A Coupled Fixed Point Theorem for Mixed Monotone Mappings on Partial Ordered G -Metric Spaces

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ABSTRACT. In this paper, we establish coupled fixed point theorems for mixed monotone mappings satisfying nonlinear contraction involving a pair of altering distance functions in ordered G -metric spaces. Via presented theorems we extend and generalize the results of Harjani et al. [J. Harjani, B. López and K. Sadarangani, Fixed point theorems for mixed monotone operators and applications to integral equations, *Nonlinear Anal.* 74 (2011) 1749-1760] and Choudhury and Maity [B.S. Choudhury and P. Maity, Coupled fixed point results in generalized metric spaces. *Math. Comput. Model.* 54 (2011), 73-79].

1. Introduction and Preliminaries

Mustafa and Sims [21] introduced the notion of G -metric spaces. The structure of G -metric spaces is a generalization of metric spaces. Mustafa and Sims [21] initiated the theory of fixed points in G -metric spaces and established the Banach contraction principle in this generalized structure. Afterwards, different authors proved several fixed point results in this space (see, e.g., [2, 3, 6, 7, 10, 11, 18, 19, 20, 22, 27, 28]).

Definition 1.1.([21]) Let X be a nonempty set. Suppose that a mapping $G : X \times X \times X \rightarrow \mathbb{R}_+ = [0, \infty)$ satisfies:

- (G1) $G(x, y, z) = 0$ if $x = y = z$;
- (G2) $G(x, x, y) > 0$ for all $x, y \in X$ with $x \neq y$;
- (G3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$;
- (G4) (symmetry in all three variables)
 $G(x, y, z) = G(x, z, y) = G(y, z, x) = \dots$;
- (G5) (rectangle inequality)
 $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$.

Then G is called a G -metric on X and (X, G) is called a G -metric space or a generalized metric space by G .

Received October 15, 2013; accepted December 13, 2013.

2010 Mathematics Subject Classification: 47H10, 54H25.

Key words and phrases: Fixed point, mixed monotone property, G -metric space.

The following are examples of G -metric spaces.

Example 1.2. Let $(\mathbb{R}, d)$ be the usual metric space. Define G_1 and G_2 by

$$\begin{aligned} G_1(x, y, z) &= d(x, y) + d(y, z) + d(x, z), \\ G_2(x, y, z) &= \max\{d(x, y), d(y, z), d(x, z)\} \end{aligned}$$

for all $x, y, z \in \mathbb{R}$. Then it is clear that $(\mathbb{R}, G_1)$ and $(\mathbb{R}, G_2)$ are G -metric spaces.

Example 1.3. Let $X = \{a, b\}$ and $G : X \times X \times X \rightarrow [0, \infty)$ be defined by

$$\begin{aligned} G(a, a, a) &= G(b, b, b) = 0, \\ G(a, a, b) &= G(a, b, a) = G(b, a, a) = 1, \\ G(a, b, b) &= G(b, a, b) = G(b, b, a) = 2. \end{aligned}$$

It is easy to show that the function G satisfies all properties of Definition .

Definition 1.4. ([21]) Let X be a G -metric space and let $\{x_n\}$ be a sequence of points of X . A point $x \in X$ is said to be the *limit of a sequence* $\{x_n\}$ if $\lim_{n, m \rightarrow \infty} G(x, x_n, x_m) = 0$ and we say in this case that the sequence $\{x_n\}$ is said to be G -convergent to x .

Thus, $x_n \rightarrow x$ in a G -metric space X if for any $\epsilon > 0$, there exists a positive integer N such that $G(x, x_n, x_m) < \epsilon$, for all $n, m \geq N$. It has been shown in [21] that the G -metric induces a Hausdorff topology and the convergence described in the above definition is relative to this topology.

Lemma 1.5. ([21]) If X is a G -metric space, then the following are equivalent:

- (i) $\{x_n\}$ is G -convergent to x ,
- (ii) $G(x_n, x_n, x) \rightarrow 0$ as $n \rightarrow \infty$,
- (iii) $G(x_n, x, x) \rightarrow 0$ as $n \rightarrow \infty$.

Definition 1.6. ([21]) Let X be a G -metric space, a sequence $\{x_n\}$ is called G -Cauchy if for every $\epsilon > 0$ there is a positive integer N such that $G(x_n, x_m, x_l) < \epsilon$ for all $n, m, l \geq N$, that is, if $G(x_n, x_m, x_l) \rightarrow 0$, as $n, m, l \rightarrow \infty$.

Lemma 1.7. ([21]) If X is a G -metric space, then the following are equivalent:

- (i) The sequence $\{x_n\}$ is G -Cauchy.
- (ii) For every $\epsilon > 0$, there exists a positive integer N such that $G(x_n, x_m, x_m) < \epsilon$ for all $n, m \geq N$.

Definition 1.8. ([21]) A G -metric space X is said to be G -complete (or a *complete G -metric space*) if every G -Cauchy sequence is G -convergent in (X, G) .

Definition 1.9. ([21]) Let (X, G) and (X', G') be two generalized metric spaces. A mapping $f : X \rightarrow X'$ is G -continuous at a point $x \in X$ if and only if it is G sequentially continuous at x , that is, whenever $\{x_n\}$ is G -convergent to x , $\{f(x_n)\}$ is G' -convergent to $f(x)$.

