Some Properties Subclasses of Analytic Functions

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ABSTRACT. The object of the present paper is to discuss some interesting properties of analytic functions $f(z)$ associated with the subclasses $\mathcal{D}(\beta_1, \beta_2, \beta_3; \lambda)$, $\mathcal{G}(\theta, \alpha)$ and $\mathcal{Q}(\theta, \alpha)$. Also, radius problems of $\frac{1}{\delta}f(\delta z)$ for $f(z)$ in the class $\mathcal{D}(\beta_1, \beta_2, \beta_3; \lambda)$, $\mathcal{G}(\theta, \alpha)$ and $\mathcal{Q}(\theta, \alpha)$ are considered.

1. Introduction and Definitions

Let $\mathcal{A}$ denote the class of the normalized functions of the form

$$(1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the open unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$. For a function $f(z)$ in the class $\mathcal{A}$, Sălăgean [1] defined the differential operator D^k , by

$$\begin{aligned} D^0 f(z) &= f(z), \\ D^1 f(z) &= Df(z) = zf'(z), \\ D^2 f(z) &= D(D^1 f(z)) = z(zf'(z))', \end{aligned}$$

and

$$D^k f(z) = D(D^{k-1} f(z)), \quad (k \in \mathbb{N}).$$

Thus

$$(2) \quad D^k f(z) = z + \sum_{n=2}^{\infty} n^k a_n z^n, \quad (k \in \mathbb{N} \cup \{0\}).$$

Received October 13, 2011; accepted March 27, 2013.

2010 Mathematics Subject Classification: 30C45.

Key words and phrases: Analytic, univalent functions, Cauchy-Schwarz inequality, radius problems.

Let $\mathcal{G}(\alpha)$ denote the subclass of $\mathcal{A}$ consisting of functions $f(z)$ which satisfy

$$(3) \quad \operatorname{Re} \left\{ \frac{D^3 f(z)}{z} \right\} > \alpha$$

for some $\alpha (0 \leq \alpha < 1)$ and for all $z \in \mathcal{U}$. Also, Let $\mathcal{Q}(\alpha)$ denote the subclass of $\mathcal{A}$ consisting of functions $f(z)$ which satisfy

$$\operatorname{Re} \left\{ \frac{D^2 f(z)}{D^1 f(z)} \right\} > \alpha$$

for some $\alpha (0 \leq \alpha < 1)$ and for all $z \in \mathcal{U}$.

For analytic functions $f(z)$, Uyanik and Owa [2], obtained some interesting properties for analytic functions in the subclass $\mathcal{A}(\beta_1, \beta_2, \beta_3; \lambda)$ defined by

$$\left| \beta_1 z \left(\frac{f(z)}{z} \right)' + \beta_2 z^2 \left(\frac{f(z)}{z} \right)'' + \beta_3 z^3 \left(\frac{f(z)}{z} \right)''' \right| \leq \lambda$$

$$(\beta_1, \beta_2, \beta_3 \in \mathbb{C}; \lambda > 0; z \in \mathcal{U}),$$

associated with close-to-convex functions and starlike functions of order α .

In this paper, we define the following subclass of analytic functions.

Definition 1.1. A function $f(z)$ belonging to $\mathcal{A}$ is said to be in the class $\mathcal{D}(\beta_1, \beta_2, \beta_3; \lambda)$ if it satisfies

$$(4) \quad \left| \beta_1 z \left(\frac{D^2 f(z)}{z} \right)' + \beta_2 z^2 \left(\frac{D^2 f(z)}{z} \right)'' + \beta_3 z^3 \left(\frac{D^2 f(z)}{z} \right)''' \right| \leq \lambda \quad (z \in \mathcal{U}),$$

for some complex numbers $\beta_1, \beta_2, \beta_3$, and for some real $\lambda > 0$.

