A Note on S -Noetherian Domains

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ABSTRACT. Let D be an integral domain, t be the so-called t -operation on D , and S be a (not necessarily saturated) multiplicative subset of D . In this paper, we study the Nagata ring of S -Noetherian domains and locally S -Noetherian domains. We also investigate the t -Nagata ring of t -locally S -Noetherian domains. In fact, we show that if S is an anti-archimedean subset of D , then D is an S -Noetherian domain (respectively, locally S -Noetherian domain) if and only if the Nagata ring $D[X]_N$ is an S -Noetherian domain (respectively, locally S -Noetherian domain). We also prove that if S is an anti-archimedean subset of D , then D is a t -locally S -Noetherian domain if and only if the polynomial ring $D[X]$ is a t -locally S -Noetherian domain, if and only if the t -Nagata ring $D[X]_{N_v}$ is a t -locally S -Noetherian domain.

1. Introduction

1.1 Star-operations

To help readers better understanding this paper, we briefly review some definitions and notation related to star-operations. Let D be an integral domain with quotient field K , and let $\mathbf{F}(D)$ be the set of nonzero fractional ideals of D . For an $I \in \mathbf{F}(D)$, set $I^{-1} := \{x \in K \mid xI \subseteq D\}$. The mapping on $\mathbf{F}(D)$ defined by $I \mapsto I_v := (I^{-1})^{-1}$ is called the v -operation on D , and the mapping on $\mathbf{F}(D)$ defined by $I \mapsto I_t := \bigcup \{J_v \mid J \text{ is a nonzero finitely generated fractional subideal of } I\}$ is called the t -operation on D ; and the mapping on $\mathbf{F}(D)$ defined by $I \mapsto I_w := \{a \in K \mid Ja \subseteq I \text{ for some finitely generated ideal } J \text{ of } D \text{ with } J_v = D\}$ is called the w -operation on D . It is easy to see that $I \subseteq I_w \subseteq I_t \subseteq I_v$ for all $I \in \mathbf{F}(D)$; and if an $I \in \mathbf{F}(D)$ is finitely generated, then $I_v = I_t$. An $I \in \mathbf{F}(D)$ is called a t -ideal (respectively, w -ideal) of D if $I_t = I$ (respectively, $I_w = I$). A

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maximal t -ideal means a t -ideal which is maximal among proper integral t -ideals. It is well known that a maximal t -ideal of D always exists if D is not a field. We say that D is *of finite character* (respectively, *of finite t -character*) if each nonzero nonunit in D belongs to only finitely many maximal ideals (respectively, maximal t -ideals) of D .

1.2 S -Noetherian domains

Let D be an integral domain and S a (not necessarily saturated) multiplicative subset of D . In [4], the authors introduced the concept of “almost finitely generated” to study Querre’s characterization of divisorial ideals in integrally closed polynomial rings. Later, the authors in [2] generalized the concept of (almost) finitely generatedness and defined a general notion of Noetherian domains. (Recall that D is a *Noetherian domain* if it satisfies the ascending chain condition on integral ideals of D , or equivalently, every (prime) ideal of D is finitely generated.) To do this, they first built the notion of S -finiteness. Let I be an ideal of D . Then I is said to be *S -finite* if there exist an element $s \in S$ and a finitely generated ideal J of D such that $sI \subseteq J \subseteq I$. Also, D is called an *S -Noetherian domain* if each ideal of D is S -finite. As mentioned above, the concept of S -Noetherian domains can be regarded as a slight generalization of that of Noetherian domains, because two notions precisely coincide when S consists of units. Hence the results on S -Noetherian domains can recover known facts for Noetherian domains.

Among other results in [2], Anderson and Dumitrescu proved the Hilbert basis theorem for S -Noetherian domains, which states that if S is an anti-archimedean subset of an S -Noetherian domain D , then the polynomial ring $D[X]$ is also an S -Noetherian domain [2, Proposition 9]. (Recall that a multiplicative subset S of D is *anti-archimedean* if $\bigcap_{n \geq 1} s^n D \cap S \neq \emptyset$ for all $s \in S$. For example, if V is a valuation domain with no height-one prime ideals, then $V \setminus \{0\}$ is an anti-archimedean subset of V [3, Proposition 2.1].) After the paper by Anderson and Dumitrescu, more properties of S -Noetherian domains have been studied further. In [14], Liu found an equivalent condition for the generalized power series ring to be an S -Noetherian domain. In [12], the authors studied the S -Noetherian properties in special pullbacks which are the so-called composite ring extensions $D + E[\Gamma^*]$ and $D + \llbracket E^{\Gamma^*, \leq} \rrbracket$. As a continuation of [12], the same authors investigated when the amalgamated algebra along an ideal has the S -Noetherian property [13]. For more results, the readers can refer to [2, 12, 13, 14].

