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The π-extending Property via Singular Quotient Submodules

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Abstract. A module is said to be π-extending provided that every projection invariant submodule is essential in a direct summand of the module. In this article, we focus on the class of modules having the π-extending property by looking at the singularity of quotient submodules. By doing so, we provide counterexamples, using hypersurfaces in projective spaces over complex numbers, to show that being generalized π-extending is not inherited by direct summands. Moreover, it is shown that the direct sums of generalized π-extending modules are generalized π-extending.

1. Introduction

All rings are associative with unity unless indicated otherwise. R and M will denote a ring and a right R-module, respectively. Recall that a module M is called *extending* or CS if every submodule of M is essential in a direct summand of M or equivalently; every complement submodule of M is a direct summand of M. There have been many generalizations of extending modules including the class of π-extending modules [15]. A module M is called π-*extending* [2] if every projection invariant submodule (i.e., every submodule which is invariant under all idempotent endomorphisms of M) is essential in a direct summand of M.

In this paper, we investigate the π-extending property of modules via singular quotient submodules. To this end, we call a module M *generalized* π-*extending* if for every projection invariant submodule N of M, there exists a direct summand K of M such that $N \leq K$ and K/N is singular. Furthermore, a ring R is called *right*

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generalized π-extending if R_R is a right generalized π-extending module. The class of extending modules and the class of π-extending modules are proper subclasses of generalized π-extending modules. We obtain fundamental results related to the notion of generalized π-extending and make connections with the notions of extending and π-extending. Moreover, we determine conditions which ensure the equivalence of the extending, π-extending and generalized π-extending properties. We then examine direct summands and direct sums properties for the aforementioned class. We provide several counterexamples including hypersurfaces in projective spaces over complex numbers to show that being generalized π-extending is not inherited by direct summands. Having done this, we deal with the question of when a direct summand of a generalized π-extending module is generalized π-extending. We prove that the class of generalized π-extending modules is closed under direct sums. Moreover, we exhibit several applications on right generalized π-extending rings including polynomial rings.

Recall from [11], a module M has C_2 (C_3) condition if for each direct summand K of M and each monomorphism $\alpha : K \to M$, the submodule $\alpha(K)$ is a direct summand of M (if for all direct summand K and L of M with $K \cap L = 0$, the submodule $K \oplus L$ is also a direct summand of M). It is clear that C_2 implies C_3 but not conversely [11]. Throughout this paper, if $X \subseteq M$, then $X \leq M$, $X \leq_e M$, $X \trianglelefteq_p M$, $Z(M)$, $Z_2(M)$ and $E(M)$ denote X is a submodule of M, X is an essential submodule of M, X is a projection invariant submodule of M, the singular submodule of M, the second singular submodule of M and the injective hull of M, respectively. A ring R is called *Abelian* if every idempotent of R is central. Let $e^2 = e \in R$. Recall from [3], e is called a *left* (*right*) *semicentral idempotent* if $xe = exe$ $(ex = exe)$ for all $x \in R$. $S_l(R)$ and $S_r(R)$ denote the set of left semicentral idempotents and right semicentral idempotents, respectively. Other terminology can be found in [5, 9, 11, 15].

2. Basic Results

In this section, we deal with the class of generalized π-extending modules. We obtain fundamental results and make connections with the notions of extending and π-extending. Moreover, we determine conditions which ensure the equivalence of the π-extending and generalized π-extending properties. Let us begin with a basic fact about projection invariant submodules.