Example 1.2. Let us consider the function $f_\gamma(z)$, $\gamma \in \mathbb{R}$, given by

$$f_\gamma(z) = z(1+z)^\gamma,$$

then, we have

$$(5) \quad D^2 f_\gamma(z) = z + \sum_{n=2}^{\infty} n^2 \binom{\gamma}{n-1} z^n$$

where

$$\binom{\gamma}{n-1} = \frac{\gamma(\gamma-1)(\gamma-2)\dots(\gamma-n+2)}{(n-1)!}.$$

From (5), it follows that

$$\begin{aligned} & \left| \beta_1 z \left(\frac{D^2 f_\gamma(z)}{z} \right)' + \beta_2 z^2 \left(\frac{D^2 f_\gamma(z)}{z} \right)'' + \beta_3 z^3 \left(\frac{D^2 f_\gamma(z)}{z} \right)''' \right| \\ &= \left| \sum_{n=2}^{\infty} n^2(n-1) \binom{\gamma}{n-1} (\beta_1 + (n-2)\beta_2 + (n-2)(n-3)\beta_3) z^{n-1} \right|. \end{aligned}$$

Therefore, if $\gamma = 1$, then

$$\left| \beta_1 z \left(\frac{D^2 f_1(z)}{z} \right)' + \beta_2 z^2 \left(\frac{D^2 f_1(z)}{z} \right)'' + \beta_3 z^3 \left(\frac{D^2 f_1(z)}{z} \right)''' \right| = 4|\beta_1 z| \leq 4|\beta_1|.$$

This implies that $f_1(z) \in \mathcal{D}(\beta_1, \beta_2, \beta_3; \lambda)$ for $\lambda \geq 4|\beta_1|$. If $\gamma = 2$, then

$$\begin{aligned} & \left| \beta_1 z \left(\frac{D^2 f_2(z)}{z} \right)' + \beta_2 z^2 \left(\frac{D^2 f_2(z)}{z} \right)'' + \beta_3 z^3 \left(\frac{D^2 f_2(z)}{z} \right)''' \right| \\ &= |8\beta_1 z + 18(\beta_1 + \beta_2)z^2| \leq 26|\beta_1| + 18|\beta_2|. \end{aligned}$$

Therefore, $f_2(z) \in \mathcal{D}(\beta_1, \beta_2, \beta_3; \lambda)$ for $\lambda \geq 26|\beta_1| + 18|\beta_2|$. Further, if $\gamma = 3$, then we have

$$\begin{aligned} & \left| \beta_1 z \left(\frac{D^2 f_3(z)}{z} \right)' + \beta_2 z^2 \left(\frac{D^2 f_3(z)}{z} \right)'' + \beta_3 z^3 \left(\frac{D^2 f_3(z)}{z} \right)''' \right| \\ &= |12\beta_1 z + 54(\beta_1 + \beta_2)z^2 + 48(\beta_1 + 2\beta_2 + 2\beta_3)z^3| \\ &\leq 114|\beta_1| + 150|\beta_2| + 96|\beta_3|. \end{aligned}$$

Thus, $f_3(z) \in \mathcal{D}(\beta_1, \beta_2, \beta_3; \lambda)$ for $\lambda \geq 114|\beta_1| + 150|\beta_2| + 96|\beta_3|$.

Now, let $\mathcal{A}_\theta$ denote the subclass of $\mathcal{A}$ consisting of functions $f(z)$ with

$$a_n = |a_n| e^{i((n-1)\theta + \pi)} \quad (n = 2, 3, \dots).$$

Also, we introduce the subclasses $\mathcal{G}(\theta, \alpha)$ and $\mathcal{Q}(\theta, \alpha)$ of $\mathcal{A}_\theta$ as follows:

$$\mathcal{G}(\theta, \alpha) = \mathcal{A}_\theta \cap \mathcal{G}(\alpha) \quad \text{and} \quad \mathcal{Q}(\theta, \alpha) = \mathcal{A}_\theta \cap \mathcal{Q}(\alpha).$$

2. Properties of the Class $\mathcal{D}(\beta_1, \beta_2, \beta_3; \lambda)$

We first prove

Theorem 2.1. *If $f(z) \in \mathcal{A}$ satisfies*

$$(6) \quad \sum_{n=2}^{\infty} n^2(n-1)(|\beta_1| + (n-2)|\beta_2| + (n-2)(n-3)|\beta_3|)|a_n| \leq \lambda$$

for some complex numbers $\beta_1, \beta_2, \beta_3$ and for some real $\lambda > 0$, then $f(z) \in \mathcal{D}(\beta_1, \beta_2, \beta_3; \lambda)$.

