KYUNGPOOK Math. J. 58(2018), 533-546
https://doi.org/10.5666/KMJ.2018.58.3.533
pISSN 1225-6951 eISSN 0454-8124
© Kyungpook Mathematical Journal

Existence of Solutions for a Class of $p(x)$-Kirchhoff Type Equation with Dependence on the Gradient

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Abstract. The object of this work is to study the existence of solutions for a class of $p(x)$-Kirchhoff type problem under no-flux boundary conditions with dependence on the gradient. We establish our results by using the degree theory for operators of (S_+) type in the framework of variable exponent Sobolev spaces.

1. Introduction

In this paper we discuss the existence of weak solutions for the following Kirchhoff type problem

$$\begin{aligned} -M\Big(\int_\Omega (A(x,\nabla u)+\frac{1}{p(x)}|u|^{p(x)})\,dx\Big)\Big[\operatorname{div}(a(x,\nabla u))-|u|^{p(x)-2}u\Big] \\ = f(x,u,\nabla u)|u|^{t(x)}_{s(x)} \quad \text{in } \Omega \\ u = constant \quad \text{on } \partial\Omega, \\ \int_{\partial\Omega} a(x,\nabla u).\nu d\Gamma = 0. \end{aligned} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$, and $N \geq 1$, $p, s, t \in C(\overline{\Omega})$ for any $x \in \overline{\Omega}$; , div $\big(a(x,\nabla u)\big)$ is a $p(x)$-Laplacian type operator and M, f are functions that satisfy conditions which will be stated later.

The problem (1.1) is related to the stationary problem of a model introduced

* Corresponding Author.
Received May 29, 2017; accepted July 18, 2018.
2010 Mathematics Subject Classification: 35J30, 35J92, 47H05, 47H30.
Key words and phrases: $p(x)$-Kirchhoff type equation, variable exponent Sobolev space, degree theory, gradient term.

by Kirchhoff [30]. Accurately, he introduced a model given by the equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \Big(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L |\frac{\partial u}{\partial x}|^2 dx\Big)\frac{\partial^2 u}{\partial x^2} = 0, \tag{1.2}$$

which extends the classical D'Alembert's wave equation by considering the effects of the changes in the length of the strings during the vibrations. After the work of Lions [31], where an abstract framework to the problem (1.1) was given, various problems of Kirchhoff type have been studied by many authors, we refer to [3, 7, 17] . The study of Kirchhoff type equations has already been extended to the case involving the p-Laplacian (see [11, 12, 18, 19]) and $p(x)$-Laplacian (see [10, 14, 15, 16, 20, 39]). For the physical and biological meaning of the nonlocal coefficients we refer the readers to [1, 8, 9, 13] and the references therein.

In a recent paper, the authors in [6], have dealt with the $p(x)$-Kirchhoff type equation

$$\begin{aligned} -M\Big(\int_\Omega (A(x,\nabla u) + \frac{1}{p(x)}|u|^{p(x)})\,dx\Big)\Big[\,\mathrm{div}(a(x,\nabla u)) - |u|^{p(x)-2}u\Big] \\ = f(x,u)|u|^{t(x)}_{s(x)} \quad \text{in } \Omega, \\ u = constant \quad \text{on } \partial\Omega, \\ \int_{\partial\Omega} a(x,\nabla u).\nu d\Gamma = 0. \end{aligned}$$

by topological methods.

We note that our problem has no variational structure because the nonlinearity f depends on the gradient of the solution, so the most usual variational methods are not applicable. For the case $M = 1$, $p :=$ constant and $t(x) = 0$ with a homogeneous Dirichlet boundary condition there have been several works , using method of sub and supersolutions, topological degree, fixed point theorems, theory of pseudomonotone operators, Morse theory and approximation techniques; see, among other papers [24, 25, 32, 35, 37]. In [26], the authors used the idea developed in [27]. This idea consists of regaining some variational aspect for (1.1) by freezing the term ∇u appearing as argument of f. After this, an iterative scheme is performed to obtain a nontrivial solution for the initial problem. Similarly, in the very recent paper [4], the authors established the existence of a positive solution. The case $M = M(s)$ and $p :=$ constant , $t(x) = 0$, was studied in [2, 28, 33, 34].