Lemma 2.1.([6, Exercise 4]) *Let M be a right R-module. Then*

(i) *Any sum or intersection of projection invariant submodule of M is projection invariant submodule of M.*

(ii) *Let X and Y be submodules of M such that $X \leq Y \leq M$. If X is projection invariant in Y and Y is projection invariant in M then X is projection invariant in M.*

(iii) *Let $M = \bigoplus_{i\in I} M_i$ and N be a projection invariant submodule of M. Then*

$N = \bigoplus_{i \in I}(\pi_i(N)) = \bigoplus_{i \in I}(N \cap M_i)$ *where* π_i *is the i-th projection map.*

It can be easily seen that any singular module satisfies generalized π-extending condition. But the converse of this fact is not true. For example, $M_{\mathbb{Z}} = \mathbb{Z}_{\mathbb{Z}}$ is generalized π-extending, but it is nonsingular.

Lemma 2.2. *Let M be a module. Consider the following statements:*

(i) *M is extending.*

(ii) *M is π-extending.*

(iii) *M is generalized π-extending.*

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii), *but the reverse implications do not hold, in general.*

Proof. (i) $\Rightarrow$ (ii). It is clear from [2, Proposition 3.7].

(ii) $\Rightarrow$ (iii). Let N be a projection invariant submodule of M. Then there exists a direct summand K of M such that $N \leq_e K$. Let $k \in K$. Then there exists an essential right ideal $I = \{r \in R \mid kr \in N\}$ of R. Hence $(k+N)I = 0$ implies that K/N is singular. Thus M is generalized π-extending.

(ii) $\not\Rightarrow$ (i). Let M be $\mathbb{Z}$-module such that $M = (\mathbb{Z}/\mathbb{Z}p) \oplus (\mathbb{Z}/\mathbb{Z}p^3)$ for any prime p. It is well known that $M_{\mathbb{Z}}$ is not extending by [13, page 1814]. However it is π-extending.

(iii) $\not\Rightarrow$ (ii). Let R be a subring of $\begin{bmatrix} F & V \\ 0 & F \end{bmatrix}$ such that

$$R = \left\{ \begin{bmatrix} f & v \\ 0 & f \end{bmatrix} : f \in F, v \in V \right\}$$

where F is a field and V_F is a vector space over F with dimension 2. It is clear that R_R is a commutative and indecomposable module. Since the dimension of V_F is 2, R_R is not uniform. Hence R_R is not π-extending by [2, Proposition 3.8]. On the other hand, let $N_1 = \begin{bmatrix} 0 & v_1F \oplus 0 \\ 0 & 0 \end{bmatrix}$, $N_2 = \begin{bmatrix} 0 & 0 \oplus v_2F \\ 0 & 0 \end{bmatrix} \leq R_R$ for $v_1, v_2 \in V$. It is clear that N_1 and N_2 are projection invariant in R_R and $N_1, N_2 \subseteq Z(R_R) = \begin{bmatrix} 0 & V \\ 0 & 0 \end{bmatrix}$. Thus N_1 and N_2 singular. It follows that R/N_1 and R/N_2 are singular which yields that R_R is a generalized π-extending module. $\square$

Proposition 2.3. *The following statements are equivalent for a nonsingular module M.*

(i) *M is π-extending.*

(ii) *M is generalized π-extending.*

(iii) *Every projection invariant essentially closed submodule of M is a direct summand.*

Proof. (i) $\Rightarrow$ (ii). It is obvious from Lemma 2.2.

(ii) $\Rightarrow$ (iii). Let X be a projection invariant essentially closed submodule of M. Then there exists a direct summand K of M such that $X \leq K$ and K/X is singular. Hence $X \leq_e K$ by [7, Proposition 1.21]. Since X has no proper essential extension, $X = K$ which gives that X is a direct summand of M.

(iii) $\Rightarrow$ (i). Let N be a projection invariant submodule of M. Then there exists a submodule K of M such that K is an essential closure of N in M. Since M is nonsingular, K is projection invariant in M by [2, Proposition 2.4]. Now K is a direct summand of M by assumption. Then M is π-extending. □

Recall that a module M is called C_{11}-*module* [13] if each submodule of M has a complement that is a direct summand of M.