Proof. We observe that

$$\begin{aligned} & \left| \beta_1 z \left(\frac{D^2 f(z)}{z} \right)' + \beta_2 z^2 \left(\frac{D^2 f(z)}{z} \right)'' + \beta_3 z^3 \left(\frac{D^2 f(z)}{z} \right)''' \right| \\ &= \left| \sum_{n=2}^{\infty} n^2(n-1)(\beta_1 + (n-2)\beta_2 + (n-2)(n-3)\beta_3)a_n z^{n-1} \right| \\ &\leq \sum_{n=2}^{\infty} n^2(n-1)(|\beta_1| + (n-2)|\beta_2| + (n-2)(n-3)|\beta_3|)|a_n||z|^{n-1} \\ &< \sum_{n=2}^{\infty} n^2(n-1)(|\beta_1| + (n-2)|\beta_2| + (n-2)(n-3)|\beta_3|)|a_n|. \end{aligned}$$

Therefore, if $f(z)$ satisfies the inequality (6), then $f(z) \in \mathcal{D}(\beta_1, \beta_2, \beta_3; \lambda)$. □

Next, we prove

Theorem 2.2. *If $f(z) \in \mathcal{D}(\beta_1, \beta_2, \beta_3; \lambda)$ with $\arg \beta_1 = \arg \beta_2 = \arg \beta_3 = \phi$ and $a_n = |a_n| e^{i((n-1)\theta - \phi)}$ ($n = 2, 3, \dots$), then we have*

$$\sum_{n=2}^{\infty} n^2(n-1)(|\beta_1| + (n-2)|\beta_2| + (n-2)(n-3)|\beta_3|)|a_n| \leq \lambda.$$

Proof. For $f(z) \in \mathcal{D}(\beta_1, \beta_2, \beta_3; \lambda)$, we see that

$$\begin{aligned} & \left| \beta_1 z \left(\frac{D^2 f(z)}{z} \right)' + \beta_2 z^2 \left(\frac{D^2 f(z)}{z} \right)'' + \beta_3 z^3 \left(\frac{D^2 f(z)}{z} \right)''' \right| \\ &= \left| \sum_{n=2}^{\infty} n^2(n-1)(\beta_1 + (n-2)\beta_2 + (n-2)(n-3)\beta_3)a_n z^{n-1} \right| \\ &= \left| \sum_{n=2}^{\infty} n^2(n-1)(|\beta_1| + (n-2)|\beta_2| + (n-2)(n-3)|\beta_3|)|a_n| e^{i(n-1)\theta} z^{n-1} \right| \\ &\leq \lambda. \end{aligned}$$

for all $z \in \mathcal{U}$. Let us consider a point $z \in \mathcal{U}$ such that $z = |z|e^{-i\theta}$. Then we have

$$\sum_{n=2}^{\infty} n^2(n-1)(|\beta_1| + (n-2)|\beta_2| + (n-2)(n-3)|\beta_3|)|a_n||z|^{n-1} \leq \lambda.$$

Letting $|z| \rightarrow 1^-$, we obtain

$$\sum_{n=2}^{\infty} n^2(n-1)(|\beta_1| + (n-2)|\beta_2| + (n-2)(n-3)|\beta_3|)|a_n| \leq \lambda.$$

□

Corollary 2.3. If $f(z) \in \mathcal{D}(\beta_1, \beta_2, \beta_3; \lambda)$ with $\arg \beta_1 = \arg \beta_2 = \arg \beta_3 = \phi$ and $a_n = |a_n|e^{i((n-1)\theta-\phi)}$ ($n = 2, 3, \dots$), then we have

$$|a_n| \leq \frac{\lambda}{n^2(n-1)(|\beta_1| + (n-2)|\beta_2| + (n-2)(n-3)|\beta_3|)} \quad (n = 2, 3, \dots).$$

Example 2.4. Let us consider the function $f(z) \in \mathcal{D}(\beta_1, \beta_2, \beta_3; \lambda)$ with $\arg \beta_1 = \arg \beta_2 = \arg \beta_3 = \phi$ and

$$a_n = \frac{\lambda e^{i((n-1)\theta-\phi)}}{n^3(n-1)^2(|\beta_1| + (n-2)|\beta_2| + (n-2)(n-3)|\beta_3|)} \quad (n = 2, 3, \dots).$$