Our aim is to establish the existence results for problem (1.1) through the degree theory. First, we will establish the existence of a weak solution via the degree theory for (S_+) type mapping in terms of the Hammerstein equation. This method was recently introduced in [29] in order to study a quasilinear equation governed by the p-Laplacian operator $-\Delta_p$ when the nonlinear term depends on the gradient of the solution, i.e. when it is of the form $f = f(x, u, \nabla u)$. Here we adapt this technique in order to consider an equation with a class of $p(x)$-Kirchhoff type operator. Next, we

will get the existence of a weak solution of the problem (1.1) by employing directly the degree theory for (S_+) type mapping.

To the best of our knowledge, problem (1.1) has not been studied via the degree theory for operators of generalized (S_+) type . The results obtained in the present paper are also new for the case that $p(x)$ is a constant.

This paper is organized as follows. In Section 2, we recall some basic facts about the variable Lebesgue–Sobolev spaces and the degree theory for operators of (S_+) type . Sections 3, is devoted to the main results of the paper.

2. Preliminaries

To discuss problem (1.1), we need some theory on $W^{1,p(x)}(\Omega)$ which is called variable exponent Sobolev space. Firstly we state some basic properties of spaces $W^{1,p(x)}(\Omega)$ which will be used later (for details, see [23]). Denote by $\mathbf{S}(\Omega)$ the set of all measurable real functions defined on Ω. Two functions in $\mathbf{S}(\Omega)$ are considered as the same element of $\mathbf{S}(\Omega)$ when they are equal almost everywhere. Write

$$C_+(\overline{\Omega}) = \{h : h \in C(\overline{\Omega}), h(x) > 1 \text{ for any } x \in \overline{\Omega}\},$$
$$h^- := \min_{\overline{\Omega}} h(x), \quad h^+ := \max_{\overline{\Omega}} h(x) \quad \text{for every } h \in C_+(\overline{\Omega}).$$

Define

$$L^{p(x)}(\Omega) = \{u \in \mathbf{S}(\Omega) : \int_\Omega |u(x)|^{p(x)}\, dx < +\infty \text{ for } p \in C_+(\overline{\Omega})\}$$

with the norm

$$|u|_{L^{p(x)}(\Omega)} = |u|_{p(x)} = \inf\{\lambda > 0 : \int_\Omega |\frac{u(x)}{\lambda}|^{p(x)}\, dx \le 1\},$$

and

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\}$$

with the norm

$$\|u\|_{W^{1,p(x)}(\Omega)} = |u|_{L^{p(x)}(\Omega)} + |\nabla u|_{L^{p(x)}(\Omega)}.$$

Proposition 2.1.([23]) *The spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ are separable, uniformly convex and reflexive Banach spaces.*

Proposition 2.2.([23]) *Set $\rho(u) = \int_\Omega |u(x)|^{p(x)}\, dx$. For any $u \in L^{p(x)}(\Omega)$, then*

(1) *for $u \neq 0$, $|u|_{p(x)} = \lambda$ if and only if $\rho(\frac{u}{\lambda}) = 1$;*

(2) *$|u|_{p(x)} < 1$ $(= 1; > 1)$ if and only if $\rho(u) < 1$ $(= 1; > 1)$;*

(3) *if $|u|_{p(x)} > 1$, then $|u|^{p^-}_{p(x)} \le \rho(u) \le |u|^{p^+}_{p(x)}$;*

(4) *if $|u|_{p(x)} < 1$, then $|u|^{p^+}_{p(x)} \le \rho(u) \le |u|^{p^-}_{p(x)}$;*

(5) $\lim_{k\to+\infty} |u_k|_{p(x)} = 0$ *if and only if* $\lim_{k\to+\infty} \rho(u_k) = 0$;

(6) $\lim_{k\to+\infty} |u_k|_{p(x)} = +\infty$ *if and only if* $\lim_{k\to+\infty} \rho(u_k) = +\infty$.