Definition 1.10.([21]) Let X be a G -metric space. A mapping $F : X \times X \rightarrow X$ is said to be *continuous* if for any two G -convergent sequences $\{x_n\}$ and $\{y_n\}$ converging to x and y , respectively, $\{F(x_n, y_n)\}$ is G -convergent to $F(x, y)$.

In recent years, there has been a lot of interest in establishing fixed point theorems on ordered metric spaces with a contractive condition which holds for all points that are related by partial ordering. This trend was initiated by Ran and Reurings in [26] where they extended the Banach contraction principle in partially ordered sets with some applications to matrix equations. Subsequently, Nieto and Rodríguez-López [24] extended the results in [26] for non-decreasing mappings and applied them to obtain a unique solution for a first-order ordinary differential equation with periodic boundary conditions. Recently, many researchers have obtained common fixed point results on partially ordered metric spaces (see, e.g., [4, 5, 8, 9, 14, 23, 24, 25]).

Bhaskar and Lakshmikantham [12] introduced the notions of a mixed monotone mapping and a coupled fixed point, and proved some coupled fixed point theorems for mixed mappings in ordered metric spaces. Afterwards, Lakshmikantham and Ćirić [16] have established coupled coincidence and coupled fixed point theorems for two mappings F and g , where F has the mixed g -monotone property. Many other results on coupled fixed point theory exist in the literatures [1, 13, 16, 17, 29, 30].

Definition 1.11.([12]) Let $(X, \leq)$ be a partial ordered set. A mapping $F : X \times X \rightarrow X$ is said to have the a *mixed monotone property* if F is monotone nondecreasing in its first argument and is monotone nonincreasing in its second argument, that is, for any $x, y \in X$

$$(1.1) \quad x_1, x_2 \in X, \quad x_1 \leq x_2 \Rightarrow F(x_1, y) \leq F(x_2, y)$$

and

$$(1.2) \quad y_1, y_2 \in X, \quad y_1 \leq y_2 \Rightarrow F(x, y_1) \geq F(x, y_2).$$

Definition 1.12.([12]) An element $(x, y) \in X \times X$ is called a *coupled fixed point* of mapping $F : X \times X \rightarrow X$ if

$$x = F(x, y) \quad \text{and} \quad y = F(y, x).$$

An altering distance function was introduced by Khan et al. in [15] where they present some fixed point theorems.

Definition 1.13. An *altering distance function* is a map $\Psi : [0, \infty) \rightarrow [0, \infty)$ satisfying:

- (i) Ψ is continuous and nondecreasing;
- (ii) $\Psi(t) = 0$ if and only if $t = 0$.

Let $(X, \leq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Harjani, López and Sadarangani [13] established some coupled fixed point theorems for the mixed monotone mapping $F : X \times X \rightarrow X$ involving a pair of altering distance functions under a contractive condition of the form

$$\phi(d(F(x, y), F(u, v))) \leq \phi(\max\{d(x, u), d(y, v)\}) - \psi(\max\{d(x, u), d(y, v)\})$$

for $x, y, u, v \in X$ with $x \geq u$ and $y \leq v$, where ϕ and ψ are altering distance functions. The purpose of this work is to extend this theorem to the set of G -metric spaces.

2. Coupled Fixed Point in G -Metric Spaces

Theorem 2.1. *Let $(X, \leq)$ be a partially ordered set such that there exists a complete G -metric on X and $F : X \times X \rightarrow X$ be a continuous mapping having the mixed monotone property. Suppose that there exist altering distance functions φ and ψ such that*

$$(2.1) \quad \begin{aligned} & \varphi(G(F(x, y), F(u, v), F(w, z))) \\ & \leq \varphi(\max\{G(x, u, w), G(y, v, z)\}) - \psi(\max\{G(x, u, w), G(y, v, z)\}) \end{aligned}$$

for all $x, y, z, u, v, w \in X$ for which $x \geq u \geq w$ and $y \leq v \leq z$ where either $x \neq u$ or $y \neq v$. If there exist $x_0, y_0 \in X$ such that

$$x_0 \leq F(x_0, y_0) \quad \text{and} \quad y_0 \geq F(y_0, x_0),$$

then F has a coupled fixed point.

Proof. We construct sequences (x_n) and (y_n) putting

$$x_{n+1} = F(x_n, y_n) \quad \text{and} \quad y_{n+1} = F(y_n, x_n) \quad \text{for } n \geq 0.$$

In order that the proof is more comprehensive, we will divide it in several steps.

Step 1. $x_n \leq x_{n+1}$ and $y_n \geq y_{n+1}$, for $n \geq 0$.

In fact, we use mathematical induction.

As $x_0 \leq F(x_0, y_0) = x_1$ and $y_0 \geq F(y_0, x_0) = y_1$ our claim is satisfied for $n = 0$.

Again by the induction hypothesis and the mixed monotone property of F , we have

$$x_{n+1} = F(x_n, y_n) \geq F(x_{n-1}, y_{n-1}) = x_n$$

and

$$y_{n+1} = F(y_n, x_n) \leq F(y_{n-1}, x_{n-1}) = y_n.$$

This proves our claim.

Step 2. $\lim_{n \rightarrow \infty} G(x_n, x_{n+1}, x_{n+1}) = \lim_{n \rightarrow \infty} G(y_n, y_{n+1}, y_{n+1}) = 0$.

