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On Semi C-Reducibility of General (α, β) Finsler Metrics

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Abstract. In this paper, we study general (α, β) Finsler metrics and prove that every general (α, β)-metric is semi C-reducible but not C2-like. As a consequence of this result we prove that every general (α, β)-metric satisfying the Ricci flow equation is Einstein.

1. Introduction

The concept of (α, β) Finsler metrics was introduced by M. Matsumoto in 1972 as a generalization of the Randers metric [10]. The Randers metric is of the form $F = \alpha + \beta$, where α is a Riemannian metric and β is a 1-form satisfying $\|\beta\| = 1$. It was first introduced by G. Randers regarding an asymmetric metric on four-dimensional space-time in general relativity [16]. The (α, β) Finsler metrics can be written as $F = \alpha\phi(s)$, where ϕ is a smooth function satisfying

$$\phi(s) > 0, \quad \phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, \quad (|s| \leq b < b_0).$$

In the study of Finsler geometry, we often encounter long and complicated calculations. However, when we consider Finsler metrics with certain symmetries, things become much easier. In general relativity, the solution of vacuum Einstein field equations describing the gravitational field, which is spherically symmetric, we obtain the Schwarzschild metric in four dimensional space-time [21]. In this process, the condition of spherical symmetry plays a very important role. In 1996, S.F. Rutz [17] introduced a special class of Finsler metrics called spherically symmetric. A

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Finsler metric F on $B^n(\delta)$ is called spherically symmetric if $F(Ax, Ay) = F(x, y)$, for all $n \times n$ orthogonal matrix A, $x = (x^i) \in B^n(\delta)$ and $y = (y^i) \in T_xB^n(\delta)$. Here $B^n(\delta)$ denotes the Euclidean ball of radius δ around the origin and $T_xB^n(\delta)$ denotes the tangent space of $B^n(\delta)$ at the point x. L. Zhou [22] proved that a Finsler metric F on $B^n(\delta)$ is a spherically symmetric if and only if there exist a function $\phi : [0, \delta) \times \mathbb{R} \to \mathbb{R}$ such that

$$F(x, y) = |y|\phi\left(|x|, \frac{\langle x, y\rangle}{|y|}\right), \tag{1.1}$$

where $|.|$ denotes the Euclidean norm and $\langle , \rangle$ denotes the Euclidean inner product on $\mathbb{R}^n$.

In 2012, Yu and Zhu [20] introduced a new class of Finsler metrics, called general (α, β)-Finsler metrics given by $F = \alpha\phi(b^2, s)$ where $\phi = \phi(b^2, s)$ is a C^∞ positive function and $b^2 := \|\beta\|^2_\alpha$. This class of Finsler metrics not only generalizes (α, β)-metrics in a natural way, but also generalizes the spherically symmetric metrics. Moreover, this class of Finsler metric also include Finsler metrics constructed by R. Bryant [4, 5, 6]. Bryant's metrics are rectilinear Finsler metrics on S^n with flag curvature $K = 1$ and given by

$$F(X, Y) = \Re\left\{\frac{\sqrt{Q(X, X)Q(Y, Y) - Q(X, Y)^2}}{Q(X, X)} - i\frac{Q(X, Y)}{Q(X, X)}\right\},$$

where $Q(X, Y) = x_0y_0 + e^{ip_1}x_1y_1 + e^{ip_2}x_2y_2 + \ldots.. + e^{ip_n}x_ny_n$ is a complex quadratic form on $\mathbb{R}^{n+1}$ for $n \geq 2$ with the parameters satisfying $0 \leq p_1 \leq p_2 \leq \ldots \leq p_n < \pi$ and $X = (x_0, \ldots, x_n) \in S^n$, $Y = (y_0, \ldots, y_n) \in T_XS^n$.