Proposition 2.3.([21, 23]) *If* $q \in C_+(\overline{\Omega})$ *and* $q(x) \leq p^*(x)$ $(q(x) < p^*(x))$ *for* $x \in \overline{\Omega}$, *then there is a continuous (compact) embedding* $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$, *where*

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N, \\ +\infty & \text{if } p(x) \geq N. \end{cases}$$

Proposition 2.4.([22, 23]) *The conjugate space of* $L^{p(x)}(\Omega)$ *is* $L^{q(x)}(\Omega)$, *where* $\frac{1}{q(x)} + \frac{1}{p(x)} = 1$ *holds a.e. in* Ω. *For any* $u \in L^{p(x)}(\Omega)$ *and* $v \in L^{q(x)}(\Omega)$, *we have the following Hölder-type inequality*

$$\left| \int_\Omega uv\, dx \right| \leq \left(\frac{1}{p^-} + \frac{1}{q^-} \right) |u|_{p(x)} |v|_{q(x)}.$$

We recall some definitions and basic properties of the the Berkovits degree theory for demicontinuous operators of generalized (S_+) type in real reflexive Banach spaces. Let X be a real reflexive Banach space with dual space X^*. We can assume that X and X^* are locally uniformly convex (Troyanski's theorem [38]). The symbol $\langle .,. \rangle_X$ denotes the usual dual paring between X^* and X in this order. In the reflexive case where the bidual space X^{**} is identified with X, we sometimes write $\langle y, x \rangle$ for $\langle x, y \rangle_{X^*}$ for $x \in X$ and $y \in X^*$.

Let Ω be a nonempty subset of X, $(u_\nu)_{\nu\geq 1} \subseteq \Omega$ and $F : \Omega \subseteq X \to X^*$. Then the mapping F is

(1) demicontinuous if $u_\nu \to u$ implies $F(u_\nu) \rightharpoonup F(u)$

(2) of class (S_+) if $u_\nu \rightharpoonup u$ and $\limsup_{\nu\to+\infty} \langle Fu_\nu, u_\nu - u \rangle \leq 0$ yield $u_\nu \to u$

(3) Quasimonotone if $u_\nu \rightharpoonup u$ implies $\limsup_{\nu\to+\infty} \langle Fu_\nu, u_\nu - u \rangle \leq 0$

For any operator $F : \Omega \subseteq X \to X$ and any bounded operator $T : \Omega_1 \subseteq X \to X^*$ such that $\Omega \subseteq \Omega_1$, we say that F satisfies condition $(S_+)_T$ if for any sequence (u_ν) in Ω with $u_\nu \rightharpoonup u$, $Tu_\nu \rightharpoonup y$ and $\limsup_{\nu\to+\infty} \langle Fu_\nu, Tu_\nu - y \rangle \leq 0$, we have $u_\nu \to u$

Remark 2.1. It is known that

(a) If a mapping is compact in a set, then it is quasi-monotone in that set.

(b) If the mapping is demi-continuous and satisfies the condition (S_+) in a set, then it is quasimonotone in that set.

Now, we consider the following sets

$$\mathfrak{F}_1(\Omega) = \Big\{F : \Omega \subseteq X \to X^*, F \quad \text{is bounded, demicontinuous and of class } (S_+)\Big\}$$
$$\mathfrak{F}_T(\Omega) = \Big\{F : \Omega \subseteq X \to X^*, F \quad \text{is demicontinuous and of class } (S_+)_T\Big\}$$

for any $\Omega \subseteq D_F$ and any $T \in \mathfrak{F}_1(\Omega)$, where D_F denotes the domain of F.

We need the following result related to the Hammerstein operator of the form $I + S \circ T$

Lemma 2.1.([29]) *Suppose that $T \in \mathfrak{F}_1(\overline{\Omega})$ is continuous and $S : D_S \subseteq X^* \to X$ demicontinuous such that $T(\overline{\Omega}) \subseteq D_S$, where Ω is a bounded open set in the real reflexive Banach space X. Then the following statements are true:*

(a) *If S is quasimonotone, then $I + S \circ T \in \mathfrak{F}_T(\overline{\Omega})$ where I denotes the identity operator.*

(b) *If S satisfies conditions (S_+), then $S \circ T \in \mathfrak{F}_T(\overline{\Omega})$.*

Next, we present a result which pertains to the existence of the topological degree for a class of demicontinuous operator satisfying condition $(S_+)_T$.