From the contractive condition (2.1) and Step 1, we obtain

$$\begin{aligned}
 & \varphi(G(x_n, x_{n+1}, x_{n+1})) \\
 &= \varphi(G(F(x_{n-1}, y_{n-1}), F(x_n, y_n), F(x_n, y_n))) \\
 (2.2) \quad &\leq \varphi(\max\{G(x_{n-1}, x_n, x_n), G(y_{n-1}, y_n, y_n)\}) \\
 &\quad - \psi(\max\{G(x_{n-1}, x_n, x_n), G(y_{n-1}, y_n, y_n)\}) \\
 &\leq \varphi(\max\{G(x_{n-1}, x_n, x_n), G(y_{n-1}, y_n, y_n)\}).
 \end{aligned}$$

Using the fact that φ is nondecreasing, we have

$$G(x_n, x_{n+1}, x_{n+1}) \leq \max\{G(x_{n-1}, x_n, x_n), G(y_{n-1}, y_n, y_n)\}.$$

Similarly, we get

$$G(y_n, y_{n+1}, y_{n+1}) \leq \max\{G(x_{n-1}, x_n, x_n), G(y_{n-1}, y_n, y_n)\}.$$

Hence, the sequence $\{\max\{G(x_n, x_{n+1}, x_{n+1}), G(y_n, y_{n+1}, y_{n+1})\}\}_{n=0}^{\infty}$ is nonnegative and decreasing. This implies that there exists $\alpha \geq 0$ such that

$$\lim_{n \rightarrow \infty} \max\{G(x_n, x_{n+1}, x_{n+1}), G(y_n, y_{n+1}, y_{n+1})\} = \alpha.$$

It is easily seen that if $\varphi : [0, \infty) \rightarrow [0, \infty)$ is nondecreasing, $\varphi(\max\{a_1, a_2\}) = \max\{\varphi(a_1), \varphi(a_2)\}$ for $a_1, a_2 \in [0, \infty)$. Taking into account this and (2.2) we get

$$\begin{aligned}
 & \varphi(\max\{G(x_n, x_{n+1}, x_{n+1}), G(y_n, y_{n+1}, y_{n+1})\}) \\
 &= \max\{\varphi(G(x_n, x_{n+1}, x_{n+1})), \varphi(G(y_n, y_{n+1}, y_{n+1}))\} \\
 &\leq \varphi(\max\{G(x_{n-1}, x_n, x_n), G(y_{n-1}, y_n, y_n)\}) \\
 &\quad - \psi(\max\{G(x_{n-1}, x_n, x_n), G(y_{n-1}, y_n, y_n)\}) \\
 &\leq \varphi(\max\{G(x_{n-1}, x_n, x_n), G(y_{n-1}, y_n, y_n)\}).
 \end{aligned}$$

Since φ is a continuous function, letting $n \rightarrow \infty$ in the above inequalities yields

$$\varphi(\alpha) \leq \varphi(\alpha) - \psi(\alpha) \leq \varphi(\alpha).$$

and this implies $\psi(\alpha) = 0$. Since ψ is an altering distance function, $\alpha = 0$ and

$$\lim_{n \rightarrow \infty} \max\{G(x_n, x_{n+1}, x_{n+1}), G(y_n, y_{n+1}, y_{n+1})\} = 0,$$

and this proves our claim.

Step 3. $\{x_n\}$ and $\{y_n\}$ are G -Cauchy sequences.

Assume that at least one of the sequences $\{x_n\}$ and $\{y_n\}$ is not a G -Cauchy sequence. By Lemma , this implies that

$$\lim_{n, m \rightarrow \infty} G(x_n, x_m, x_m) \neq 0 \quad \text{or} \quad \lim_{n, m \rightarrow \infty} G(y_n, y_m, y_m) \neq 0$$

and, consequently,

$$\lim_{n, m \rightarrow \infty} \max\{G(x_n, x_m, x_m), G(y_n, y_m, y_m)\} \neq 0.$$

This means that there exists an $\varepsilon > 0$ for which we can find two subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_k\}$ such that $n(k)$ is the smallest index for which

$$(2.3) \quad \max \{G(x_{n(k)}, x_{m(k)}, x_{m(k)}), G(y_{n(k)}, y_{m(k)}, y_{m(k)})\} \geq \varepsilon$$

for $n(k) > m(k) > k$.

This means that

$$(2.4) \quad \max \{G(x_{n(k)-1}, x_{m(k)}, x_{m(k)}), G(y_{n(k)-1}, y_{m(k)}, y_{m(k)})\} < \varepsilon.$$

The rectangle inequality and (2.4) give us, for each k ,

$$(2.5) \quad \begin{aligned} & G(x_{n(k)}, x_{m(k)}, x_{m(k)}) \\ & \leq G(x_{n(k)}, x_{n(k)-1}, x_{n(k)-1}) + G(x_{n(k)-1}, x_{m(k)}, x_{m(k)}) \\ & < G(x_{n(k)}, x_{n(k)-1}, x_{n(k)-1}) + \varepsilon \end{aligned}$$

and

$$(2.6) \quad \begin{aligned} & G(y_{n(k)}, y_{m(k)}, y_{m(k)}) \\ & \leq G(y_{n(k)}, y_{n(k)-1}, y_{n(k)-1}) + G(y_{n(k)-1}, y_{m(k)}, y_{m(k)}) \\ & < G(y_{n(k)}, y_{n(k)-1}, y_{n(k)-1}) + \varepsilon \end{aligned}$$

Using (2.3), (2.5) and (2.6), we get

$$\begin{aligned} \varepsilon & \leq \max \{G(x_{n(k)}, x_{m(k)}, x_{m(k)}), G(y_{n(k)}, y_{m(k)}, y_{m(k)})\} \\ & < \max \{G(x_{n(k)}, x_{n(k)-1}, x_{n(k)-1}), G(y_{n(k)}, y_{n(k)-1}, y_{n(k)-1})\} + \varepsilon. \end{aligned}$$