In 1978, Matsumoto and Shibata [12] introduced a special class of Finsler metric called semi-C-reducible. The concept of "semi-C-reducibility" is a generalization of the well-known C-reducibility. In 1992, Matsumoto and Shibata [11] proved that every (α, β)-metric is semi-C-reducible. In this paper, we prove the following results:

Theorem 1.1. *Every general (α, β)-metric is semi C-reducible.*

Corollary 1.2. *A general (α, β)-metric can not be C2-like.*

Recently, Finsler metrics satisfying Ricci flow equation has been an important topic of research. The Ricci flow equation introduced by R.S. Hamilton in 1981 [8, 9] which is an intrinsic flow that deforms the metric of a Riemannian manifold, in a way formally analogous to the diffusion of heat, smoothing out irregularities in the metric. Though it is a primary tool used in Grigori Perelman's solution of the Poincare conjecture [13, 14, 15], it has various applications to dynamical systems, mathematical physics and even in cosmology. Sadegzadeh and Razavi studied C-reducible metrics satisfying Ricci flow equation [18], where as Tayebi, Payghan and Najafi studied semi C-reducible Finsler metrics satisfying Ricci flow equation [19]. In this paper we obtain the following:

Theorem 1.3. *A general (α, β)-metric satisfying normal or unnormal Ricci flow equation is Einstein.*

2. Preliminaries

Let M be an n-dimensional C^∞-manifold. T_xM denotes the tangent space of M at x. The tangent bundle of M is the union of tangent spaces $TM := \bigcup_{x\in M} T_xM$. We denote the elements of TM by (x, y) where $y \in T_xM$ and define $TM_0 := TM \setminus \{0\}$.

Definition 2.1. A *Finsler metric* on M is a function $F : TM \to [0, \infty)$ satisfying the following conditions:

(i) F is C^∞ on TM_0,

(ii) F is a positively 1-homogeneous on the fibers of tangent bundle TM,

(iii) The Hessian of $\frac{F^2}{2}$ with element $g_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$ is positive definite on TM_0.

The pair (M, F) is called a *Finsler space*, F is called the *fundamental function* and g_{ij} is called the *fundamental tensor*.

Let (M, F) be a Finsler space. For a vector $y \in T_xM \setminus \{0\}$, let

$$C_y(u, v, w) := \frac{1}{4}\frac{\partial^3}{\partial s \partial t \partial r}[F^2(y + su + tv + rw)]_{s=t=r=0},$$

where $u, v, w \in T_xM$. Each C_y is a symmetric trilinear form on T_xM. We call the family $C := \{C_y : y \in T_xM \setminus \{0\}\}$ the Cartan torsion. Denote the components of Cartan torsion C by C_{ijk}. Therefore, we have

$$C_{ijk} = \frac{1}{4}[F^2]_{y^i y^j y^k} = \frac{1}{2}\frac{\partial}{\partial y^k}(g_{ij}).$$

The mean Cartan torsion I at $x \in M$ is defined by

$$I := \{I_y | y \in T_xM \setminus \{0\}\},$$

where, $I_y(u) := g^{ij}(y)C_y(u, \partial_i, \partial_j), u \in T_xM$. Denote the components of mean Cartan torsion I by I_i and therefore, we have $I_i = g^{jk}C_{ijk}$.

Roughly speaking the Cartan torsion measures how much a Finsler metric is far from being a Riemannian one. In particular, if $C = 0$, a Finsler metric reduces to a Riemannian metric. In general, the calculation with general form of Cartan torsion is very tedious. However, when we consider some special form of it sometimes we deduce some very interesting geometric properties of the Finsler space. To simplify the calculation and geometrical objects M. Matsumoto studied various special Finsler spaces [12]. For instance, C2-like Finsler spaces, C-reducible Finsler spaces, semi-C-reducible Finsler spaces etc.

Definition 2.2. A Finsler metric F is called *Semi C-reducible* if its Cartan torsion is given by

$$C_{ijk} = \frac{p}{n+1}\left\{I_i h_{jk} + I_j h_{ki} + I_k h_{ij}\right\} + \frac{q}{I^2} I_i I_j I_k, \tag{2.1}$$

where h_{ij} is angular metric tensor given by $h_{ij} = F\frac{\partial^2 F}{\partial y^i \partial y^j}$, $p = p(x,y)$ and $q = q(x,y)$ are scalar functions on TM with $p + q = 1$ and $I^2 = I_i I^i$.

Remark 2.3. If $p = 0$, then F is called C2-like Finsler metric and if $q = 0$ then F is called C-reducible Finsler metric.

Remark 2.4. A two dimensional Finsler space is always C2-like as well as C-reducible where as, a three dimensional Finsler space is always semi C-reducible.