Theorem 2.1.([29]) *Let $\mathcal{M} = \Big\{(F, \Omega, h)/\Omega \in \mathcal{O}, T \in \mathfrak{F}_1(\overline{\Omega}), F \in \mathfrak{F}_T(\overline{\Omega}), h \notin F(\partial\Omega)\Big\}$ where $\mathcal{O}$ denotes the collection of all bounded open sets in X. There exists one (Hammerstein type)degree function*

$$d : \mathcal{M} \to \mathbb{Z}$$

which has the following properties:

(a) *(Existence) If $d(F, \Omega, h) \neq 0$ then the equation $Fu = h$ has a solution in Ω*

(b) *(Additivity) Let $F \in \mathfrak{F}_T(\overline{\Omega})$. For every disjoint open sets $\Omega_1, \Omega_2 \subseteq \Omega$ and every $h \notin F(\overline{\Omega} \setminus (\Omega_1 \cup \Omega_2))$ we have*

$$d(F, \Omega, h) = d(F, \Omega_1, h) + d(F, \Omega_2, h)$$

(c) *(Normalization) $d(I, \Omega, h) = 1$, for every $h \in \Omega$.*

(d) *(Invariance under homotopy) Let $T \in \mathfrak{F}_1(\overline{\Omega})$ be continuous and $F, S \in \mathfrak{F}_T(\overline{\Omega})$. An admissible affine homotopy (from $\overline{\Omega}$ into X) is a mapping*

$$H : [0, 1] \times \overline{\Omega} \to X$$

defined by

$$H(t, u) = (1 - t)Fu + tSu; \qquad (t, u) \in [0, 1] \times \overline{\Omega}$$

(so H satisfies condition $(S_+)_T$ by Lemma 2.5 in [29]).

If $H : [0,1] \times \overline{\Omega} \to X$ is an admissible affine homotopy, with the common "support" T and $h : [0,1] \to X$ is a continuous curve in X such that $h(t) \notin H(t, \partial\Omega)$ for all $t \in [0,1]$, the value $d(H(t,.), \Omega, h(t))$ is constant for all $t \in [0,1]$

Throughout this paper, let

$$Y = \{u \in W^{1,p(x)}(\Omega) : u|_{\partial\Omega} = \text{constant}\}.$$

The space Y is a closed subspace of the separable and reflexive Banach space $W^{1,p(x)}(\Omega)$ (See [5]) , so Y is also separable and reflexive Banach space with the usual norm of $W^{1,p(x)}(\Omega)$. The space Y is the space where we will try to find weak solutions for problem (1.1).

For presenting our main results, we first have to describe the data involved in our problem. We assume that $a(x,\xi) : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}^N$ is the continuous derivative with respect to ξ of the mapping $A : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}$,$A = A(x,\xi)$, i.e. $a(x,\xi) = \nabla_\xi A(x,\xi)$. Furthermore, we suppose that a and A satisfy the following hypotheses:

(A1) a satisfies the growth condition $|a(x,\xi)| \le c_0(a_0(x) + |\xi|^{p(x)-1})$ for all $x \in \Omega$, $\xi \in \mathbb{R}^N$, for some constant $c_0 > 0$; $a_0 \in L^{p'(x)}(\Omega)$ is a nonnegative function.

(A2) $A(x,0) = 0$ for all $x \in \Omega$;

(A3) $|\xi|^{p(x)} \le a(x,\xi)\xi \le p(x)A(x,\xi)$ for all $x \in \Omega$, $\xi \in \mathbb{R}^N$.

(A4) The monotonicity condition $0 \le [a(x,\eta_1) - a(x,\eta_2)(\eta_1 - \eta_2)]$, for all $x \in \Omega$ and all $\eta_1, \eta_2 \in \mathbb{R}^N$,with equality if and only if $\eta_1 = \eta_2$.

(M_0) $M : [0, +\infty[\to]m_0, +\infty[$ is a continuous and nondecreasing function with $m_0 > 0$.

Next, we give the properties of the $p(x)$-Kirchhoff type operator

$$-M\Big(\int_\Omega (A(x,\nabla u) + \frac{1}{p(x)}|u|^{p(x)})\,dx\Big)\Big[\operatorname{div}(a(x,\nabla u)) - |u|^{p(x)-2}u\Big]$$

Consider the following functional

$$\Phi(u) = \widehat{M}\Big(L(u)\Big) + \int_\Omega \frac{1}{p(x)}|u|^{p(x)}\,dx \quad \text{for all } u \in Y$$

where

$$L(u) = \int_\Omega A(x,\nabla u)dx \quad \text{for all } u \in Y$$

and $\widehat{M}(s) = \int_0^s M(t)\,dt$.