Let $\mathbf{P}$ denote one of the properties “Noetherian” or “ S -Noetherian”. We say that D is *locally $\mathbf{P}$* (respectively, *t -locally $\mathbf{P}$*) if D_M is $\mathbf{P}$ for all maximal ideals (respectively, maximal t -ideals) M of D .

The purpose of this paper is to study the Nagata ring of S -Noetherian domains and locally S -Noetherian domains, and to investigate the t -Nagata ring of t -locally S -Noetherian domains. (The concepts of Nagata rings and t -Nagata rings will be reviewed in Section .) More precisely, we show that if S is an anti-archimedean subset of D , then D is an S -Noetherian domain (respectively, locally S -Noetherian domain) if and only if the Nagata ring $D[X]_N$ is an S -Noetherian domain (respec-

tively, locally S -Noetherian domain); a locally S -Noetherian domain with finite character is an S -Noetherian domain; and if S is an anti-archimedean subset of D , then D is a t -locally S -Noetherian domain if and only if the polynomial ring $D[X]$ is a t -locally S -Noetherian domain, if and only if the t -Nagata ring $D[X]_{N_v}$ is a t -locally S -Noetherian domain.

2. Main Results

We start this section with a simple result for a quotient ring of S -Noetherian domains. This also recovers the fact that any quotient ring of a Noetherian domain is Noetherian [5, Proposition 7.3].

Lemma 1. *Let D be an integral domain and S a (not necessarily saturated) multiplicative subset of D . If D is an S -Noetherian domain and T is a (not necessarily saturated) multiplicative subset of D , then D_T is an S -Noetherian domain.*

Proof. Let A be an ideal of D_T . Then $A = ID_T$ for some ideal I of D . Since D is an S -Noetherian domain, there exist an element $s \in S$ and a finitely generated ideal J of D such that $sI \subseteq J \subseteq I$. Therefore we obtain

$$sA = sID_T \subseteq JD_T \subseteq ID_T = A,$$

and hence A is S -finite. Thus D_T is an S -Noetherian domain. $\square$

The next result is an S -Noetherian version of well-known facts that a Noetherian domain is locally Noetherian; and a locally Noetherian domain with finite character is Noetherian [5, Section 7, Exercise 9].

Theorem 2. *The following statements hold.*

- (1) *An S -Noetherian domain is locally S -Noetherian.*
- (2) *A locally S -Noetherian domain with finite character is S -Noetherian.*

Proof. (1) This is an immediate consequence of Lemma 1.

(2) Assume that D is a locally S -Noetherian domain which is of finite character, and let I be an ideal of D . If $I \cap S \neq \emptyset$, then for any $s \in I \cap S$, $sI \subseteq (s) \subseteq I$; so I is S -finite. Next, we consider the case when I does not intersect S . Choose any $0 \neq a \in I$. Since D has finite character, a belongs to only a finite number of maximal ideals of D , say $M_1, \dots, M_n$. Fix an $i \in \{1, \dots, n\}$. Since D_{M_i} is S -Noetherian, there exist an element $s_i \in S$ and a finitely generated subideal F_i of I such that $s_i ID_{M_i} \subseteq F_i D_{M_i}$. By letting $s = s_1 \cdots s_n$ and setting $C = (a) + F_1 + \cdots + F_n$, we obtain that $sID_{M_i} \subseteq CD_{M_i}$. Let M' be a maximal ideal of D which is distinct from $M_1, \dots, M_n$. Then a is a unit in $D_{M'}$; so $ID_{M'} = D_{M'} = CD_{M'}$. Therefore

$sID_M \subseteq CD_M$ for all maximal ideals M of D . Hence we have

$$\begin{aligned} sI &= \bigcap_{M \in \text{Max}(D)} sID_M \\ &\subseteq \bigcap_{M \in \text{Max}(D)} CD_M \\ &= C, \end{aligned}$$

where $\text{Max}(D)$ denotes the set of maximal ideals of D and the equalities follow from [9, Proposition 2.8(3)]. Note that C is a finitely generated subideal of I . Therefore I is S -finite, and thus D is an S -Noetherian domain. $\square$

Recall that an integral domain D is an *almost Dedekind domain* if D_M is a Noetherian valuation domain for all maximal ideals M of D .