3. General (α, β)- Finsler Metrics

Definition 3.1.([20]) A Finsler metric F on a manifold M is called a *general* (α,β)-*metric*, if F can be expressed in the form $F = \alpha\phi\left(b^2, \frac{\beta}{\alpha}\right)$ for some C^∞ function $\phi := \phi(b^2, s)$, where α is a Riemannian metric and β is a 1-form. In particular, if ϕ only depends on s, i.e., $\phi = \phi(\frac{\beta}{\alpha})$, then the Finsler metric $F = \alpha\phi\left(\frac{\beta}{\alpha}\right)$ is called an (α,β)-*metric*.

You and Zhu [20] proved that the function ϕ in the general (α,β)-metric $F = \alpha\phi\left(b^2, \frac{\beta}{\alpha}\right)$ satisfies

$$\phi - s\phi' > 0,\ \phi - s\phi' + (b^2 - s^2)\phi'' > 0,\ (\text{for } n \geq 3)$$

or

$$\phi - s\phi' + (b^2 - s^2)\phi'' > 0,\ (\text{for } n = 2)$$

where s and b are arbitrary numbers with $|s| \leq b < b_0$. Here ϕ' denotes the differentiation of ϕ with respect to s.

For a general (α,β)- metric $F = \alpha\phi\left(b^2, \frac{\beta}{\alpha}\right)$, the fundamental tensor g_{ij} is given in [20] as:

$$g_{ij} = \rho a_{ij} + \rho_0 b_i b_j + \rho_1\left(b_i \alpha_{y^j} + b_j \alpha_{y^i}\right) - s\rho_1 \alpha_{y^i}\alpha_{y^j}, \tag{3.1}$$

where $\rho = \phi\left(\phi - s\phi'\right)$, $\rho_0 = \phi\phi'' + \phi'\phi'$, $\rho_1 = \left(\phi - s\phi'\right)\phi' - s\phi\phi''$. Moreover,

$$det\left(g_{ij}\right) = \phi^{n+1}\left(\phi - s\phi'\right)^{n-2}\left(\phi - s\phi' + \left(b^2 - s^2\right)\phi''\right) det\left(a_{ij}\right),$$

and

$$g^{ij} = \rho^{-1}\left\{a^{ij} + \eta b^i b^j + \eta_0 \alpha^{-1}\left(b^i y^j + b^j y^i\right) + \eta_1 \alpha^{-2} y^i y^j\right\}, \tag{3.2}$$

where $\left(g^{ij}\right)=\left(g_{ij}\right)^{-1}$, $\left(a^{ij}\right)=\left(a_{ij}\right)^{-1}$, $b^i=a^{ij}b_j$, $\eta=-\frac{\phi''}{\phi-s\phi'+(b^2-s^2)\phi''}$,
$\eta_0=-\frac{\left(\phi-s\phi'\right)\phi'-s\phi\phi''}{\phi(\phi-s\phi'+(b^2-s^2)\phi'')}$, $\eta_1=\frac{\left(s\phi+\left(b^2-s^2\right)\phi'\right)\left(\left(\phi-s\phi'\right)\phi'-s\phi\phi''\right)}{\phi^2(\phi-s\phi'+(b^2-s^2)\phi'')}$.

Proposition 3.1. *The Cartan torsion C_{ijk} of a general (α,β)-metric $F=\alpha\phi\left(b^2,\frac{\beta}{\alpha}\right)$ is given by*

$$C_{ijk}=\frac{1}{2\alpha}\left[\left\{\rho_1\left(b_k-s\frac{y_k}{\alpha}\right)a_{ij}-sTb_ib_j\frac{y_k}{\alpha}+\left(s^2T-\rho_1\right)b_i\frac{y_jy_k}{\alpha^2}+(i\to j\to k\to i)\right\}\right.$$
$$\left.+Tb_ib_jb_k+s\left(3\rho_1-s^2T\right)\frac{y_iy_jy_k}{\alpha^3}\right], \tag{3.3}$$

where $(i\to j\to k\to i)$ denotes cyclic permutation of indices i,j,k in the preceeding terms.