Letting $k \rightarrow \infty$ in the last inequality and using Step 2, we obtain that

$$(2.7) \quad \lim_{k \rightarrow \infty} \max \{G(x_{n(k)}, x_{m(k)}, x_{m(k)}), G(y_{n(k)}, y_{m(k)}, y_{m(k)})\} = \varepsilon.$$

Again, the rectangle inequality and (2.4) give us, for each k ,

$$(2.8) \quad \begin{aligned} & G(x_{n(k)-1}, x_{m(k)-1}, x_{m(k)-1}) \\ & \leq G(x_{n(k)-1}, x_{m(k)}, x_{m(k)}) + G(x_{m(k)}, x_{m(k)-1}, x_{m(k)-1}) \\ & < \varepsilon + G(x_{m(k)}, x_{m(k)-1}, x_{m(k)-1}) \end{aligned}$$

and

$$(2.9) \quad \begin{aligned} & G(y_{n(k)-1}, y_{m(k)-1}, y_{m(k)-1}) \\ & \leq G(y_{n(k)-1}, y_{m(k)}, y_{m(k)}) + G(y_{m(k)}, y_{m(k)-1}, y_{m(k)-1}) \\ & < \varepsilon + G(y_{m(k)}, y_{m(k)-1}, y_{m(k)-1}). \end{aligned}$$

By (2.8) and (2.9) we get

$$(2.10) \quad \begin{aligned} & \max \{G(x_{n(k)-1}, x_{m(k)-1}, x_{m(k)-1}), G(y_{n(k)-1}, y_{m(k)-1}, y_{m(k)-1})\} \\ & < \max \{G(x_{m(k)}, x_{m(k)-1}, x_{m(k)-1}), G(y_{m(k)}, y_{m(k)-1}, y_{m(k)-1})\} + \varepsilon. \end{aligned}$$

Using the rectangle inequality we have

$$\begin{aligned} & G(x_{n(k)}, x_{m(k)}, x_{m(k)}) \\ & \leq G(x_{n(k)}, x_{n(k)-1}, x_{n(k)-1}) + G(x_{n(k)-1}, x_{m(k)-1}, x_{m(k)-1}) \\ & \quad + G(x_{m(k)-1}, x_{m(k)}, x_{m(k)}) \end{aligned}$$

and

$$\begin{aligned} & G(y_{n(k)}, y_{m(k)}, y_{m(k)}) \\ & \leq G(y_{n(k)}, y_{n(k)-1}, y_{n(k)-1}) + G(y_{n(k)-1}, y_{m(k)-1}, y_{m(k)-1}) \\ & \quad + G(y_{m(k)-1}, y_{m(k)}, y_{m(k)}) . \end{aligned}$$

By the two last inequalities and (2.3) we get

$$\begin{aligned} (2.11) \quad \varepsilon & \leq \max \{ G(x_{n(k)}, x_{m(k)}, x_{m(k)}), G(y_{n(k)}, y_{m(k)}, y_{m(k)}) \} \\ & \leq \max \{ G(x_{n(k)}, x_{n(k)-1}, x_{n(k)-1}), G(y_{n(k)}, y_{n(k)-1}, y_{n(k)-1}) \} \\ & \quad + \max \{ G(x_{n(k)-1}, x_{m(k)-1}, x_{m(k)-1}), G(y_{n(k)-1}, y_{m(k)-1}, y_{m(k)-1}) \} \\ & \quad + \max \{ G(x_{m(k)-1}, x_{m(k)}, x_{m(k)}), G(y_{m(k)-1}, y_{m(k)}, y_{m(k)}) \} . \end{aligned}$$

By (2.10) and (2.11) we have

$$\begin{aligned} & \varepsilon + \max \{ G(x_{m(k)}, x_{m(k)-1}, x_{m(k)-1}), G(y_{m(k)}, y_{m(k)-1}, y_{m(k)-1}) \} \\ & > \max \{ G(x_{n(k)-1}, x_{m(k)-1}, x_{m(k)-1}), G(y_{n(k)-1}, y_{m(k)-1}, y_{m(k)-1}) \} \\ & \geq \varepsilon - \max \{ G(x_{n(k)}, x_{n(k)-1}, x_{n(k)-1}), G(y_{n(k)}, y_{n(k)-1}, y_{n(k)-1}) \} \\ & \quad - \max \{ G(x_{m(k)-1}, x_{m(k)}, x_{m(k)}), G(y_{m(k)-1}, y_{m(k)}, y_{m(k)}) \} . \end{aligned}$$

Letting $k \rightarrow \infty$ in the last inequality, and by Step 2, we obtain that

$$(2.12) \quad \max \{ G(x_{n(k)-1}, x_{m(k)-1}, x_{m(k)-1}), G(y_{n(k)-1}, y_{m(k)-1}, y_{m(k)-1}) \} = \varepsilon .$$