Proposition 2.4. *Let M be an indecomposable nonsingular R-module. Then the following statements are equivalent.*

(i) *M is uniform.*

(ii) *M is extending.*

(iii) *M is a C_{11}-module.*

(iv) *M is π-extending.*

(v) *M is generalized π-extending.*

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv). These implications are obvious from [2, Proposition 3.7].

(iv) $\Rightarrow$ (v) It is clear from Lemma 2.2.

(v) $\Rightarrow$ (i). Let $0 \neq X \leq M$. Since M is indecomposable, every submodule of M is projection invariant so $X \trianglelefteq_p M$. Then there exists a direct summand K of M such that $X \leq K$ and K/X is singular. It follows that $X \leq_e K$ by [7, Proposition 1.21]. Hence $K = M$, so M is uniform. □

Observe that if R is an indecomposable right generalized π-extending ring, then for all nonzero right ideal I of R, R/I is singular. It follows that there exists an essential right ideal J of R such that $\bar{x}J \subseteq I$ for all $\bar{x} \in R/I$. Hence I is an essential right ideal of R which gives that R_R is uniform. Therefore Proposition 2.4 is true without nonsingularity condition for an indecomposable ring R.

The following result provides a number of characterizations for generalized π-extending modules.

Proposition 2.5. *The following statements are equivalent for a module M.*

(i) *M is a generalized π-extending module.*

(ii) *For any projection invariant submodule N of M, there exists a direct summand K of M such that $N \leq K$ and $M/(K' \oplus N)$ is singular where $M = K \oplus K'$ for some submodule K' of M.*

(iii) *For any projection invariant submodule N, M/N has a decomposition $M/N = (K/N) \oplus (K'/N)$ such that K is a singular direct summand of M where $M = K \oplus K'$ for some submodule K' of M.*

(iv) *For any projection invariant submodule N of M, there exists a direct summand K of M such that $N \leq K$ and for any $x \in K$, there is an essential right ideal I of R such that $xI \leq N$.*

Proof. (i) $\Rightarrow$ (ii). Let $N \trianglelefteq_p M$. Then there exists a direct summand K of M such that $N \leq K$ and K/N is singular. Hence $M = K \oplus K'$ for some submodule K' of M. It is clear that $N \cap K' = 0$. Thus $K/N \cong M/(K' \oplus N)$ is singular.

(ii) $\Rightarrow$ (iii). Take $N = 0$ in (ii) which yields the result.

(iii) $\Rightarrow$ (iv). Let $N \trianglelefteq_p M$. Then $M/K' \cong K$ is singular by the condition (iii). Let $x \in K$. Then there exists an essential right ideal I of R such that $xI = 0$, as K is singular. Hence we get the result.

(iv) $\Rightarrow$ (i). Let $N \trianglelefteq_p M$. Then there exists a direct summand K of M such that $N \leq K$ and for any $x \in K$, there is an essential right ideal I of R such that $xI \leq N$. We need to show that K/N is singular. Since $xI \leq N$, we have $(x + N)I \leq N$. Thus $x + N \in Z(K/N)$, which yields that K/N is singular. □

Any submodule of generalized π-extending modules need not to be generalized π-extending as shown in the following example.

Example 2.6. Let M be the Specker group, $M_{\mathbb{Z}} = \prod_{i=1}^{\infty} A_i$ with $A_i = \mathbb{Z}$ for any positive integer i. Then $M_{\mathbb{Z}}$ is not π-extending by [6], but $M_{\mathbb{Z}}$ is nonsingular by [7, Proposition 1.22]. Hence $M_{\mathbb{Z}}$ is not generalized π-extending module by Proposition 2.3. However $M_{\mathbb{Z}}$ is a submodule of its injective hull $E(M_{\mathbb{Z}})$ while $E(M_{\mathbb{Z}})$ is a generalized π-extending module.

We focus whether the generalized π-extending property is inherited by submodules.

