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On Generalized ϕ-recurrent Kenmotsu Manifolds with respect to Quarter-symmetric Metric Connection

Shyamal Kumar Hui
Department of Mathematics, The University of Burdwan, Burdwan –713104, West Bengal, India
e-mail : shyamal_hui@yahoo.co.in

Richard Santiago Lemence*
Institute of Mathematics, College of Science, University of the Philippines, Diliman, Quezon City 1101, Philippines
e-mail : rslemence@math.upd.edu.ph

Abstract. A Kenmotsu manifold $M^n(\phi, \xi, \eta, g)$, $(n = 2m+1 > 3)$ is called a generalized ϕ-recurrent if its curvature tensor R satisfies

$$\phi^2((\nabla_W R)(X,Y)Z) = A(W)R(X,Y)Z + B(W)G(X,Y)Z$$

for all X, Y, Z, $W \in \chi(M)$, where ∇ denotes the operator of covariant differentiation with respect to the metric g, i.e. ∇ is the Riemannian connection, A, B are non-vanishing 1-forms and G is given by $G(X,Y)Z = g(Y,Z)X - g(X,Z)Y$. In particular, if $A = 0 = B$ then the manifold is called a ϕ-symmetric. Now, a Kenmotsu manifold $M^n(\phi, \xi, \eta, g), (n = 2m+1 > 3)$ is said to be generalized ϕ-Ricci recurrent if it satisfies

$$\phi^2((\nabla_W Q)(Y)) = A(X)QY + B(X)Y$$

for any vector field X, $Y \in \chi(M)$, where Q is the Ricci operator, i.e., $g(QX,Y) = S(X,Y)$ for all X, Y. In this paper, we study generalized ϕ-recurrent and generalized ϕ-Ricci recurrent Kenmotsu manifolds with respect to quarter-symmetric metric connection and obtain a necessary and sufficient condition of a generalized ϕ-recurrent Kenmotsu manifold with respect to quarter symmetric metric connection to be generalized Ricci recurrent Kenmotsu manifold with respect to quarter symmetric metric connection.

*Corresponding Author.
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1. Introduction

Tanno, in [43], classified connected almost contact metric manifolds whose automorphism groups possess the maximum dimension. For such a manifold, the sectional curvature of plane sections containing ξ is a constant, say c. He has shown that these manifolds could be divided into three classes: (i) homogeneous normal contact Riemannian manifolds with $c > 0$, (ii) global Riemannian products of a line or a circle with a Kähler manifold of constant holomorphic sectional curvature if $c = 0$ and (iii) a warped product space $\mathbb{R} \times_f \mathbb{C}^n$ if $c < 0$. It is known that the manifolds of class (i) are characterized by admitting a Sasakian structure. The manifolds of class (ii) are characterized by a tensorial relation admitting a cosymplectic structure. In [24], Kenmotsu characterized the differential geometric properties of the manifolds of class (iii) which are nowadays called Kenmotsu manifolds and later studied by several authors.

As a generalization of both Sasakian and Kenmotsu manifolds, In [29], Oubiña introduced the notion of trans-Sasakian manifolds, which are closely related to the locally conformal Kähler manifolds. A trans-Sasakian manifold of type (0, 0), $(\alpha, 0)$ and $(0, \beta)$ are called the cosympletic, α-Sasakian and β-Kenmotsu manifolds respectively, α, β being scalar functions. In particular, if $\alpha = 0, \beta = 1$; and $\alpha = 1, \beta = 0$ then a trans-Sasakian manifold will be a Kenmotsu and Sasakian manifold respectively.

The study of Riemann symmetric manifolds began with the work of Cartan (see [8]). A Riemannian manifold (M^n, g) is said to be locally symmetric due to Cartan [8] if its curvature tensor R satisfies the relation $\nabla R = 0$, where ∇ denotes the operator of covariant differentiation with respect to the metric tensor g.

During the last five decades, the notion of locally symmetric manifolds has been weakened by many authors in several ways to a different extent such as recurrent manifold by Walker [46], semisymmetric manifold by Szabó [41], pseudosymmetric manifold in the sense of Deszcz [14], pseudosymmetric manifold in the sense of Chaki [9], generalized recurrent manifold by Dubey [16].

A Riemannian manifold $(M^n, g)(n > 2)$ is called generalized recurrent [16] if its curvature tensor R satisfies the condition

$$(\nabla_W R)(X, Y)Z = A(W)R(X, Y)Z + B(W)G(X, Y)Z, \tag{1.1}$$

where A and B are non-vanishing 1-forms defined by $A(X) = g(X, \rho_1)$, $B(X) = g(X, \rho_2)$ and the tensor G is defined by

$$G(X, Y)Z = g(Y, Z)X - g(X, Z)Y \tag{1.2}$$

for all X, Y, $Z \in \chi(M)$; $\chi(M)$ being the Lie algebra of smooth vector fields on M and ∇ denotes the operator of covarient differentiation with respect to the metric g. The 1-forms A and B are called the associated 1-forms of the manifold. In particular if $B = 0$ then the notion of (1.1) turns into recurrent manifold introduced by Walker [46].