Since $n(k) > m(k) > k$, by Step 1

$$x_{n(k)-1} \leq x_{n(k)} \quad \text{and} \quad y_{n(k)-1} \geq y_{n(k)}$$

and using the contractive condition we can obtain

$$\begin{aligned} & \varphi(G(x_{n(k)}, x_{m(k)}, x_{m(k)})) \\ & = \varphi(G(F(x_{n(k)-1}, y_{n(k)-1}), F(x_{m(k)-1}, y_{m(k)-1}), F(x_{m(k)-1}, y_{m(k)-1}))) \\ & \leq \varphi(\max \{ G(x_{n(k)-1}, x_{m(k)-1}, x_{m(k)-1}), G(y_{n(k)-1}, y_{m(k)-1}, y_{m(k)-1}) \}) \\ & \quad - \psi(\max \{ G(x_{n(k)-1}, x_{m(k)-1}, x_{m(k)-1}), G(y_{n(k)-1}, y_{m(k)-1}, y_{m(k)-1}) \})) \end{aligned}$$

and

$$\begin{aligned} & \varphi(G(y_{n(k)}, y_{m(k)}, y_{m(k)})) \\ & = \varphi(G(F(y_{n(k)-1}, x_{n(k)-1}), F(y_{m(k)-1}, x_{m(k)-1}), F(y_{m(k)-1}, x_{m(k)-1}))) \\ & \leq \varphi(\max \{ G(x_{n(k)-1}, x_{m(k)-1}, x_{m(k)-1}), G(y_{n(k)-1}, y_{m(k)-1}, y_{m(k)-1}) \}) \\ & \quad - \psi(\max \{ G(x_{n(k)-1}, x_{m(k)-1}, x_{m(k)-1}), G(y_{n(k)-1}, y_{m(k)-1}, y_{m(k)-1}) \})) . \end{aligned}$$

Thus,

$$(2.13) \quad \begin{aligned} & \varphi \left(\max \left\{ G \left(x_{n(k)}, x_{m(k)}, x_{m(k)} \right), G \left(y_{n(k)}, y_{m(k)}, y_{m(k)} \right) \right\} \right) \\ & \leq \varphi \left(\max \left\{ G \left(x_{n(k)-1}, x_{m(k)-1}, x_{m(k)-1} \right), G \left(y_{n(k)-1}, y_{m(k)-1}, y_{m(k)-1} \right) \right\} \right) \\ & \quad - \psi \left(\max \left\{ G \left(x_{n(k)-1}, x_{m(k)-1}, x_{m(k)-1} \right), G \left(y_{n(k)-1}, y_{m(k)-1}, y_{m(k)-1} \right) \right\} \right). \end{aligned}$$

Finally, letting $k \rightarrow \infty$ in (2.13) and using (2.7), (2.12), and the continuity of φ and ψ , we get

$$\varphi(\varepsilon) \leq \varphi(\varepsilon) - \psi(\varepsilon)$$

and, consequently, $\psi(\varepsilon) = 0$. Since ψ is an altering distance function, $\varepsilon = 0$, and this is a contradiction. This proves our claim.

Since (X, G) is a complete G -metric space there exist $x, y \in X$ such that the sequences $\{x_k\}$ and $\{y_k\}$ are G -convergent to x and y , respectively.

In fact, using the continuity of F we have

$$\begin{aligned} x &= \lim_{k \rightarrow \infty} x_{k+1} = \lim_{k \rightarrow \infty} F(x_k, y_k) = F \left(\lim_{k \rightarrow \infty} x_k, \lim_{k \rightarrow \infty} y_k \right) = F(x, y) \\ y &= \lim_{k \rightarrow \infty} y_{k+1} = \lim_{k \rightarrow \infty} F(y_k, x_k) = F \left(\lim_{k \rightarrow \infty} y_k, \lim_{k \rightarrow \infty} x_k \right) = F(y, x). \end{aligned}$$

This proves that (x, y) is a coupled fixed point F . $\square$

In the following result, the continuity of F is not required.

Theorem 2.2. *Let $(X, \leq)$ be a partially ordered set such that there exists a complete G -metric on X and $F : X \times X \rightarrow X$ be a mapping having the mixed monotone property. Suppose that there exist altering distance functions φ and ψ such that*

$$(2.14) \quad \begin{aligned} & G(F(x, y), F(u, v), F(w, z)) \\ & \leq \varphi(\max\{G(x, u, w), G(y, v, z)\}) - \psi(\max\{G(x, u, w), G(y, v, z)\}) \end{aligned}$$

for all $x, y, z, u, v, w \in X$ for which $x \geq u \geq w$ and $y \leq v \leq z$ where either $x \neq u$ or $y \neq v$. If there exists $x_0, y_0 \in X$ such that

$$x_0 \leq F(x_0, y_0) \quad \text{and} \quad y_0 \geq F(y_0, x_0)$$

and X has the following property:

- (i) if a nondecreasing sequence $\{x_n\}$ is G -convergent to x , then $x_n \leq x$ for all $n \in N$,
- (ii) if a nonincreasing sequence $\{y_n\}$ is G -convergent to y , then $y_n \geq y$ for all $n \in N$,

then F has a coupled fixed point.

Proof. Following the proof of Theorem we only have to check that (x, y) is a coupled fixed point of F .

In fact, since $\{x_n\}$ is nondecreasing and $x_n \rightarrow x$, and $\{y_n\}$ is nonincreasing and $y_n \rightarrow y$, by our assumption, $x_n \leq x$ and $y_n \geq y$ for every $n \in N$.