Lemma 2.7. *Let M be a generalized π-extending module and N any projection invariant submodule of M. Then N is a generalized π-extending module.*

Proof. Let $X \trianglelefteq_p N$ and $N \trianglelefteq_p M$. Hence $X \trianglelefteq_p M$ by Lemma 2.1. Then there exists a direct summand K of M such that $X \leq K$ and K/X is singular. Hence $M = K \oplus K'$ for some submodule K' of M. Since $N \trianglelefteq_p M$, $N = (N \cap K) \oplus (N \cap K')$ by Lemma 2.1. It is clear that $X \leq N \cap K$ where $N \cap K$ is a direct summand of N and $(N \cap K)/X \leq K/X$. Thus $(N \cap K)/X$ is singular, hence N is generalized π-extending module. □

Proposition 2.8. *Let R be a nonsingular right R-module. Then M is generalized π-extending if and only if for any projection invariant submodule N of M, there exists $e^2 = e \in \mathrm{End}(E(M))$ such that $N \leq e(E(M))$, $e(E(M))/N$ is singular and $e(M) \leq M$.*

Proof. Let M be generalized π-extending and $N \trianglelefteq_p M$. Then there exists a direct summand K of M such that $N \leq K$ and K/N is singular. Hence $M = K \oplus K'$

for some submodule K' of M. Let $\tau : E(M) \to E(K)$ be a projection map. Then $\tau(M) \leq M$ and $K/N \leq E(K)/N = \tau(E(M))/N$. Since K/N is singular and $Z(R_R) = 0$, $\tau(E(M))/N$ is singular by [7, Proposition 1.23]. Conversely, let $N \trianglelefteq_p M$. Then there exists $e^2 = e \in \mathrm{End}(E(M))$ such that $N \leq e(E(M))$, $e(E(M))/N$ is singular and $e(M) \leq M$ by hypothesis. Since $e(M) \leq M$, $e(M)$ is a direct summand of M. It is clear that $N \leq M \cap e(E(M)) \leq e(M)$ and $e(M) \leq e(E(M))$. Thus $e(M)/N \leq e(E(M))/N$ gives that $e(M)/N$ singular. Therefore M is generalized π-extending. □

3. Direct Sums and Direct Summands

In this section, we examine direct summands and direct sums properties of generalized π-extending modules. It is shown that a direct summand of generalized π-extending modules need not to be generalized π-extending. Further, we deal with when a direct summand of a generalized π-extending module is generalized π-extending. Moreover, we are able to show that the class of generalized π-extending modules is closed under direct sums.

It is well known that a direct summand of any extending module is extending. In contrast to extending modules, generalized π-extending property is not inherited by direct summands. The following results illustrate this fact.

Example 3.1.([2, Example 5.5] or [14, Example 4]) Let $\mathbb{R}$ be the real field and n any odd integer with $n \geq 3$. Let S be the polynomial ring $\mathbb{R}[x_1, ..., x_n]$ with indeterminates $x_1, ..., x_n$ over $\mathbb{R}$. Let R be the ring S/Ss where $s = x_1^2 + ... + x_n^2 - 1$. Then the free R-module $M = \oplus_{i=1}^n R$ is generalized π-extending, but contains a direct summand K_R which is not generalized π-extending.

Proof. M_R is a π-extending module which contains a direct summand K_R is not π-extending by [2, Example 5.5]. Hence M_R is generalized π-extending module by Lemma 2.2. It is clear that R is a commutative Noetherian domain. Then M_R is a nonsingular module, so is K_R. Therefore K_R is not generalized π-extending by Proposition 2.3. □

We can construct more examples which based on hypersurfaces in projective spaces, $\mathbb{P}_{\mathbb{C}}^{n+1}$ over complex numbers.