Then we see that

$$\begin{aligned} & \sum_{n=2}^{\infty} n^2(n-1)(|\beta_1| + (n-2)|\beta_2| + (n-2)(n-3)|\beta_3|)|a_n| \\ &= \lambda \sum_{n=2}^{\infty} \frac{1}{n(n-1)} = \lambda \sum_{n=2}^{\infty} \left(\frac{1}{n-1} - \frac{1}{n} \right) = \lambda. \end{aligned}$$

Corollary 2.5. If $f(z) \in \mathcal{D}(\beta_1, \beta_2, \beta_3; \lambda)$ with $\arg \beta_1 = \arg \beta_2 = \arg \beta_3 = \phi$ and $a_n = |a_n|e^{i((n-1)\theta-\phi)}$ ($n = 2, 3, \dots$), then we have

$$|z| - \sum_{n=2}^j |a_n||z|^n - A_j |z|^{j+1} \leq |f(z)| \leq |z| + \sum_{n=2}^j |a_n||z|^n + A_j |z|^{j+1}$$

with

$$A_j = \frac{\left(\lambda - \sum_{n=2}^j n^2(n-1)(|\beta_1| + (n-2)|\beta_2| + (n-2)(n-3)|\beta_3|)|a_n| \right)}{j(j+1)^2(|\beta_1| + (j-1)|\beta_2| + (j-1)(j-2)|\beta_3|)}$$

and

$$1 - \sum_{n=2}^j |a_n| |z|^{n-1} - B_j |z|^j \leq |f'(z)| \leq 1 + \sum_{n=2}^j |a_n| |z|^{n-1} + B_j |z|^j$$

with

$$B_j = \frac{\left(\lambda - \sum_{n=2}^j n^2(n-1)(|\beta_1| + (n-2)|\beta_2| + (n-2)(n-3)|\beta_3|) |a_n| \right)}{j(j+1)(|\beta_1| + (j-1)|\beta_2| + (j-1)(j-2)|\beta_3|)}.$$

Proof. In view of Theorem 2.1, we know that

$$\begin{aligned} & \sum_{n=j+1}^{\infty} n^2(n-1)(|\beta_1| + (n-2)|\beta_2| + (n-2)(n-3)|\beta_3|) |a_n| \\ & \leq \lambda - \sum_{n=2}^j n^2(n-1)(|\beta_1| + (n-2)|\beta_2| + (n-2)(n-3)|\beta_3|) |a_n|. \end{aligned}$$

Further, we note that

$$\begin{aligned} & j(j+1)^2(|\beta_1| + (j-1)|\beta_2| + (j-1)(j-2)|\beta_3|) \sum_{n=j+1}^{\infty} |a_n| \\ & \leq \sum_{n=j+1}^{\infty} n^2(n-1)(|\beta_1| + (n-2)|\beta_2| + (n-2)(n-3)|\beta_3|) |a_n|, \end{aligned}$$

which is equivalent to

$$\begin{aligned} \sum_{n=j+1}^{\infty} |a_n| & \leq \frac{\left(\lambda - \sum_{n=2}^j n^2(n-1)(|\beta_1| + (n-2)|\beta_2| + (n-2)(n-3)|\beta_3|) |a_n| \right)}{j(j+1)^2(|\beta_1| + (j-1)|\beta_2| + (j-1)(j-2)|\beta_3|)} \\ & = A_j. \end{aligned}$$

Thus, we have

$$|f(z)| \leq |z| + \sum_{n=2}^j |a_n| |z|^n + \sum_{n=j+1}^{\infty} |a_n| |z|^n \leq |z| + \sum_{n=2}^j |a_n| |z|^n + A_j |z|^{j+1}$$

and

$$|f(z)| \geq |z| - \sum_{n=2}^j |a_n| |z|^n - \sum_{n=j+1}^{\infty} |a_n| |z|^n \geq |z| - \sum_{n=2}^j |a_n| |z|^n - A_j |z|^{j+1}.$$