Remark 3. The converse of Theorem 2(1) does not generally hold. (This also indicates that the condition being finite character in Theorem 2(2) is essential.) For example, if D is an almost Dedekind domain which is not Noetherian, then D is a locally S -Noetherian domain which is not S -Noetherian. (This is the case when S consists of units in D .) For a concrete illustration, see [8, Example 42.6].

Let D be an integral domain and $D[X]$ be the polynomial ring over D . For an $f \in D[X]$, $c(f)$ denotes the content ideal of f , i.e., the ideal of D generated by the coefficients of f , and for an ideal I of $D[X]$, $c(I)$ stands for the ideal of D generated by the coefficients of polynomials in I , i.e., $c(I) = \sum_{f \in I} c(f)$. Let $N = \{f \in D[X] \mid c(f) = D\}$. Then N is a saturated multiplicative subset of $D[X]$ and the quotient ring $D[X]_N$ is called the *Nagata ring* of D . It was shown that D is a Noetherian domain if and only if $D[X]$ is a Noetherian domain [5, Theorem 7.5 (Hilbert basis theorem)] (or [10, Theorem 69]), if and only if $D[X]_N$ is a Noetherian domain (cf. [1, Theorem 2.2(2)]). We now give the S -Noetherian analogue of these equivalences.

Theorem 4. Let D be an integral domain, S an anti-archimedean subset of D , and $N := \{f \in D[X] \mid c(f) = D\}$. Then the following statements are equivalent.

- (1) D is an S -Noetherian domain.
- (2) $D[X]$ is an S -Noetherian domain.
- (3) $D[X]_N$ is an S -Noetherian domain.

Proof. (1) $\Rightarrow$ (2) This implication appears in [2, Proposition 9].

(2) $\Rightarrow$ (3) This was shown in Lemma 1.

(3) $\Rightarrow$ (1) Let I be an ideal of D . Then $ID[X]_N$ is an ideal of $D[X]_N$. Since $D[X]_N$ is an S -Noetherian domain, we can find an element $s \in S$ and a finitely generated subideal J of $ID[X]$ such that $sID[X]_N \subseteq JD[X]_N$; so $sID[X]_N \subseteq c(J)D[X]_N$. Let $a \in I$. Then $sag \in c(J)D[X]$ for some $g \in N$; so $sa \in c(J)$. Hence $sI \subseteq c(J)$. Note that $c(J)$ is a finitely generated subideal of I . Therefore I is S -finite, and thus D is an S -Noetherian domain. $\square$

Let D be an integral domain and let $N_v = \{f \in D[X] \mid c(f)_v = D\}$. Then N_v is a saturated multiplicative subset of $D[X]$ and the quotient ring $D[X]_{N_v}$ is called the t -Nagata ring of D . It was shown that D is t -locally Noetherian if and only if $D[X]$ is t -locally Noetherian, if and only if $D[X]_{N_v}$ is t -locally Noetherian [6, Theorem 1.4]. To investigate the (t) -Nagata ring of (t) -locally S -Noetherian domains, we need the following lemma.

Lemma 5. *Let D be a quasi-local domain with unique maximal ideal M , S a (not necessarily saturated) multiplicative subset of D , and I an ideal of D . Then I is S -finite if and only if $ID[X]_{MD[X]}$ is S -finite.*