Theorem 3.2.([10, Theorem 1.5]) *Let X be the hypersurface in $\mathbb{P}_{\mathbb{C}}^{n+1}$, $n \geq 2$, defined by the equation $x_0^m + x_1^m + \cdots + x_{n+1}^m = 0$. Let*

$$R = \mathbb{C}[x_1, \ldots, x_{n+1}]/(\sum_{i=1}^{n+1} x_i^m + 1)$$

be the coordinate ring of X. Then there exist generalized π-extending R-modules but contain direct summands which are not generalized π-extending for $m \geq n+2$.

Proof. There are indecomposable projective R-modules of rank n over R by [12]. Then there exists a free R-module F_R such that $F_R = K \oplus K'$ where K is indecomposable and projective R-module of rank n. From Theorem 3.9, F_R is generalized

π-extending. However K_R is not uniform. Thus K_R is not generalized π-extending by Proposition 2.4. □

The next proposition gives a condition which ensures that a direct summand of a module is generalized π-extending module.

Proposition 3.3. *Let $M = M_1 \oplus M_2$. Then M_1 is generalized π-extending if and only if for every projection invariant submodule N of M_1 there exists a direct summand K of M such that $M_2 \subseteq K$, $K \cap N = 0$ and $M/(K \oplus N)$ is singular.*

Proof. Let N be a projection invariant submodule of M_1. Then there exists a direct summand L of M_1 such that $M_1 = L \oplus L'$ with $N \leq L$ and $M_1/(L' \oplus N)$ is singular by Proposition 2.5. It is clear that $L' \oplus M_2$ is a direct summand of M, $M_2 \subseteq L' \oplus M_2$ and $(L' \oplus M_2) \cap N = 0$. Moreover $M_1/(L' \oplus N) \cong M/(L' \oplus N \oplus M_2)$ is singular. Conversely, let M_1 holds the assumptions. Let T be a projection invariant submodule of M_1. By hypothesis there exists a direct summand K of M such that $M_2 \subseteq K$, $K \cap T = 0$ and $M/(K \oplus T)$ is singular. Now $K = K \cap (M_1 \oplus M_2) = M_2 \oplus (K \cap M_1)$ yields that $K \cap M_1$ is a direct summand of M_1. Hence there exists a submodule X of M_1 such that $M_1 = (K \cap M_1) \oplus X$. Since T is a projection invariant submodule of M_1, $T = (T \cap K \cap M_1) \oplus (T \cap X)$ by Lemma 2.1. Note that $K \cap T = 0$, hence we get $T \leq X$. Furthermore it can be easily seen that $M/(K \oplus T) \cong M_1/[(K \cap M_1) \oplus T]$ is singular. Thus Proposition 2.5 yields the result. □

Theorem 3.4. *Let $M = M_1 \oplus M_2$ be a generalized π-extending module. If M_2 is a projection invariant direct summand and for every direct summand K of M with $K \cap M_2 = 0$ and $K \oplus M_2$ is a direct summand of M, then M_1 and M_2 are generalized π-extending.*

Proof. It is clear that M_2 is generalized π-extending by Lemma 2.7. Now, let N be a projection invariant submodule of M_1. Then $N \oplus M_2$ is a projection invariant submodule of M by [2, Lemma 4.13]. By Proposition 2.5, there exists a direct summand K of M such that $N \oplus M_2 \leq K$ and $M/(K' \oplus N \oplus M_2)$ is singular where $M = K \oplus K'$ for some submodule K' of M. Since $K' \cap M_2 \subseteq K' \cap (N \oplus M_2) = 0$, $K' \oplus M_2$ is a direct summand of M.
Now the result follows from Proposition 3.3. □

Corollary 3.5. *Let $M = M_1 \oplus M_2$ be a generalized π-extending module with C_3 condition. If M_2 is a projection invariant direct summand, then M_1 and M_2 are generalized π-extending.*

Proof. It is clear from Theorem 3.4. □

The next result characterizes the direct summand of a generalized π-extending module in terms of relative injectivity.