Next, we observe that

$$\begin{aligned}
 & j(j+1)(|\beta_1| + (j-1)|\beta_2| + (j-1)(j-2)|\beta_3|) \sum_{n=j+1}^{\infty} n |a_n| \\
 & \leq \sum_{n=j+1}^{\infty} n^2(n-1)(|\beta_1| + (n-2)|\beta_2| + (n-2)(n-3)|\beta_3|) |a_n| \\
 & \leq \lambda - \sum_{n=2}^j n^2(n-1)(|\beta_1| + (n-2)|\beta_2| + (n-2)(n-3)|\beta_3|) |a_n|,
 \end{aligned}$$

which implies that

$$\begin{aligned}
 \sum_{n=j+1}^{\infty} n |a_n| & \leq \frac{\left(\lambda - \sum_{n=2}^j n^2(n-1)(|\beta_1| + (n-2)|\beta_2| + (n-2)(n-3)|\beta_3|) |a_n| \right)}{j(j+1)(|\beta_1| + (j-1)|\beta_2| + (j-1)(j-2)|\beta_3|)} \\
 & = B_j.
 \end{aligned}$$

Therefore, we obtain that

$$|f'(z)| \leq 1 + \sum_{n=2}^j n |a_n| |z|^{n-1} + \sum_{n=j+1}^{\infty} n |a_n| |z|^{n-1} \leq 1 + \sum_{n=2}^j |a_n| |z|^{n-1} + B_j |z|^j$$

and

$$|f'(z)| \geq 1 - \sum_{n=2}^j n |a_n| |z|^{n-1} - \sum_{n=j+1}^{\infty} n |a_n| |z|^{n-1} \geq 1 - \sum_{n=2}^j |a_n| |z|^{n-1} - B_j |z|^j.$$

□

3. Radius Problem for the Class $\mathcal{G}(\theta, \alpha)$

To obtain the radius problem for the class $\mathcal{G}(\theta, \alpha)$, we need the following lemma.

Lemma 3.1. *If $f(z) \in \mathcal{G}(\theta, \alpha)$, then*

$$(7) \quad \sum_{n=2}^{\infty} n^3 |a_n| \leq 1 - \alpha.$$

Proof. Let $f(z) \in \mathcal{G}(\theta, \alpha)$. Then, we have

$$\begin{aligned}
 \operatorname{Re} \left\{ \frac{D^3 f(z)}{z} \right\} &= \operatorname{Re} \left\{ 1 + \sum_{n=2}^{\infty} n^3 a_n z^{n-1} \right\} = \operatorname{Re} \left\{ 1 + \sum_{n=2}^{\infty} n^3 |a_n| e^{i((n-1)\theta + \pi)} z^{n-1} \right\} \\
 &= \operatorname{Re} \left\{ 1 - \sum_{n=2}^{\infty} n^3 |a_n| e^{i((n-1)\theta)} z^{n-1} \right\} > \alpha
 \end{aligned}$$

for all $z \in \mathcal{U}$. Let us consider a point $z \in \mathcal{U}$ such that $z = |z|e^{-i\theta}$. Then we have

$$1 - \sum_{n=2}^{\infty} n^3 |a_n| |z|^{n-1} > \alpha$$

Letting $|z| \rightarrow 1^-$, we obtain the inequality (7). $\square$

Corollary 3.2. If $f(z) \in \mathcal{G}(\theta, \alpha)$, then

$$|a_n| \leq \frac{1-\alpha}{n^3} \quad (n = 2, 3, \dots).$$

Remark 3.3. By Lemma 3.1, we observe that if $f(z) \in \mathcal{G}(\theta, \alpha)$, then

$$\sum_{n=2}^{\infty} n^2(n-1) |a_n| \leq \sum_{n=2}^{\infty} n^3 |a_n| \leq 1 - \alpha.$$

Applying Theorem 2.1 and Lemma 3.1, we derive

Theorem 3.4. If $f(z) \in \mathcal{G}(\theta, \alpha)$, and $\delta \in \mathbb{C}$ ($0 < |\delta| < 1$). Then the function $\frac{1}{\delta}f(\delta z) \in \mathcal{D}(\beta_1, \beta_2, \beta_3; \lambda)$ for $(0 < |\delta| \leq |\delta_0(\lambda)|)$, where $|\delta_0(\lambda)|$ is the smallest positive root of the equation