It is well known that Φ is well defined and continuously Gâteaux differentiable whose Gâteaux derivatives at point $u \in Y$ is the functional $\Phi'(u) \in Y'$ given by

$$\langle \Phi'(u), v\rangle = \langle T(u), v\rangle \quad \text{for all } u \in Y.$$

where the operator $T : Y \to Y'$ is defined by

$$\langle Tu, v\rangle = M\Big(\int_\Omega A(x, \nabla u)dx\Big)\int_\Omega a(x, \nabla u)\nabla v\, dx + \int_\Omega |u|^{p(x)-2}uv\, dx \tag{2.1}$$

for all $u, v \in Y$.

Proposition 2.5.([6]) *Assume that* (M_0) *holds. Then*

(i) $\Phi' : Y \to Y^*$ *is a continuous, bounded and strictly monotone operator;*

(ii) Φ' *is of type* (S_+)*, i.e. if* $u_\nu \rightharpoonup u$ *in* Y *and*

$$\limsup_{\nu\to+\infty}\langle \Phi'(u_n) - \Phi'(u), u_\nu - u\rangle = 0,$$

then $u_\nu \to u$ *in* Y*;*

(iii) Φ' *is coercive. Furthermore,* Φ' *is a homeomorphism.*

From now on we deal with the properties for the superposition operator induced by the function f in (1.1). We assume that

(F_1) $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfy the Carathéodory condition in the sense that $f(., u, \xi)$ is measurable for all $(u, \xi) \in \mathbb{R} \times \mathbb{R}^N$ and $f(x, ., .)$ is continuous for almost all $x \in \Omega$

(F_2) $|f(x, u, \xi)| \le k(x) + |u|^{\eta(x)} + |\xi|^{\delta(x)}$ a.e. $x \in \Omega$, all $(u, \xi) \in \mathbb{R} \times \mathbb{R}^N$, where $k : \mathbb{R} \to \mathbb{R}^+$, $k \in L^{p'(x)}(\Omega)$ and $0 \le \eta(x) < p^- - 1, 0 \le \delta(x) < (p^- - 1)/p'^+$.

Lemma 2.2 *Under assumption* $(F1)$ *the operator* $S : Y \to Y^*$ *given by*

$$\langle Su, v\rangle = -\int_\Omega f(x, u, \nabla u)|u|^{t(x)}_{s(x)}v\, dx$$

is continuous and compact.

Proof. Here, we follow the lines of Theorem 3.1 in [6]. We split the proof in two steps.

Step 1: S is well defined. Indeed, using (F_1) and $t \in C(\overline{\Omega})$, for all u, v in Y we

have
(2.2)

$$\begin{aligned}|\langle Su, v\rangle| &\le \int_\Omega |f(x,u,\nabla u)||u|_{s(x)}^{t(x)}|v|\,dx\\ &\le C|f(x,u,\nabla u)|_{p'(x)}|v|_{p(x)}\\ &\le C\Big(\int_\Omega |k(x)|^{p'(x)}\,dx + \int_\Omega |u(x)|^{\eta(x)p'(x)}\,dx + \int_\Omega |\nabla u(x)|^{\delta(x)p'(x)}\,dx\Big)^{1/\alpha}\|v\|\\ &\le C\Big(|k|_{p'(x)}^{\tau} + |u|_{\eta(x)p'(x)}^{\beta} + |\nabla u|_{\delta(x)p'(x)}^{\theta}\Big)^{1/\alpha}\|v\|\\ &\le C(1+\|u\|^{\beta}+\|u\|^{\theta})^{1/\alpha}\|v\| < \infty,\end{aligned}$$

where

$$\alpha = \begin{cases} p'^{-}, & \text{if } |f(x,u,\nabla u)|_{p'(x)} > 1,\\ p'^{+}, & \text{if } |f(x,u,\nabla u)|_{p'(x)} \le 1,\end{cases} \qquad \tau = \begin{cases} p'^{-}, & \text{if } |k|_{p'(x)} > 1,\\ p'^{+}, & \text{if } |k|_{p'(x)} \le 1,\end{cases}$$

$$\beta = \begin{cases} (\eta p')^{+}, & \text{if } |u|_{\eta(x)p'(x)} > 1,\\ (\eta p')^{-}, & \text{if } |u|_{\eta(x)p'(x)} \le 1,\end{cases} \quad \text{and } \theta = \begin{cases} (\delta p')^{+}, & \text{if } |\nabla u|_{\delta(x)p'(x)} > 1,\\ (\delta p')^{-}, & \text{if } |\nabla u|_{\delta(x)p'(x)} \le 1.\end{cases}$$