Proof. If I is S -finite, then there exist an element $s \in S$ and a finitely generated subideal J of I such that $sI \subseteq J$; so we obtain

$$sID[X]_{MD[X]} \subseteq JD[X]_{MD[X]} \subseteq ID[X]_{MD[X]}.$$

Thus $ID[X]_{MD[X]}$ is S -finite. Conversely, if $ID[X]_{MD[X]}$ is S -finite, then there exist suitable elements $s \in S$ and $f_1, \dots, f_n \in ID[X]$ such that $sID[X]_{MD[X]} \subseteq (f_1, \dots, f_n)D[X]_{MD[X]}$; so we obtain

$$sID[X]_{MD[X]} \subseteq (c(f_1) + \dots + c(f_n))D[X]_{MD[X]}.$$

Note that $JD[X]_{MD[X]} \cap D = J$ for all ideals J of D , because D is quasi-local. Hence we obtain

$$\begin{aligned} sI &= sID[X]_{MD[X]} \cap D \\ &\subseteq (c(f_1) + \dots + c(f_n))D[X]_{MD[X]} \cap D \\ &= c(f_1) + \dots + c(f_n). \end{aligned}$$

Note that $c(f_1) + \dots + c(f_n)$ is a finitely generated subideal of I . Thus I is S -finite. $\square$

We are ready to study the polynomial extension and the t -Nagata ring of t -locally S -Noetherian domains.

Theorem 6. *Let D be an integral domain, S an anti-archimedean subset of D , and $N_v := \{f \in D[X] \mid c(f)_v = D\}$. Then the following statements are equivalent.*

- (1) D is a t -locally S -Noetherian domain.
- (2) $D[X]$ is a t -locally S -Noetherian domain.
- (3) $D[X]_{N_v}$ is a locally S -Noetherian domain.
- (4) $D[X]_{N_v}$ is a t -locally S -Noetherian domain.

Proof. (1) $\Rightarrow$ (2) Let M be a maximal t -ideal of $D[X]$ and let K be the quotient field of D . If $M \cap D = (0)$, then $D[X]_M$ is a quotient ring of $K[X]$; so $D[X]_M$ is a principal ideal domain. Hence $D[X]_M$ is an S -Noetherian domain. Next, we assume that $M \cap D \neq (0)$, and let $P = M \cap D$. Then $M = PD[X]$ and P is a

maximal t -ideal of D [7, Proposition 2.2]. Since D is t -locally S -Noetherian, D_P is S -Noetherian. Also, since S is an anti-archimedean subset of D_P , $D_P[X]$ is S -Noetherian [2, Proposition 9]; so by Lemma 1, $D_P[X]_{PD_P[X]}$ is S -Noetherian. Note that $D[X]_M = D_P[X]_{PD_P[X]}$; so $D[X]_M$ is an S -Noetherian domain. From both cases, we conclude that $D[X]$ is a t -locally S -Noetherian domain.

(2) $\Rightarrow$ (3) Let Q be a maximal ideal of $D[X]_{N_v}$. Then $Q = MD[X]_{N_v}$ for some maximal t -ideal M of D [9, Proposition 2.1(2)]. Note that $(D[X]_{N_v})_Q = (D[X]_{N_v})_{MD[X]_{N_v}} = D[X]_{MD[X]}$ and $MD[X]$ is a maximal t -ideal of $D[X]$ [7, Proposition 2.2]. Since $D[X]$ is t -locally S -Noetherian, $D[X]_{MD[X]}$, and hence $(D[X]_{N_v})_Q$ is S -Noetherian. Thus $D[X]_{N_v}$ is a locally S -Noetherian domain.

(3) $\Rightarrow$ (1) Let M be a maximal t -ideal of D . Then $MD[X]_{N_v}$ is a maximal ideal of $D[X]_{N_v}$ [9, Proposition 2.1(2)]. Note that $(D[X]_{N_v})_{MD[X]_{N_v}} = D[X]_{MD[X]} = D_M[X]_{MD_M[X]}$; so $D_M[X]_{MD_M[X]}$ is S -Noetherian, because $D[X]_{N_v}$ is locally S -Noetherian. Let I be an ideal of D_M . Then $ID_M[X]_{MD_M[X]}$ is S -finite. Since D_M is quasi-local, Lemma 5 forces I to be S -finite. Hence D_M is S -Noetherian, and thus D is a t -locally S -Noetherian domain.

(3) $\Leftrightarrow$ (4) This equivalence follows directly from the fact that the set of maximal t -ideals of $D[X]_{N_v}$ is precisely the same as that of maximal ideals of $D[X]_{N_v}$ (cf. [9, Propositions 2.1(2) and 2.2(3)]). $\square$

We next study locally S -Noetherian domains in terms of the Nagata ring.