Proof. The Cartan torsion of a Finsler metric is given by

$$C_{ijk}=\frac{1}{2}\frac{\partial g_{ij}}{\partial y^k}. \tag{3.4}$$

Now differentiating ρ, ρ_0, ρ_1, ρ_2 with respect to s we have

$$\rho_s=\rho_1,\quad (\rho_1)_s=-sT,\quad (\rho_0)_s=T, \tag{3.5}$$

where $T=3\phi'\phi''+\phi\phi'''$.
Moreover, differentiating α and s with respect to y^i we have respectively

$$\alpha_{y^i}=\frac{y_i}{\alpha}=\frac{a_{ij}y^j}{\alpha},\quad s_{y^i}=\frac{\alpha b^i-sy^i}{\alpha^2}. \tag{3.6}$$

Now differentiating equation (3.1) with respect to y^k and using equations (3.5) and (3.6) we have equation (3.3). □

Proposition 3.2. *The mean Cartan torsion I_i of a general (α,β)-metric $F=\alpha\phi\left(b^2,\frac{\beta}{\alpha}\right)$ is given by*

$$I_i=\frac{1}{2\alpha}\left[\rho^{-1}\rho_1\left\{(n+1)+3\eta(b^2-s^2)\right\}+(b^2-s^2)\rho^{-1}T\left\{1+\eta(b^2-s^2)\right\}\right]$$
$$(b_i-\frac{sy_i}{\alpha}). \tag{3.7}$$

Proof. The mean Cartan torsion of a Finsler metric is given by

$$I_i=g^{jk}C_{ijk}$$

Using equations (3.2) and (3.3) we obtain equation (3.7). □

Proposition 3.3. *The angular metric tensor h_{ij} of a general (α, β)-metric $F = \alpha\phi\left(b^2, \frac{\beta}{\alpha}\right)$ is given by*

$$h_{ij} = \phi^2(a_{ij} - \frac{y_i y_j}{\alpha}) + \frac{\phi\phi'}{\alpha^2}(s y_i y_j - s\alpha^2 a_{ij}) + \frac{\phi\phi''}{\alpha^2}(\alpha^2 b_i b_j - 2s\alpha b_i y_j + s^2 y_i y_j). \tag{3.8}$$

Proof. Differentiating expressions in the equations (3.6) with respect to y^j, we have

$$\alpha_{y^i y^j} = \frac{1}{\alpha}(a_{ij} - \frac{y_i y_j}{\alpha^2}), \quad s_{y^i y^j} = -\frac{1}{\alpha^2}[s a_{ij} + \frac{1}{\alpha}(b_i y_j + b_j y_i) - \frac{3s}{\alpha^2} y^i y^j]. \tag{3.9}$$

The angular metric tensor h_{ij} of a Finsler metric F is given by

$$h_{ij} = F F_{y^i y^j}. \tag{3.10}$$

In view of (3.9) the angular metric tensor of a general (α, β)-metric is given by

$$h_{ij} = \alpha\phi[\phi\alpha_{y^i y^j} + \phi'(\alpha_{y^i} s_{y^j} + \alpha_{y^j} s_{y^i}) + \phi''\alpha s_{y^i} s_{y^j} + \phi'\alpha s_{y^i y^j}]. \tag{3.11}$$

Using equation (3.6) and equation (3.9) in equation (3.11), we get

$$h_{ij} = \phi^2(a_{ij} - \frac{y_i y_j}{\alpha^2}) + \phi\phi'(s\frac{y_i y_j}{\alpha^2} - s a_{ij}) + \phi\phi''(b_i b_j - 2s b_i \frac{y_j}{\alpha} + s^2 \frac{y_i y_j}{\alpha^2}). \qquad \square$$

4. Proof of Theorem 1.1.

For a semi C-reducible Finsler metric F, the Cartan torsion is given by equation (2.1). Equation (3.7) can be rewritten as

$$I_i = \frac{P}{2\alpha}(b_i - s\frac{y_i}{\alpha}), \tag{4.1}$$

where $P = \frac{1}{\rho}[\rho_1 \left\{(n+1) + 3\eta(b^2 - s^2)\right\} + (b^2 - s^2)T \left\{1 + \eta(b^2 - s^2)\right\}]$. Using equation (3.2) and (3.7) we obtain

$$I^2 = \frac{P^2(b^2 - s^2)}{4\alpha^2\phi(\phi - s\phi' + (b^2 - s^2)\phi'')}. \tag{4.2}$$