We note that $0 \le \eta(x) < p(x) - 1$ implies $0 \le \eta(x)p'(x) < p(x)$ and $0 \le \delta(x) < (p(x)-1)/p'(x)$ implies $0 \le \delta(x)p'(x) < p(x) - 1 < p(x)$. So, there exist $C_1, C_2 > 0$ such that

$$|u|_{\eta(x)p'(x)} \le C_1|u|_{p(x)} \quad \text{and} \quad |\nabla u|_{\delta(x)p'(x)} \le C_2|\nabla u|_{p(x)}$$

respectively.
Step 2: S is continuous on Y.
Let $u_\nu \to u$ in Y. So, up to a subsequence we deduce

$$u_\nu \to u \quad \text{and} \quad \nabla u_\nu \to \nabla u \quad , and \tag{2.3}$$

$$|u_\nu(x)|^{p(x)} \le \hat{k}(x)\ ,\ |\partial u_\nu(x)/\partial x_j|^{p(x)} \le \hat{w}_j(x) \quad \text{for some} \quad \hat{k}, \hat{w}_j \in L^1(\Omega) \tag{2.4}$$

for almost all $x \in \Omega$.

Since $t \in C(\overline{\Omega})$

$$|u_\nu|_{s(x)}^{t(x)} \to |u|_{s(x)}^{t(x)} \quad \text{a.e.} \quad x \in \Omega.$$

Furthermore, since the function f is a Caratheodory function

$$f(x,u_\nu,\nabla u_\nu) \to f(x,u,\nabla u) \quad \text{a.e.} \quad x \in \Omega,$$

Thus , we get

$$fx,u_\nu,\nabla u_\nu)|u_\nu|_{s(x)}^{t(x)} \to f(x,u,\nabla u)|u|_{s(x)}^{t(x)} \quad \text{a.e.} \quad x \in \Omega.$$

It follows from (F_1) and (2.4) that

$$\begin{aligned}&\left|f(x,u_\nu,\nabla u_\nu)|u_\nu|_{s(x)}^{t(x)}-f(x,u,\nabla u)|u|_{s(x)}^{t(x)}\right|^{p'(x)}\\&\le C2^{p'^+}\left[|f(x,u_\nu,\nabla u_\nu)|^{p'^+}+|f(x,u_\nu,\nabla u_\nu)|^{p'^+}\right]\\&\le C(1+\hat{k}(x)+|\hat{w}|^{\alpha(x)}),\qquad \hat{w}=(\hat{w}_1,\hat{w}_2,\cdots,\hat{w}_n)\end{aligned}$$

But, $C(1+k(x)+|\hat{w}|^{\alpha(.)})\in L^1(\Omega)$; thus, applying the Dominated Convergence Theorem with (2.3), we obtain

$$\lim_{\nu\to\infty}\int_\Omega\left|f(x,u_\nu,\nabla u_\nu)|u_\nu|_{s(x)}^{t(x)}-f(x,u,\nabla u)|u|_{s(x)}^{t(x)}\right|^{p'(x)}dx=0$$

This implies that

$$\lim_{\nu\to\infty}\left|f(x,u_\nu,\nabla u_\nu)|u_\nu|_{s(x)}^{t(x)}-f(x,u,\nabla u)|u|_{s(x)}^{t(x)}\right|_{p'(x)}=0 \tag{2.5}$$

Hence by direct computations we get

$$\begin{aligned}|\langle Su_\nu,v\rangle-\langle Su,v\rangle|&\le\int_\Omega\left|f(x,u_\nu,\nabla u_\nu)|u_\nu|_{s(x)}^{t(x)}-f(x,u,\nabla u)|u|_{s(x)}^{t(x)}\right||v|\,dx\\&\le C\left|f(x,u_\nu,\nabla u_\nu)|u_\nu|_{s(x)}^{t(x)}-f(x,u,\nabla u)|u|_{s(x)}^{t(x)}\right|_{p'(x)}\|v\|\end{aligned}$$

therefore, from (2.5)

$$|Su_\nu-Su|\le C\left|f(x,u_\nu,\nabla u_\nu)|u_\nu|_{s(x)}^{t(x)}-f(x,u,\nabla u)|u|_{s(x)}^{t(x)}\right|_{p'(x)}\to 0 \tag{2.6}$$

Therefore, by the convergence principle in Banach spaces , $Su_\nu\to Su$ in Y^*.