$$\begin{aligned} & |\beta_1| \frac{|\delta| \sqrt{2(|\delta|^2 + 2)(1 - \alpha)}}{(1 - |\delta|^2)^2} \\ (8) \quad & + |\beta_2| \frac{|\delta|^2 \sqrt{6(3|\delta|^4 + 14|\delta|^2 + 3)(1 - \alpha - 4|a_2|^2)}}{(1 - |\delta|^2)^3} \\ & + |\beta_3| \frac{|\delta|^3 \sqrt{48(6|\delta|^6 + 52|\delta|^4 + 43|\delta|^2 + 4)(1 - \alpha - 4|a_2|^2 - 18|a_3|^2)}}{(1 - |\delta|^2)^4} \\ & = \lambda \\ & \text{in } 0 < |\delta| < 1. \end{aligned}$$

Proof. For $f(z) \in \mathcal{G}(\theta, \alpha)$, we see that

$$\frac{1}{\delta}f(\delta z) = z + \sum_{n=2}^{\infty} \delta^{n-1} a_n z^n$$

and

$$\sum_{n=2}^{\infty} n^2(n-1) |a_n|^2 \leq 1 - \alpha.$$

Thus, to show that $\frac{1}{\delta}f(\delta z) \in \mathcal{D}(\beta_1, \beta_2, \beta_3; \lambda)$, from Theorem 2.1, it is sufficient to prove that

$$\sum_{n=2}^{\infty} n^2(n-1)(|\beta_1| + (n-2)|\beta_2| + (n-2)(n-3)|\beta_3|) |\delta|^{n-1} |a_n| \leq \lambda.$$

Applying Cauchy-Schwarz inequality, we note that

$$\begin{aligned} & \sum_{n=2}^{\infty} n^2(n-1)(|\beta_1| + (n-2)|\beta_2| + (n-2)(n-3)|\beta_3|) |\delta|^{n-1} |a_n| \\ & \leq \frac{|\beta_1|}{|\delta|} \left(\sum_{n=2}^{\infty} n^2(n-1) |\delta|^{2n} \right)^{\frac{1}{2}} \left(\sum_{n=2}^{\infty} n^2(n-1) |a_n|^2 \right)^{\frac{1}{2}} \\ & \quad + \frac{|\beta_2|}{|\delta|} \left(\sum_{n=3}^{\infty} n^2(n-1)(n-2)^2 |\delta|^{2n} \right)^{\frac{1}{2}} \left(\sum_{n=3}^{\infty} n^2(n-1) |a_n|^2 \right)^{\frac{1}{2}} \\ & \quad + \frac{|\beta_3|}{|\delta|} \left(\sum_{n=4}^{\infty} n^2(n-1)(n-2)^2(n-3)^2 |\delta|^{2n} \right)^{\frac{1}{2}} \left(\sum_{n=4}^{\infty} n^2(n-1) |a_n|^2 \right)^{\frac{1}{2}} \\ & \leq \frac{|\beta_1|}{|\delta|} \left(\sum_{n=2}^{\infty} n^2(n-1) |\delta|^{2n} \right)^{\frac{1}{2}} \sqrt{1-\alpha} \\ & \quad + \frac{|\beta_2|}{|\delta|} \left(\sum_{n=3}^{\infty} n^2(n-1)(n-2)^2 |\delta|^{2n} \right)^{\frac{1}{2}} \sqrt{1-\alpha-4|a_2|^2} \\ (9) \quad & + \frac{|\beta_3|}{|\delta|} \left(\sum_{n=4}^{\infty} n^2(n-1)(n-2)^2(n-3)^2 |\delta|^{2n} \right)^{\frac{1}{2}} \sqrt{1-\alpha-4|a_2|^2-18|a_3|^2}. \end{aligned}$$

Making use of

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad (|x| < 1),$$

we have

$$(10) \quad \sum_{n=2}^{\infty} n^2(n-1)x^n = \frac{2x^2(x+2)}{(1-x)^4}.$$

Since

$$\sum_{n=3}^{\infty} (n-2)x^{n-1} = x^2 \left(\sum_{n=3}^{\infty} (n-2)x^{n-3} \right) = x^2 \left(\sum_{n=3}^{\infty} x^{n-2} \right)' = \frac{x^2}{(1-x)^2},$$

we see that

$$\sum_{n=3}^{\infty} (n-1)(n-2)^2 x^n = x^3 \left(\frac{x^2}{(1-x)^2} \right)'' = \frac{2x^3 + 4x^4}{(1-x)^4}.$$

and thus, we obtain

$$\sum_{n=3}^{\infty} n(n-1)(n-2)^2 x^n = \frac{6x^3 + 18x^4}{(1-x)^5}$$

which yields

$$(11) \quad \sum_{n=3}^{\infty} n^2(n-1)(n-2)^2 x^n = \frac{6x^3(3x^2 + 14x + 3)}{(1-x)^6}.$$

