-Ricci solitons*, Math. Ann., **364**(2016), 777—792.

[11] U. C. De and A. Sarkar, *On (ε)-Kenmotsu manifolds*, Hadronic J., **32(2)**(2009), 231–242.

[12] U. C. De and A. Sarkar, *On three-dimensional Trans-Sasakian Manifolds*, Extracta Math., **23**(2008), 265–277.

[13] A. Friedmann and J. Schouten, *Uber die Geometric der halbsymmetrischen*, Ubertragung, Math. Z., **21**(1924), 211–223.

[14] A. Gray and L. M. Harvella, *The sixteen classes of almost Hermitian manifolds and their linear invariants*, Ann. Mat. Pura Appl., **123(4)**(1980), 35–58.

[15] H. Gill and K. K. Dube, *Generalized CR-Submanifolds of a trans Lorentzian para Sasakian manifold*, Proc. Nat. Acad. Sci. India Sec. A Phys. Sci., **76**(2006), 119-124.

[16] H. A. Hayden, *Sub-spaces of a space with torsion*, Proc. London Math. Soc., **34**(1932), 27–50.

[17] I. E. Hirica and L. Nicolescu, *Conformal connections on Lyra manifolds*, Balkan J. Geom. Appl., **13**(2008), 43–49.

[18] I. E. Hirica and L. Nicolescu, *On Weyl structures*, Rend. Circ. Mat. Palermo (2), **53**(2004), 390–400.

[19] R. S. Hamilton, *The Ricci flow on surfaces*, Mathematics and general relativity (Santa Cruz. CA, 1986), 237-262, Contemp. Math. **71**, Amer. Math. Soc., Providence, RI, 1988.

[20] T. Ikawa and M. Erdogan, *Sasakian manifolds with Lorentzian metric*, Kyungpook Math. J., **35**(1996), 517–526.

[21] J. B. Jun, U. C. De and G. Pathak, *On Kenmotsu manifolds*, J. Korean Math. Soc., **42(3)**(2005), 435–445.

[22] H. Levy, *Symmetric tensors of the second order whose covariant derivatives vanish*, Ann. Math., **27(2)**(1925), 91–98.

[23] J. C. Marrero, *The local structure of Trans-Sasakian manifolds*, Ann. Mat. Pura Appl., **162**(1992), 77–86.

[24] K. Matsumoto, *On Lorentzian paracontact manifolds*, Bull. Yamagata Univ. Natur. Sci., **12**(1989), 151–156.

[25] H. G. Nagaraja and C.R. Premalatha, *Ricci solitons in Kenmotsu manifolds*, J. Math. Anal., **3 (2)**(2012), 18–24.

[26] J. A. Oubina, *New classes of almost contact metric structures*, Publ. Math. Debrecen, **32**(1985), 187–193.

[27] G. Pathak and U. C. De, *On a semi-symmetric metric connection in a Kenmotsu manifold*, Bull. Calcutta Math. Soc., **94(4)**(2002), 319–324.

[28] S. S. Pujar and V. J. Khairnar, *On Lorentzian trans-Sasakian manifold-I*, Int. J. of Ultra Sciences of Physical Sciences, **23(1)**(2011), 53–66.

[29] S. S. Pujar, *On Lorentzian Sasakian manifolds*, Antactica J. Math., **8**(2012), 30–38.

[30] R. Sharma, *Certain results on K-contact and (k,μ)-contact manifolds*, J. Geom., **89(1-2)**(2008), 138–147.

[31] A. Sharfuddin and S. I. Hussain, *Semi-symmetric metric connections in almost contact manifolds*, Tensor (N.S.), **30**(1976), 133–139.

[32] S. S. Shukla and D. D. Singh, *On (ε)-trans-Sasakian manifolds*, Int. J. Math. Anal., **4(49-52)**(2010), 2401–2414.

[33] M. D. Siddiqi, A. Haseeb and M. Ahmad, *On generalized Ricci-recurrent (ε,δ)-trans-Sasakian manifolds*, Palest. J. Math., **4(1)**(2015), 156–163.

[34] M. M. Tripathi, *On a semi-symmetric metric connection in a Kenmotsu manifold*, J. Pure Math., **16**(1999), 67–71.

[35] M. M. Tripathi, E. Kilic, S. Y. Perktas and S. Keles, *Indefnite almost para-contact metric manifolds*, Int. J. Math. Math. Sci., (2010), Art. ID 846195, 19 pp.

[36] T. Takahashi, *Sasakian manifold with Pseudo-Riemannian metric*, Tohoku Math. J., **21**(1969), 271–290.

[37] S. Tanno, *The automorphism groups of almost contact Riemannian manifolds*, Tohoku Math. J., **21**(1969), 21–38.

[38] K. Venu and H.G. Nagaraja, *η-Ricci solitons in trans-Sasakian manifolds*, Commun. Fac. sci. Univ. Ank. Ser. A1 Math. Stat., **66 (2)**(2017), 218–224.

[39] X. Xufeng and C. Xiaoli, *Two theorems on ε-Sasakian manifolds*, Internat. J. Math. Math. Sci., **21**(1998), 249–254.

[40] A. F. Yaliniz, A. Yildiz and M. Turan, *On three-dimensional Lorentzian β- Kenmotsu manifolds*, Kuwait J. Sci. Engrg., **36**(2009), 51–62.

[41] A. Yildiz, M. Turan, M. and C. Murathan, *A class of Lorentzian α-Sasakian manifolds*, Kyungpook Math. J., **49**(2009), 789 –799.

[42] K. Yano, *On semi-symmetric metric connections*, Rev. Roumaine Math. Pures Appl., **15**(1970), 1579–1586.

[43] K. Yano and M. Kon, *Structures on Manifolds*, Series in Pure Mathematics **3**, World Scientific Publishing, Singapore, 1984.