Proposition 3.6. *Let R be a nonsingular right R-module and M a generalized π-extending module. Then $M = Z(M) \oplus T$ for some submodule T of M and T is $Z(M)$-injective.*

Proof. If $Z(M) = 0$ or $Z(M) = M$, then the result holds trivially. Assume $Z(M) \neq 0$ and $Z(M) \neq M$. Since $Z(M)$ is a fully invariant submodule of M, it is also projection invariant submodule of M. Hence there exists a direct summand K of M such that $Z(M) \leq K$ and $K/Z(M)$ is singular where $M = K \oplus T$ for some $T \leq M$. Thus K is singular by [7, Proposition 1.23]. It follows that $K = Z(M)$. Thus $M = Z(M) \oplus T$ for some $T \leq M$. Now, let N be a submodule of $Z(M)$. Note that $Z(T) = 0$, so $\mathrm{Hom}_R(N, T) = 0$ by [7, Proposition 1.20]. Therefore T is $Z(M)$-injective. □

Proposition 3.7. *Let M be a π-extending module and K a projection invariant submodule of M such that K is essentially closed in M. If M/K is nonsingular, then $M = Z_2(M) \oplus X \oplus Y$ where $K = Z_2(M) \oplus X$ and Y are generalized π-extending.*

Proof. Let M be π-extending, $K \trianglelefteq_p M$ and K essentially closed in M. Then $M = K \oplus N$ for some $N \leq M$ by [2, Corollary 3.2]. Since $K \trianglelefteq_p M$, K and N are π-extending by [2, Proposition 4.14] and hence K and N are generalized π-extending by Lemma 2.2. Note that $Z_2(M)$ is projection invariant closed submodule of M which follows that $Z_2(M) = eM$ for some $e \in S_l(End_R(M))$ by [2, Proposition 4.12] and [2, Corollary 3.2]. Since M/K is nonsingular, $Z_2(M) \subseteq K$. Thus $K = Z_2(M) \oplus X$ where $X = (1-e)M \cap K$. Now, $M = K \oplus N = Z_2(M) \oplus X \oplus N$. So let $N = Y$, Y is the desired direct summand. □

Proposition 3.8. *Let M be a generalized π-extending module with Abelian endomorphism ring. Then every direct summand of M is generalized π-extending.*

Proof. Let M be generalized π-extending module and K be a direct summand of M. Let $S = \mathrm{End}(M_R)$ and $\pi : M \to K'$ be the canonical projection where $K' \leq M$ such that $M = K \oplus K'$. It is clear that $\ker\pi = K$. Since S is Abelian, $f(\ker\pi) \subseteq \ker\pi$ for all $f^2 = f \in S$. Hence K is a projection invariant submodule of M. Therefore apply Lemma 2.7 to get the result. □

It is well known that a direct sum of extending modules (even, for uniform modules) need not to be an extending module, in general. For example, let M be the $\mathbb{Z}$-module $(\mathbb{Z}/\mathbb{Z}p) \oplus (\mathbb{Z}/\mathbb{Z}p^3)$ for any prime p, and let $R = \mathbb{Z}[x]$ be the polynomial ring. Now, let us think of the free R-module $T = R \oplus R$. Then both $M_{\mathbb{Z}}$ and T_R is not extending (see, [15]). However, generalized π-extending property yields the following result.

Theorem 3.9. *Any direct sum of generalized π-extending modules is generalized π-extending.*

Proof. Let $M = \bigoplus_{i \in I} M_i$ where M_i is generalized π-extending for all $i \in I$. Let N be projection invariant submodule of M. Then $N = \bigoplus_{i \in I}(M_i \cap N)$ by Lemma 2.1. Note that $N \cap M_i \trianglelefteq_p M_i$ for all $i \in I$. Hence there exists a direct summand H_i of M_i such that $N \cap M_i \leq H_i$ and $H_i/(N \cap M_i)$ is singular. Then $H = \bigoplus_{i \in I} H_i$ is a direct summand of M such that $N \leq H$. It is clear that H/N is singular. Thus M is generalized π-extending. □