Theorem 7. *Let D be an integral domain, S an anti-archimedean subset of D , and $N := \{f \in D[X] \mid c(f) = D\}$. Then the following statements are equivalent.*

- (1) D is a locally S -Noetherian domain.
- (2) $D[X]_N$ is a locally S -Noetherian domain.

Proof. (1) $\Rightarrow$ (2) Let Q be a maximal ideal of $D[X]_N$. Then $Q = MD[X]_N$ for some maximal ideal M of D [9, Proposition 2.1(2)]. Since D is locally S -Noetherian, D_M is S -Noetherian. Also, since S is an anti-archimedean subset of D_M , $D_M[X]$ is S -Noetherian [2, Proposition 9]. Hence by Lemma 1, $D_M[X]_{MD_M[X]}$ is an S -Noetherian domain. Note that $(D[X]_N)_Q = D[X]_{MD[X]} = D_M[X]_{MD_M[X]}$; so $(D[X]_N)_Q$ is S -Noetherian. Thus $D[X]_N$ is a locally S -Noetherian domain.

(2) $\Rightarrow$ (1) Let M be a maximal ideal of D . Then $MD[X]_N$ is a maximal ideal of $D[X]_N$ [9, Proposition 2.1(2)]. Since $D[X]_N$ is locally S -Noetherian, $(D[X]_N)_{MD[X]_N}$ is S -Noetherian. Note that $(D[X]_N)_{MD[X]_N} = D[X]_{MD[X]} = D_M[X]_{MD_M[X]}$; so $D_M[X]_{MD_M[X]}$ is S -Noetherian. Let I be an ideal of D_M . Then $ID_M[X]_{MD_M[X]}$ is S -finite. Since D_M is quasi-local, I is S -finite by Lemma 5. Hence D_M is S -Noetherian, and thus D is a locally S -Noetherian domain. $\square$

We are closing this article by comparing our results with recent researches related to S -Noetherian domains. In [11], the authors defined an integral domain D to be an S -strong Mori domain (S -SM-domain) if for each nonzero ideal I of D , there exist an element $s \in S$ and a finitely generated ideal J of D such that

$sI \subseteq J_w \subseteq I_w$. This concept generalizes the notions of both S -Noetherian domains and strong Mori domains. (Recall from [15, Definition 4] that D is a *strong Mori domain* (SM-domain) if it satisfies the ascending chain condition on integral w -ideals of D , or equivalently, for each (prime) w -ideal I of D , $I = J_w$ for some finitely generated ideal J of D [15, Theorem 4.3].) It was shown that if D is a t -locally S -Noetherian domain with finite t -character, then D is an S -SM-domain [11, Proposition 2.1(2)]; and that if S is an anti-archimedean subset of D , then D is an S -SM-domain if and only if $D[X]_{N_v}$ is an S -SM-domain [11, Theorem 2.10].

Lemma 8. *Let D be an integral domain, $N := \{f \in D[X] \mid c(f) = D\}$, and $N_v := \{f \in D[X] \mid c(f)_v = D\}$. Then the following assertions hold.*

- (1) *D is of finite character if and only if $D[X]_N$ is of finite character.*
- (2) *D is of finite t -character if and only if $D[X]_{N_v}$ is of finite character.*

Proof. The equivalence is an immediate consequence of the fact that $\{MD[X]_N \mid M \text{ is a maximal ideal of } D\}$ (respectively, $\{MD[X]_{N_v} \mid M \text{ is a maximal } t\text{-ideal of } D\}$) is the set of maximal ideals of $D[X]_N$ (respectively, $D[X]_{N_v}$) [9, Proposition 2.1(2)]. $\square$

By Theorems 6 and 7 and Lemma 8, we obtain

Corollary 9. *Let D be an integral domain, S an anti-archimedean subset of D , $N := \{f \in D[X] \mid c(f) = D\}$, and $N_v := \{f \in D[X] \mid c(f)_v = D\}$. Then the following assertions hold.*

- (1) *D is a locally S -Noetherian domain with finite character if and only if $D[X]_N$ is a locally S -Noetherian domain with finite character.*
- (2) *D is a t -locally S -Noetherian domain with finite t -character if and only if $D[X]_{N_v}$ is a locally S -Noetherian domain with finite character.*

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