Applying the contractive condition of altering distance functions φ and ψ we have

$$\begin{aligned} & \varphi(G(F(x, y), F(x_n, y_n), F(x_n, y_n))) \\ & \leq \varphi(\max\{G(x, x_n, x_n)G(y, y_n, y_n)\}) - \psi(\max\{G(x, x_n, x_n)G(y, y_n, y_n)\}) \\ & \leq \varphi(\max\{G(x, x_n, x_n)G(y, y_n, y_n)\}). \end{aligned}$$

and, since φ is nondecreasing, we obtain

$$(2.15) \quad G(F(x, y), F(x_n, y_n), F(x_n, y_n)) \leq \max\{G(x, x_n, x_n)G(y, y_n, y_n)\}.$$

On the other hand, by the rectangle inequality and (2.15) we get

$$\begin{aligned} & G(x, F(x, y), F(x, y)) \\ & \leq G(x, x_{n+1}, x_{n+1}) + G(x_{n+1}, F(x, y), F(x, y)) \\ & = G(x, x_{n+1}, x_{n+1}) + G(F(x_n, y_n), F(x, y), F(x, y)) \\ & \leq G(x, x_{n+1}, x_{n+1}) + \max\{G(x_n, x, x), G(y_n, y, y)\}. \end{aligned}$$

Taking $n \rightarrow \infty$ in the last inequality, Lemma 1.3 yields

$$G(x, F(x, y), F(x, y)) = 0$$

and hence, $x = F(x, y)$.

Using a similar argument it can be proved that $y = F(y, x)$ and this finished the proof. $\square$

Now, we will show that many results can be deduced from our previously obtained results.

Corollary 2.3. If in Theorem 2.1 (resp. Theorem 2.3) we replace the contractive condition by

there exists $\alpha \in [0, 1)$ such that

$$G(F(x, y), F(u, v), F(w, z)) \leq \alpha \cdot \max\{G(x, u, w), G(y, v, z)\}$$

for all $x \geq u \geq w$ and $y \leq v \leq z$ where either $x \neq u$ or $y \neq v$,

then F has a coupled fixed point of F .

Proof. Taking as $\varphi = \text{identity}$ and $\psi = (1 - \alpha)\varphi$, we obtain the corollary. $\square$

Corollary 2.4. If in Theorem 2.1 (resp. Theorem 2.2) we replace the contractive condition by

there exist $\delta_1, \delta_2 \in [0, 1)$ and $\delta_1 + \delta_2 < 1$ such that

$$G(F(x, y), F(u, v), F(w, z)) \leq \delta_1 G(x, u, w) + \delta_2 G(y, v, z)$$

for all $x \geq u \geq w$ and $y \leq v \leq z$ where either $x \neq u$ or $y \neq v$,

then F has a coupled fixed point of F .

Proof. We have

$$\begin{aligned} G(F(x, y), F(u, v), F(w, z)) &\leq \delta_1 G(x, u, w) + \delta_2 G(y, v, z) \\ &\leq (\delta_1 + \delta_2) \max\{G(x, u, w), G(y, v, z)\}. \end{aligned}$$

Therefore, applying Corollary 2.3 we obtain the desired result. $\square$

Remark 2.5. Taking $\delta_1 = \delta_2 = \frac{k}{2}$ in Corollary 2.4, we can obtain Theorem 3.2 of Choudhury and Maity [10].

3. Uniqueness of Coupled Fixed Point in G -Metric Spaces

In this section, we consider some additional conditions to ensure the uniqueness of a coupled fixed point in the setting of partially ordered G -metric spaces. Furthermore, we study appropriate conditions to ensure that for a coupled fixed point (x, y) we have $x = y$.

Notice that if $(X, \leq)$ is a partially ordered set, we endow the product space $X \times X$ with the partial order relation given by

$$(u, v) \leq (x, y) \Leftrightarrow x \geq u \text{ and } y \leq v.$$

We say that two pairs (x, y) and (u, v) are *comparable*.

Theorem 3.1. *In addition to the hypotheses of Theorem 2.1, suppose that, for every $(a, b), (c, d) \in X \times X$, there exists a pair $(u, v) \in X \times X$ such that (u, v) is comparable to (a, b) and (c, d) . Then F has a unique coupled fixed point.*

Proof. Suppose that (x, y) and (z, t) are coupled fixed point of F , that is, $x = F(x, y)$, $y = F(y, x)$, $z = F(z, t)$ and $t = F(t, z)$.

Let (u, v) be an element of $X \times X$ and comparable to (x, y) and (z, t) . Suppose that $(x, y) \geq (u, v)$ (the proof is similar in other cases).

We construct the sequences $\{u_n\}$ and $\{v_n\}$ defined by

$$u_0 = u, \quad v_0 = v, \quad u_{n+1} = F(u_n, v_n), \quad v_{n+1} = F(v_n, u_n).$$

We claim that $(x, y) \geq (u_n, v_n)$ for each $n \in \mathbb{N}$.

We will use the induction.

For $n = 0$, as $(x, y) \geq (u, v)$, this means $u_0 = u \leq x$ and $v_0 = v \geq y$ and, thus, $(u_0, v_0) \leq (x, y)$.

Suppose that $(x, y) \geq (u_n, v_n)$ for some $n \in N$. Then using the mixed monotone property of F , we get

$$\begin{aligned} u_{n+1} &= F(u_n, v_n) \leq F(x, y) = x, \\ v_{n+1} &= F(v_n, u_n) \geq F(y, x) = y \end{aligned}$$

and this proves our claim.