Furthermore, we have

$$\begin{aligned} \sum_{n=4}^{\infty} (n-1)(n-2)^2(n-3)^2 x^n &= x^4 \left(\sum_{n=4}^{\infty} (n-1)(n-2)^2(n-3)^2 x^{n-4} \right) \\ &= x^4 \left(\sum_{n=4}^{\infty} (n-2)(n-3)x^{n-1} \right)''' , \end{aligned}$$

but

$$\sum_{n=4}^{\infty} (n-2)(n-3)x^{n-1} = x^3 \left(\sum_{n=4}^{\infty} (n-2)(n-3)x^{n-4} \right) = \frac{2x^3}{(1-x)^3}$$

thus, we have

$$\sum_{n=4}^{\infty} (n-1)(n-2)^2(n-3)^2 x^n = \frac{12x^4 + 72x^5 + 36x^6}{(1-x)^6},$$

which yields

$$\sum_{n=4}^{\infty} n(n-1)(n-2)^2(n-3)^2 x^n = \frac{48x^4 + 384x^5 + 288x^6}{(1-x)^7}.$$

This gives us that

$$(12) \quad \sum_{n=4}^{\infty} n^2(n-1)(n-2)^2(n-3)^2 x^n = \frac{48x^4(6x^3 + 52x^2 + 43x + 4)}{(1-x)^8}.$$

Therefore, from (9)- (12) with $|\delta|^2 = x$, we obtain

$$\begin{aligned}
 & \sum_{n=2}^{\infty} n^2(n-1)(|\beta_1| + (n-2)|\beta_2| + (n-2)(n-3)|\beta_3|)|\delta|^{n-1}|a_n| \\
 \leq & |\beta_1| \frac{|\delta| \sqrt{2(|\delta|^2 + 2)(1-\alpha)}}{(1-|\delta|^2)^2} \\
 & + |\beta_2| \frac{|\delta|^2 \sqrt{6(3|\delta|^4 + 14|\delta|^2 + 3)(1-\alpha - 4|a_2|^2)}}{(1-|\delta|^2)^3} \\
 & + |\beta_3| \frac{|\delta|^3 \sqrt{48(6|\delta|^6 + 52|\delta|^4 + 43|\delta|^2 + 4)(1-\alpha - 4|a_2|^2 - 18|a_3|^2)}}{(1-|\delta|^2)^4}
 \end{aligned}$$

Now, let us consider the complex number δ ($0 < |\delta| < 1$) such that

$$\begin{aligned}
 & |\beta_1| \frac{|\delta| \sqrt{2(|\delta|^2 + 2)(1-\alpha)}}{(1-|\delta|^2)^2} \\
 & + |\beta_2| \frac{|\delta|^2 \sqrt{6(3|\delta|^4 + 14|\delta|^2 + 3)(1-\alpha - 4|a_2|^2)}}{(1-|\delta|^2)^3} \\
 & + |\beta_3| \frac{|\delta|^3 \sqrt{48(6|\delta|^6 + 52|\delta|^4 + 43|\delta|^2 + 4)(1-\alpha - 4|a_2|^2 - 18|a_3|^2)}}{(1-|\delta|^2)^4} \\
 = & \lambda
 \end{aligned}$$