Corollary 3.10. *Let $M = \bigoplus_{i\in I} M_i$ where M_i is projection invariant in M for all $i \in I$. Then M is generalized π-extending if and only if M_i is generalized π-extending for all $i \in I$.*

Proof. The result follows from Lemma 2.7 and Theorem 3.9. □

Corollary 3.11. *Let M has an Abelian endomorphism ring. Then $M = \bigoplus_{i\in I} M_i$ is generalized π-extending if and only if M_i is generalized π-extending for all $i \in I$.*

Proof. It is a consequence of Proposition 3.8 and Theorem 3.9. □

Corollary 3.12. *Let M be a nonsingular module. Then M is π-extending module if and only if $M = Z_2(M) \oplus X$ where X and $Z_2(M)$ are generalized π-extending.*

Proof. Let M be a π-extending module. Then take $K = Z_2(M)$ and apply Proposition 3.7 to get the result. Conversely, assume that M has stated property. It is clear from Theorem 3.9, M is generalized π-extending. Hence M is π-extending by Proposition 2.3. □

Recall that extending property is not closed under essential extensions (see, [11, page 19]). To this end, the following example shows that π-extending and generalized π-extending modules behave same as extending modules with respect to the essential extensions.

Example 3.13. Let R be a principal ideal domain. If R is not a complete discrete valuation ring then there exists an indecomposable torsion-free R-module M of rank 2 by [8, Theorem 19]. Hence there exist uniform submodules U_1, U_2 of M such that $U_1 \oplus U_2$ is essential in M. Then $U_1 \oplus U_2$ is generalized π-extending by Theorem 3.9. However M_R is not generalized π-extending by Proposition 2.4.

Recall that a ring R is *right generalized π-extending* in case for every projection invariant right ideal I of R, there exists $e^2 = e \in R$ such that $I \leq eR$ and eR/I is singular. Finally, we obtain the following applications on generalized π-extending rings.

Proposition 3.14. *Let R be a right generalized π-extending ring. Then every cyclic R-module is generalized π-extending.*

Proof. Let R be a right generalized π-extending ring and M a cyclic R-module. Then there exists a right ideal I of R such that $M \cong R/I$. Let $J/I \trianglelefteq_p R/I$ where $I \leq J \leq R$. Then it is clear that $J \trianglelefteq_p R$. Hence there exists $e^2 = e \in R$ such that $J \leq eR$ and eR/J is singular. Since $J/I \leq eR/I$ and $(eR/I)/(J/I) \cong eR/J$ is singular, M is generalized π-extending. □

Theorem 3.15. *R is a right generalized π-extending ring if and only if $R[x]$ is a right generalized π-extending ring.*

Proof. Let R be a right generalized π-extending ring and $I[x]$ be a projection invariant right ideal of $R[x]$. Then I is a projection invariant right ideal of R by [4, Lemma 4.1]. Thus there exists $e^2 = e \in R$ such that $I \leq eR$ and eR/I is singular. Notice that $I = eI$, so $I[x] = eI[x]$. It is clear that $eR[x]$ is a direct summand of $R[x]$ and $I[x] = eI[x] \leq eR[x]$. It is easy to see that $eR[x]/I[x] \cong (eR/eI)[x]$. Observe that $Z_{R[x]}(eR[x]/I[x]) \cong (Z_R(eR/eI))[x] = (eR/eI)[x] \cong eR[x]/I[x]$ which shows that $eR[x]/I[x]$ is singular. Hence $R[x]$ is right generalized π-extending.