Now, repeating the same arguments as in steps 1 and 2, we get that the superposition operator $\Upsilon: L^{p(x)}(\Omega)\times L^{p(x)}(\Omega,\mathbb{R}^n)\to L^{p'(x)}(\Omega)$ defined by

$$\Upsilon(u,v)(x)=f(x,u,v)|u|_{s(x)}^{t(x)}$$

is bounded and continuous. Moreover, the linear operator $J: Y\to L^{p(x)}(\Omega)\times L^{p(x)}(\Omega,\mathbb{R}^n)$ defined by

$$J(u)=(u,\nabla u)$$

is bounded. The canonical linear embedding $I: Y\to L^{p(x)}(\Omega)$ is compact by Rellich's embedding theorem, so the adjoint operator $I^*: L^{p'(x)}(\Omega)\to Y^*$ given by

$$(I^*v)(u)=\int_\Omega vu\,dx$$

is a compact embedding. From the relation $S=I^*\circ\Upsilon\circ J$ it follows that $S: Y\to Y^*$ is compact. Thus, S is quasimonotone. □

3. Main Results

We shall prove the solvability of problem (1.1) by using of the degree theory. Define the mappings $F, S : Y \to Y^*$ by

$$\langle Fu, v\rangle = M\Big(\int_\Omega A(x,\nabla u)dx\Big)\int_\Omega a(x,\nabla u)\nabla v\,dx + \int_\Omega |u|^{p(x)-2}uv\,dx$$

and

$$\langle Su, v\rangle = -\int_\Omega f(x,u,\nabla u)|u|^{t(x)}_{s(x)}v\,dx$$

for any $u, v \in Y$.

Then $u \in Y$ is a weak solution of (1.1) if and only if

$$Fu = -Su.$$

Theorem 3.1. *Let (M_0), (F_1) and (F_2) hold. Then problem (1.1) has a weak solution.*

Proof. First, we consider the problem

$$-M\Big(\int_\Omega (A(x,\nabla u) + \frac{1}{p(x)}|u|^{p(x)})\,dx\Big)\Big[\operatorname{div}(a(x,\nabla u)) - |u|^{p(x)-2}u\Big] = v \quad \text{in } \Omega$$
$$u = constant \quad \text{on } \partial\Omega,$$
$$\int_{\partial\Omega} a(x,\nabla u).\nu d\Gamma = 0.$$

From theorem 3.1 in [6], this problem admits a unique weak solution for each $v \in Y^*$. By Proposition 2.5 the operator $F : Y \to Y^*$ is bounded, continuous, uniformly monotone, coercive and satisfies condition (S_+).

Set $X = Y^*$ and identify X^* with Y thanks to the reflexivity. Hence the Minty-Browder theorem implies that the inverse operator $T := F^{-1} : X \to X^*$ exists and is bounded . Besides, it is continuous and satisfies condition (S_+) because F is continuous and satisfies condition (S_+), and T is bounded. Furthermore, from Lemma 2.2 $S : X^* \to X$ is bounded, continuous and quasimonotone. Therefore $u \in Y$ is a weak solution of (1.1) if and only if

$$u = Tv \quad \text{and } v + (S\circ T)v = 0 \quad \text{for each } v \in Y^*. \tag{3.1}$$

We shall use the topological degree theory in Section 2 to solve (3.1). Let us consider the set

$$B = \Big\{v \in X \ : \exists \lambda \in [0,1] \text{ such that } v + \lambda(S\circ T)v = 0\Big\}$$

Next, we prove that B is bounded in X.

For $v \in B$, i.e. $v \in X$and $v + \lambda(S\circ T)v = 0$ for some $\lambda \in [0,1]$ we have

$$\langle Fu, u\rangle = \langle v, Tv\rangle = -\lambda\langle (S\circ T)v, Tv\rangle$$

Hence

$$m_0 C_1 \int_\Omega |\nabla u|^{p(x)}\,dx \le \lambda \int_\Omega f(x,u,\nabla u)|u|_{s(x)}^{t(x)} u\,dx$$
$$\le \int_\Omega \big(k(x)|u| + |u|^{\eta(x)+1} + |\nabla u|^{\delta(x)}|u|\big)\,dx$$