Since $(x, y) \geq (u_n, v_n)$, using the contractive condition we have

$$\begin{aligned} &\varphi(G(x, u_{n+1}, u_{n+1})) \\ &= \varphi(G(F(x, y), F(u_n, v_n), F(u_n, v_n))) \\ &\leq \varphi(\max\{G(x, u_n, u_n), G(y, v_n, v_n)\}) - \psi(\max\{G(x, u_n, u_n), G(y, v_n, v_n)\}) \\ &\leq \varphi(\max\{G(x, u_n, u_n), G(y, v_n, v_n)\}) \end{aligned}$$

and

$$\begin{aligned} &\varphi(G(y, v_{n+1}, v_{n+1})) \\ &= \varphi(G(F(y, x), F(v_n, u_n), F(v_n, u_n))) \\ &\leq \varphi(\max\{G(x, u_n, u_n), G(y, v_n, v_n)\}) - \psi(\max\{G(x, u_n, u_n), G(y, v_n, v_n)\}) \\ &\leq \varphi(\max\{G(x, u_n, u_n), G(y, v_n, v_n)\}). \end{aligned}$$

By the last two equation and using the fact that φ is nondecreasing, we obtain

$$\begin{aligned} (3.1) \quad &\varphi(\max\{G(x, u_{n+1}, u_{n+1}), G(y, v_{n+1}, v_{n+1})\}) \\ &= \max\{\varphi(G(x, u_{n+1}, u_{n+1})), \varphi(G(y, v_{n+1}, v_{n+1}))\} \\ &\leq \varphi(\max\{G(x, u_n, u_n), G(y, v_n, v_n)\}) - \psi(\max\{G(x, u_n, u_n), G(y, v_n, v_n)\}) \\ &\leq \varphi(\max\{G(x, u_n, u_n), G(y, v_n, v_n)\}). \end{aligned}$$

This last inequality implies that

$$\max\{G(x, u_{n+1}, u_{n+1}), G(y, v_{n+1}, v_{n+1})\} \leq \max\{G(x, u_n, u_n), G(y, v_n, v_n)\}.$$

Consequently, the sequence $(\max\{G(x, u_n, u_n), G(y, v_n, v_n)\})$ is decreasing and nonnegative, and so, for certain $\alpha \geq 0$

$$\lim_{n \rightarrow \infty} \max\{G(x, u_n, u_n), G(y, v_n, v_n)\} = \alpha.$$

Letting $n \rightarrow \infty$ in (3.1) we have

$$\varphi(\alpha) \leq \varphi(\alpha) - \psi(\alpha) \leq \varphi(\alpha),$$

and this implies $\psi(\alpha) = 0$ and, thus, $\alpha = 0$.

Finally, as $\lim_{n \rightarrow \infty} \max\{G(x, u_n, u_n), G(y, v_n, v_n)\} = 0$, this gives us that the sequences $\{u_n\}$ and $\{v_n\}$ are G -convergent to x and y , respectively. This means that

$$(3.2) \quad \begin{aligned} \lim_{n \rightarrow \infty} G(x, u_n, u_n) &= \lim_{n \rightarrow \infty} G(x, x, u_n) = 0, \\ \lim_{n \rightarrow \infty} G(y, v_n, v_n) &= \lim_{n \rightarrow \infty} G(y, y, v_n) = 0. \end{aligned}$$

Using a similar argument for a coupled fixed point (z, t) , we can obtain $\{u_n\}$ and $\{v_n\}$ are G -convergent to z and t , respectively, that is,

$$(3.3) \quad \begin{aligned} \lim_{n \rightarrow \infty} G(z, u_n, u_n) &= \lim_{n \rightarrow \infty} G(z, z, u_n) = 0, \\ \lim_{n \rightarrow \infty} G(t, v_n, v_n) &= \lim_{n \rightarrow \infty} G(t, t, v_n) = 0. \end{aligned}$$

By the rectangle inequality, for any $n \in N$, we have

$$\begin{aligned} G(x, z, z) &\leq G(x, u_n, u_n) + G(u_n, z, z), \\ G(y, t, t) &\leq G(y, v_n, v_n) + G(v_n, t, t) \end{aligned}$$

Letting $n \rightarrow \infty$ in the last inequalities, and using (3.2) and (3.3) we get

$$G(x, z, z) = G(y, t, t) = 0$$

and, consequently, $(x, y) = (z, t)$. $\square$

Theorem 3.2. *In addition to the hypotheses of Theorem 2.1, if x_0 and y_0 are comparable, then the coupled fixed point $(x, y) \in X \times X$ satisfies $x = y$.*

Proof. Assume $x_0 \leq y_0$ (a similar argument applies for $y_0 \leq x_0$).

Then by using the mathematical induction

$$x_{n+1} = F(x_n, y_n) \leq F(y_n, x_n) = y_{n+1}.$$

Taking $n \rightarrow \infty$, we have

$$x = \lim_{n \rightarrow \infty} x_n \leq \lim_{n \rightarrow \infty} y_n = y.$$

From the contractive condition, we get

$$\begin{aligned} &\varphi(G(x, y, y)) \\ &= \varphi(G(F(x, y), F(y, x), F(y, x))) \\ &\leq \varphi(\max\{G(x, y, y), G(x, x, y)\}) - \psi(\max\{G(x, y, y), G(x, x, y)\}) \\ &\leq \varphi(\max\{G(x, y, y), G(x, x, y)\}) \end{aligned}$$

and

$$\begin{aligned} &\varphi(G(x, x, y)) \\ &= \varphi(G(F(x, y), F(x, y), F(y, x))) \\ &\leq \varphi(\max\{G(x, y, y), G(x, x, y)\}) - \psi(\max\{G(x, y, y), G(x, x, y)\}) \\ &\leq \varphi(\max\{G(x, y, y), G(x, x, y)\}). \end{aligned}$$