If we define the function $h(|\delta|)$ by

$$\begin{aligned}
 h(|\delta|) = & |\beta_1| |\delta| (1-|\delta|^2)^2 \sqrt{2(|\delta|^2 + 2)(1-\alpha)} \\
 & + |\beta_2| |\delta|^2 (1-|\delta|^2) \sqrt{6(3|\delta|^4 + 14|\delta|^2 + 3)(1-\alpha - 4|a_2|^2)} \\
 & + |\beta_3| |\delta|^3 \sqrt{48(6|\delta|^6 + 52|\delta|^4 + 43|\delta|^2 + 4)(1-\alpha - 4|a_2|^2 - 18|a_3|^2)} \\
 & - \lambda (1-|\delta|^2)^4,
 \end{aligned}$$

then we have $h(0) = -\lambda < 0$ and $h(1) = 4\sqrt{3}|\beta_3| \sqrt{105(1-\alpha - 4|a_2|^2 - 18|a_3|^2)} > 0$. This means that there exists some δ_0 such that $h(|\delta_0|) = 0$ ($0 < |\delta_0| < 1$). This completes the proof of the theorem. $\square$

4. Radius Problem for the Class $\mathcal{Q}(\theta, \alpha)$

For the class $\mathcal{Q}(\theta, \alpha)$, we prove the following lemma.

Lemma 4.1. *If $f(z) \in \mathcal{Q}(\theta, \alpha)$, then*

$$(13) \quad \sum_{n=2}^{\infty} n(n-\alpha) |a_n| \leq 1 - \alpha.$$

Proof. Let $f(z) \in \mathcal{Q}(\theta, \alpha)$. Then, we have

$$\begin{aligned} \operatorname{Re} \left\{ \frac{D^2 f(z)}{D^1 f(z)} \right\} &= \operatorname{Re} \left\{ \frac{1 + \sum_{n=2}^{\infty} n^2 a_n z^{n-1}}{1 + \sum_{n=2}^{\infty} n a_n z^{n-1}} \right\} \\ &= \operatorname{Re} \left\{ \frac{1 - \sum_{n=2}^{\infty} n^2 |a_n| e^{i(n-1)\theta} z^{n-1}}{1 - \sum_{n=2}^{\infty} n |a_n| e^{i(n-1)\theta} z^{n-1}} \right\} > \alpha \end{aligned}$$

for all $z \in \mathcal{U}$. Let us consider a point $z \in \mathcal{U}$ such that $z = |z| e^{-i\theta}$. Then we have

$$\frac{1 - \sum_{n=2}^{\infty} n^2 |a_n| |z|^{n-1}}{1 - \sum_{n=2}^{\infty} n |a_n| |z|^{n-1}} > \alpha$$

Letting $|z| \rightarrow 1^-$, we obtain the inequality (13). $\square$

Corollary 4.2. If $f(z) \in \mathcal{Q}(\theta, \alpha)$, then

$$|a_n| \leq \frac{1 - \alpha}{n(n - \alpha)} \quad (n = 2, 3, \dots).$$

Remark 4.3. If $f(z) \in \mathcal{Q}(\theta, \alpha)$, then

$$\sum_{n=2}^{\infty} n^2(n-1) |a_n| \leq \sum_{n=2}^{\infty} n(n-\alpha) |a_n| \leq 1 - \alpha.$$

Applying Theorem 2.1, Lemma 4.1 and using the same technique as in the proof of Theorem 3.4, we derive

Theorem 4.4. *If $f(z) \in \mathcal{Q}(\theta, \alpha)$, and $\delta \in \mathbb{C}$ ($0 < |\delta| < 1$). Then the function $\frac{1}{\delta} f(\delta z) \in \mathcal{D}(\beta_1, \beta_2, \beta_3; \lambda)$ for $(0 < |\delta| \leq |\delta_0(\lambda)|)$, where $|\delta_0(\lambda)|$ is the smallest positive*

root of the equation

$$\begin{aligned}
 & |\beta_1| \frac{|\delta| \sqrt{2(|\delta|^2 + 2)(1 - \alpha)}}{(1 - |\delta|^2)^2} \\
 (14) \quad & + |\beta_2| \frac{|\delta|^2 \sqrt{6(3|\delta|^4 + 14|\delta|^2 + 3)(1 - \alpha - 4|a_2|^2)}}{(1 - |\delta|^2)^3} \\
 & + |\beta_3| \frac{|\delta|^3 \sqrt{48(6|\delta|^6 + 52|\delta|^4 + 43|\delta|^2 + 4)(1 - \alpha - 4|a_2|^2 - 18|a_3|^2)}}{(1 - |\delta|^2)^4} \\
 & = \lambda \\
 & \text{in } 0 < |\delta| < 1.
 \end{aligned}$$

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