Then

$$\|u\|^\sigma \le C\big(\|u\| + \|u\|^\beta + \|u\|^{(\vartheta/\gamma)+1}\big) \tag{3.2}$$

where

$$\sigma = \begin{cases} p^-, & \text{if } \|u\| > 1, \\ p^+, & \text{if } \|u\| \le 1, \end{cases} \qquad \gamma = \begin{cases} (p')^-, & \text{if } ||\nabla u|^{\delta(x)}|_{p'(x)} > 1, \\ (p')^+, & \text{if } ||\nabla u|^{\delta(x)}|_{p'(x)} \le 1. \end{cases}$$

From hypothesis (F_2) and (3.2), it follows that $\{u = Tv : v \in B\}$ is bounded. Since the operator S is bounded, we infer that there exists R such that

$$\|v\|_X \le R$$

So, $Iv + \lambda(S \circ T)v \neq 0 \quad \text{on} \quad \partial B(0,R)\ \ \forall \lambda \in [0,1]$

Furthermore, from Lemma 2.1 we can claim that

$$I + \lambda(S \circ T) \in \mathfrak{F}_T(\overline{B(0,R)}) \quad \text{and} \quad I = F \circ T \in \mathfrak{F}_T(\overline{B(0,R)})$$

Now, we define the homotopy $H : [0,1] \times \overline{B(0,R)} \to X$ given by

$$H(\lambda, v) = Iv + \lambda(S \circ T)v \quad \text{for} \quad (\lambda, v) \in [0,1] \times \overline{B(0,R)}$$

Hence, by the homotopy invariance and normalization properties in Theorem 2.1 we see that

$$d(I + (S \circ T), \overline{B(0,R)}, 0) = d(I, \overline{B(0,R)}, 0) = 1$$

Thus, by Theorem 2.1 (a), we can find $v \in \overline{B(0,R)}$ such that

$$v + (S \circ T)v = 0$$

Therefore, we have that $u = Tv$ is a weak solution of (1.1). The proof is now complete. □

Next, we use the degree theory of (S_+) type mappings to prove the second result of this paper.

For (S_+) mapping theory, including the degree theory and the surjection theorem, we refer the reader to [36, 40].

Theorem 3.2. *Assume that* (M_0)*and* (H_0) *hold. If* $\beta < p^-$ *and* $\frac{\theta}{\gamma} + 1 < p^-$ *problem* (1.1) *has a weak solution .*

Proof. The mappings $F, S : Y \to Y^*$ defined by

$$\langle Fu, v\rangle = M\Big(\int_\Omega A(x,\nabla u)dx\Big)\int_\Omega a(x,\nabla u)\nabla v\,dx + \int_\Omega |u|^{p(x)-2}uv\,dx$$
$$\langle Su, v\rangle = -\int_\Omega f(x,u,\nabla u)|u|^{t(x)}_{s(x)}v\,dx, \quad u, v \in Y.$$

are respectively of type (S_+) and compact.

It is clear that $u \in Y$ is a solution of (1.1) if an only if $I(u) := F(u) + S(u) = 0$. From the above analysis, it is obvious that $I : Y \to Y^*$ is continuous and bounded. Noting that the sum of an (S_+) type mapping and a compact mapping is of type (S_+) , it follows that the mapping $I = F + S$ is of type (S_+) .

Then, proceeding similarly as in the proof of Lemma 2.2, for $\|u\| > 1$ we have that

$$\langle I(u), u\rangle \geq m_0\|u\|^{p^-}\Big(1 - c_1\|u\|^{1-p^-} + c_2\|u\|^{\beta-p^-} + c_3\|u\|^{(\vartheta/\gamma)+1-p^-}\Big) \tag{3.3}$$

Choosing $\|u\| = R$ large enough, we deduce from (3.3) that

$$\langle I(u), u\rangle > 0 \text{ for all } u \in Y \text{ such that } \|u\| = R.$$

By the topological degree theory for (S_+) type mappings, for $R > 0$ we conclude that

$$deg(L, B(0,R), 0) = 1$$

Therefore the equation $I(u) = 0$ has at least one solution $u \in B(0,R)$. Furthermore

$$\lim_{\|u\|\to+\infty}\frac{\langle I(u), u\rangle}{\|u\|} \geq c_9\|u\|^{p^- - 1} = +\infty$$

So, the mapping I is coercive, and hence, by the surjection theorem for the pseudomonotone mappings (see [40, theorem 27.A]), the mapping I is surjective. This completes the proof. □

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