Since $\varphi : [0, \infty) \rightarrow [0, \infty)$ is nondecreasing, $\varphi(\max\{a, b\}) = \max\{\varphi(a), \varphi(b)\}$ for $a, b \in [0, \infty)$. Taking into account this and the last two inequalities we get

$$\psi(\max\{G(x, y, y), G(x, x, y)\}) = 0.$$

Using the fact that ψ is nondecreasing, we have

$$G(x, y, y) = G(x, y, y) = 0$$

and, consequently, $x = y$. $\square$

Example 3.3. Let $X = [0, \frac{1}{2}]$. Then $(X, \leq)$ is a partially ordered set with a natural ordering of real numbers. Let $G(x, y, z) = |x - y| + |y - z| + |z - x|$ for all $x, y, z \in X$. Let $F : X \times X \rightarrow X$ be defined as

$$F(x, y) = \begin{cases} \frac{x^2 - y^2 + 1}{3}, & x \leq y \\ \frac{1}{3}, & x > y. \end{cases}$$

Then

- (1) (X, G) is a complete G -metric space;
- (2) F has the mixed monotone property;
- (3) F is continuous;
- (4) $0 \leq F(0, \frac{1}{2})$ and $\frac{1}{2} \geq F(\frac{1}{2}, 0)$;
- (5) there exist two altering distance functions φ and ψ such that

$$\begin{aligned} & \varphi(G(F(x, y), F(u, v), F(w, z))) \\ & \leq \varphi(\max\{G(x, u, w), G(y, v, z)\}) - \psi(\max\{G(x, u, w), G(y, v, z)\}) \end{aligned}$$

for all $(x, y), (u, v), (w, z) \in X \times X$ with $x \leq u \leq w$ and $y \geq v \geq z$.

Thus by Theorem , F has a coupled fixed point. Moreover, $(\frac{1}{3}, \frac{1}{3})$ is the unique coupled fixed point of F .

Proof. The proofs of (1)-(4) are clear.

For any $x \leq u \leq w$ and $y \geq v \geq z$, we have

$$G(x, u, w) = 2(w - x), \quad G(y, v, z) = 2(y - z).$$

The proof of (5) is divided into the following cases.

Case 1. If $w \leq z$. In this case, we have $x \leq u \leq w \leq z \leq v \leq y$, and so

$$F(x, y) = \frac{x^2 - y^2 + 1}{3}, \quad F(u, v) = \frac{u^2 - v^2 + 1}{3}, \quad F(w, z) = \frac{w^2 - z^2 + 1}{3}.$$

Hence, we get

$$\begin{aligned} G(F(x, y), F(u, v), F(w, z)) &= G\left(\frac{x^2 - y^2 + 1}{3}, \frac{u^2 - v^2 + 1}{3}, \frac{w^2 - z^2 + 1}{3}\right) \\ &= \frac{2}{3}(y^2 - x^2 + w^2 - z^2) \\ &\leq \frac{1}{3} \max\{2(y^2 - z^2), 2(w^2 - x^2)\} \\ &\leq \frac{1}{3} \max\{2(y - z), 2(w - x)\}. \end{aligned}$$

Case 2. $w > z$. We divide the study in two sub-cases:

(a) If $u \leq v$, then $x \leq u \leq v \leq y$. Therefore, we get

$$F(x, y) = \frac{x^2 - y^2 + 1}{3}, \quad F(u, v) = \frac{u^2 - v^2 + 1}{3}, \quad F(w, z) = \frac{1}{3}.$$

Hence, we get

$$\begin{aligned} G(F(x, y), F(u, v), F(w, z)) &= G\left(\frac{x^2 - y^2 + 1}{3}, \frac{u^2 - v^2 + 1}{3}, \frac{1}{3}\right) \\ &= \frac{2}{3}(y^2 - x^2) \\ &\leq \frac{2}{3}(y^2 - x^2 + w^2 - z^2) \\ &\leq \frac{1}{3} \max\{2(y^2 - z^2), 2(w^2 - x^2)\} \\ &\leq \frac{1}{3} \max\{2(y - z), 2(w - x)\}. \end{aligned}$$

(b) If $u > v$, hence $F(u, v) = \frac{1}{3} = F(w, z)$; the case where $x > y$ is obvious because we get $F(x, y) = \frac{1}{3}$. If $x \leq y$, we have $F(x, y) = \frac{x^2 - y^2 + 1}{3}$. Therefore

$$\begin{aligned} G(F(x, y), F(u, v), F(w, z)) &= G\left(\frac{x^2 - y^2 + 1}{3}, \frac{1}{3}, \frac{1}{3}\right) \\ &= \frac{2}{3}(y^2 - x^2) \\ &\leq \frac{2}{3}(y^2 - x^2 + w^2 - z^2) \\ &\leq \frac{1}{3} \max\{2(y^2 - z^2), 2(w^2 - x^2)\} \\ &\leq \frac{1}{3} \max\{2(y - z), 2(w - x)\}. \end{aligned}$$

In all the above cases, the condition (5) is satisfied for the altering distance functions $\varphi = I$ and $\psi = \frac{2}{3}I$ (where I is an identity mapping). Since $X = [0, \frac{1}{2}]$ is a totally ordered set, by Theorem 3.2, $(\frac{1}{3}, \frac{1}{3})$ is the the unique coupled fixed point of F . $\square$

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