Conversely, let $R[x]$ be right generalized π-extending and J a projection invariant right ideal of R. Then $J[x]$ is a projection invariant right ideal of $R[x]$ by [4, Lemma 4.1]. It follows that $J[x] \leq fR[x]$ and $fR[x]/J[x]$ is singular for some $f^2 = f \in R[x]$. Note that $fJ[x] = J[x]$, and let $g^2 = g \in R[x]$. Then $g(J[x]) \subseteq J[x]$, as $J[x]$ is projection invariant right ideal of $R[x]$. Hence we obtain that $fgf = gf$, so $f \in S_l(R[x])$. Observe from [1, Proposition 2.4] that $fR[x] = f_0R[x]$ for some $f_0 \in S_l(R)$. Since $J[x] \leq fR[x] = f_0R[x]$, so $J \leq f_0R$. Further, $Z_R(f_0R/J) \leq Z_{R[x]}((f_0R/J)[x]) \cong Z_{R[x]}(f_0R[x]/J[x]) = Z_{R[x]}(fR[x]/J[x])$. Hence, $Z_R(f_0R/J)$ is singular, as $Z_{R[x]}(fR[x]/J[x])$ is singular. Therefore R is right generalized π-extending. □

Corollary 3.16. *Let R be a right generalized π-extending (or uniform) ring and $R[x]$ the polynomial ring. Then every free right $R[x]$-module is generalized π-extending.*

Proof. It is clear from Theorem 3.15 and Theorem 3.9. □

Proposition 3.17. *Let R be a right nonsingular generalized π-extending ring and F_R a free right R-module. Then the endomorphism ring $End(F_R)$ of F_R is generalized π-extending.*

Proof. Let R be a right nonsingular generalized π-extending ring and F_R a free right R-module. Then F_R is generalized π-extending by Theorem 3.9. Since R is right nonsingular, we obtain that $End(F_R)$ is right π-extending by [15, Theorem 4.157], so is generalized π-extending by Lemma 2.2. □

References

[1] G. F. Birkenmeier, J. Y. Kim, J. K. Park, *On polynomial extensions of principally quasi-baer rings*, Kyungpook Math. J., **40**(2000), 247–253.

[2] G. F. Birkenmeier, A. Tercan and C. C. Yücel, *The extending condition relative to sets of submodules*, Comm. Algebra, **42**(2014), 764–778.

[3] G. F. Birkenmeier, J. K. Park and T. Rizvi, *Extensions of rings and modules*, Birkhäuser, New York, 2013.

[4] G. F. Birkenmeier, Y. Kara and A. Tercan, *π-Baer rings*, J. Algebra Appl., **17(2)**(2018), 1850029, 19 pp.

[5] N. V. Dung, D. V. Huynh, P. F. Smith and R. Wisbauer, *Extending modules*, Pitman, London, 1994.

[6] L. Fuchs, *Infinite abelian groups I*, Academic Press, New York, 1970.

[7] K. R. Goodearl, *Ring theory: Nonsingular rings and modules*, Marcel Dekker, New York, 1976.

[8] I. Kaplansky, *Infinite abelian groups*, University of Michigan Press, 1969.

[9] T. Y. Lam, *Lectures on modules and rings*, Springer-Verlag, Berlin, 1999.

[10] Y. Kara, A. Tercan and R. Yasar, *PI-extending modules via nontrivial complex bundles and abelian endomorphism rings*, Bull. Iranian Math. Soc., **43(1)**(2017), 121–129.

[11] S. H. Mohamed and B. J. Müller, *Continuous and discrete modules*, London Mathematical Society Lecture Note Series **147**, Cambridge University Press, Cambridge, 1990.

[12] M. P. Murthy, *Zero cycles and projective modules*, Ann. Math., **140**(1994), 405–434.

[13] P. F. Smith and A. Tercan, *Generalizations of CS-modules*, Comm. Algebra, **21**(1993), 1809–1847.

[14] P. F. Smith and A. Tercan, *Direct summands of modules which satisfy C_{11}*, Algebra Colloq., **21**(2004), 231–237.

[15] A. Tercan and C. C. Yücel, *Module theory, Extending modules and generalizations*, Birkhäuser, Basel, 2016.