From equation (3.7) and equation (3.8) we get

$$\begin{aligned} \frac{1}{n+1}(I_i h_{jk} + I_j h_{ik} + I_k h_{ij}) = \frac{P\phi}{2\alpha}\Big[&(\phi - s\phi')(b_k - s\frac{y_k}{\alpha})a_{ij} - 3s\phi'' b_i b_j \frac{y_k}{\alpha} \\ &+ (3s^2\phi'' + s\phi' - \phi)b_i \frac{y_j y_k}{\alpha^2} + (i \to j \to k \to i) \\ &+ 3\phi'' b_i b_j b_k + 3(s\phi - s^2\phi' - s^3\phi'')\frac{y_i y_j y_k}{\alpha^3}\Big]. \end{aligned} \tag{4.3}$$

Further from equations (4.1) and (4.2) we have

$$\begin{aligned}(4.4)\quad \frac{1}{I^2} I_i I_j I_k &= \frac{4\alpha^2\phi(\phi - s\phi' + (b^2 - s^2)\phi'')}{P^2(b^2 - s^2)} \frac{P^3}{8\alpha^3}(b_i - s\frac{y_i}{\alpha})(b_j - s\frac{y_j}{\alpha})(b_k - s\frac{y_k}{\alpha}) \\ &= \frac{P\phi(\phi - s\phi' + (b^2 - s^2)\phi'')}{2\alpha(b^2 - s^2)} \Big[b_i b_j b_k - s^3 \frac{y_i y_j y_k}{\alpha^3} \\ &\quad + \Big\{ s^2 b_k \frac{y_i y_j}{\alpha^2} - s b_i b_j \frac{y_k}{\alpha} + (i \to j \to k \to i) \Big\} \Big],\end{aligned}$$

where

$$Q = \frac{\phi - s\phi' + (b^2 - s^2)\phi''}{b^2 - s^2} \tag{4.5}$$

Using equations (4.3) and (4.4), we have

$$\begin{aligned}(4.6)\quad & \frac{p}{n+1} \{ I_i h_{jk} + I_j h_{ki} + I_k h_{ij} \} + \frac{q}{I^2} I_i I_j I_k \\ &= \frac{P\phi}{2\alpha} \Big[\Big\{ (\phi - s\phi')(b_k - s\frac{y_k}{\alpha}) \frac{p}{n+1} a_{ij} - (\frac{3p\phi'' s}{n+1} + qQs) b_i b_j \frac{y_k}{\alpha} \\ &\quad + (\frac{p}{n+1}(3s^2\phi'' + s\phi' - \phi) + qQs^2) b_i \frac{y_j y_k}{\alpha^2} + (i \to j \to k \to i) \\ &\quad + (\frac{3p\phi'}{n+1} + qQ)' b_i b_j b_k + (\frac{3p}{n+1}(s\phi - s^2\phi' - s^3\phi'') - qQs^3) \frac{y_i y_j y_k}{\alpha^3} \Big\} \Big].\end{aligned}$$

The general (α, β)-metric will be semi C-reducible if equations (3.3) and (4.6) are identical.

Now comparing the coefficients of these two equations we have,

$$\frac{pP\phi}{1+n}(\phi - s\phi') = \rho_1, \tag{4.7}$$

$$\frac{pP\phi}{1+n} 3s\phi'' + PqQs\phi = sT, \tag{4.8}$$

$$\frac{pP\phi}{1+n}(3s^2\phi'' + s\phi' - \phi) + P\phi qQs^2 = s^2 T - \rho_1, \tag{4.9}$$

$$\frac{p}{1+n} 3P\phi\phi'' + P\phi qQ = T, \tag{4.10}$$

$$\frac{3pP\phi}{1+n}(s\phi - s^2\phi' - s^3\phi'') - P\phi qQs^3 = s(3\rho_1 - s^2 T). \tag{4.11}$$

Equations (4.8), (4.9) and (4.11) can be obtained from equations (4.7) and (4.10). So, a general (α,β)-metric will be semi C-reducible if it satisfies equations (4.7) and (4.10). Dividing equation (4.7) by equation (4.10), we have

$$\frac{\frac{p}{1+n}(\phi - s\phi')}{\frac{3p\phi''}{1+n} + (1-p)Q} = \frac{\rho_1}{T}. \tag{4.12}$$

Now solving for p we have,

$$p = \frac{(1+n)\rho_1 Q}{T(\phi - s\phi') + \rho_1\left\{(1+n)Q - 3\phi''\right\}}. \tag{4.13}$$

As $p+q=1$, using (4.13) we have

$$q = \frac{T(\phi - s\phi') - 3\rho_1\phi''}{T(\phi - s\phi') + \rho_1\left\{(1+n)Q - 3\phi''\right\}}. \tag{4.14}$$

Since we can find the value of p and q uniquely, the theorem follows. □

Proof of corollary 1.2. For C2-like Finsler metric we have, $p = 0$. In view of equation (4.13) and (4.5), since Q is non-zero, we have $\rho_1 = 0$. Then from equation (3.7) we get $T = 0$ and hence we have $I_i = 0$. This is a contradiction to the assumption that F is non-Riemannian. So the result follows. □

5. General (α,β)- Finsler Metrics Satisfying Ricci Flow Equation

The geometric evolution equation

$$\frac{\partial}{\partial t} g_{ij} = -2Ric_{ij}, \quad g(t=0) = g_0, \tag{5.1}$$

is known as the un-normalized Ricci flow equation in Riemannian geometry. In principle, the same equation can be used in Finsler setting, because both g_{ij} and Ric_{ij} have been generalised to the broader framework, albeit gaining a y dependence in the process.

A deformation of Finsler metric means a 1-parameter family of metrics $g_{ij}(x,y,t)$, with parameter $t \in [-\epsilon,\epsilon]$ and ϵ is sufficiently small positive number. For such a metric $\omega = F_{y^i}dx^i$, the volume element as well as connections attached to it depend on t. Instead of the above tensor evolution equation, we use the following form of it. Contracting equation (5.1) with y^i and y^j respectively and using Euler's theorem, we get

$$\frac{\partial F^2}{\partial t} = -2F^2 Ric,$$

where Ric is the Ricci scalar function. That is,

$$\frac{\partial}{\partial t}(\log F) = -Ric, \qquad F(t=0) = F_0.$$

This scalar equation directly addresses the evolution of the Finsler metric F and makes geometrical sense on both the manifold of nonzero tangent vectors TM_0 and the manifold of rays. It is therefore suitable as an un-normalized Ricci flow in Finsler geometry.

If M is compact, then so is SM and we can normalize the above equation by requiring that the flow keeps the volume of SM constant. Recalling the Hilbert form $\omega := F_{y^i} dx^i$, the volume of SM is given by

$$Vol_{SM} := \int_{SM} \frac{(-1)^{\frac{(n-1)(n-2)}{2}}}{(n-1)!} \omega \wedge (d\omega)^{(n-1)} = \int_{SM} dV_{SM}.$$

During the evolution, the Finsler metric F, the Hilbert form ω, the volume form dV_{SM} and consequently the volume Vol_{SM}, all depend on t. On the other hand, the domain of integration SM; being the quotient space of TM_0 under the equivalence relation $z \sim y$, if and only if, $z = \lambda y$ for some $\lambda > 0$; is totally independent of any Finsler metric and hence does not depend on t. We have from [1]

$$\frac{\partial}{\partial t}(dV_{SM}) = [g^{ij} - g'_{ij} - n \frac{\partial}{\partial t} \log F] dV_{SM}.$$

A normalized Ricci flow for Finsler metrics is introduced by Bao [2] and given by

$$\frac{\partial}{\partial t} \log F = -Ric + \frac{1}{Vol(SM)} \int_{SM} Ric dV, \qquad F(t=0) = F_0, \tag{5.2}$$

where the given manifold M is compact. In differential geometry and mathematical physics, an Einstein manifold is a Riemannian manifold whose Ricci tensor is proportional to the metric tensor and this condition is equivalent to saying that the metric is a solution of the vacuum Einstein field equations. In Finsler geometry the Einstein metric is defined as follows:

Definition 5.1. A Finsler metric is said to be an *Einstein metric* if the Ricci scalar function Ric depends only on x, i.e, $Ric_{ij} = Ric(x) g_{ij}$.

In general semi C-reducible Finsler metrics are not Einstein though they becomes Einsten when they satisfies the Ricci flow equation. More precisely, we have the following:

Lemma 5.2.([19])

(i) *Every semi C-reducible Finsler metric satisfying Un-normalize Ricci flow equation is Einstein.*

(ii) *Every semi C-reducible Finsler metric satisfying Normalize Ricci flow equation is Einstein.*

Proof of Theorem 1.3. From Theorem (1.1) and Lemma (5.2) the result follows immediately. □

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