A Riemannian manifold is said to be Ricci symmetric if its Ricci tensor S of type (0,2) satisfies $\nabla S = 0$, where ∇ denotes the Riemannian connection. During the last five decades, the notion of Ricci symmetry has been weakened by many authors in several ways to a different extent such as Ricci recurrent manifold [31],Ricci semi symmetric manifold [41], pseudo Ricci symmetric manifolds [15].

Again, the notion of generalized Ricci-recurrent manifolds has been introduced and studied by De, Guha and Kamilya [11]. A Riemannian manifold $(M^n, g)(n > 2)$, is called generalized Ricci-recurrent [11] if its Ricci tensor S of type (0,2) satisfies the condition

$$\nabla S = A \otimes S + B \otimes g,$$

where A and B are non-vanishing 1-forms defiened in (1.1).

As a weaker version of local symmetry, the notion of locally ϕ-symmetric Sasakian manifolds was introduced by Takahashi [42]. Generalizing the notion of locally ϕ-symmetric Sasakian manifolds, De, Shaikh and Biswas [12] introduced the notion of ϕ-recurrent Sasakian manifolds. In this connection De [10] introduced and studied ϕ-symmetric Kenmotsu manifolds and in [13] De, Yildiz and Yaliniz introduced and studied ϕ-recurrent Kenmotsu manifolds. Also Shaikh and Hui [38] studied locally ϕ-symmetric and extended generalized ϕ-recurrent β-Kenmotsu manifolds. In [32] Prakash studied concircularly ϕ-recurrent Kenmotsu manifolds. Recently Hui [21] studied ϕ-pseudo symmetric and ϕ-pseudo Ricci symmetric Kenmotsu manifolds.

In [30], Özgür studied generalized recurrent Kenmotsu manifolds. Generalizing the notion of Özgür [30], and De, Yildiz and Yaliniz [13], Basari and Murathan [5] introduced the notion of generalized ϕ-recurrent Kenmotsu manifolds.

Definition 1. A Kenmotsu manifold $M^n(\phi, \xi, \eta, g)$, $(n = 2m + 1 > 3)$ is called a *generalized ϕ-recurrent* [5] if its curvature tensor R satisfies

$$\phi^2((\nabla_W R)(X, Y)Z) = A(W)R(X, Y)Z + B(W)G(X, Y)Z \tag{1.3}$$

for all X, Y, Z, $W \in \chi(M)$, where ∇ denotes the operator of covariant differentiation with respect to the metric g, i.e. ∇ is the Riemannian connection, A, B are defined in (1.1) and G is defined in (1.2).

In particular, if $A = 0 = B$ then the manifold is called a ϕ-symmetric [10].

Definition 2. A Kenmotsu manifold $M^n(\phi, \xi, \eta, g), (n = 2m + 1 > 3)$ is said to be *generalized ϕ-Ricci recurrent* if it satisfies

$$\phi^2((\nabla_W Q)(Y)) = A(X)QY + B(X)Y \tag{1.4}$$

for any vector field X, $Y \in \chi(M)$, where A, B are non zero 1-forms defined in (1.1) and Q is the Ricci operator, i.e., $g(QX, Y) = S(X, Y)$ for all X, Y.

In particular if $A = 0 = B$ then (1.4) turns into the notion of ϕ-Ricci symmetric Kenmotsu manifold introduced by Shukla and Shukla [40].

Friedmann and Schouten, in [17], introduced the notion of semisymmetric linear connection on a differentiable manifold. Then in 1932 Hayden [19] introduced the idea of metric connection with torsion on a Riemannian manifold. A systematic study of the semisymmetric metric connection on a Riemannian manifold has been given by Yano in 1970 [47]. In 1975, Golab introduced the idea of a quarter symmetric linear connection in differentiable manifolds.

A linear connection $\overline{\nabla}$ in an n-dimensional differentiable manifold M is said to be a quarter symmetric connection [18] if its torsion tensor τ of the connection $\overline{\nabla}$ is of the form

$$\begin{aligned} \tau(X,Y) &= \overline{\nabla}_X Y - \overline{\nabla}_Y X - [X,Y] \\ &= \eta(Y)\phi X - \eta(X)\phi Y, \end{aligned} \tag{1.5}$$

where η is a 1-form and ϕ is a tensor of type (1,1). In particular, if $\phi X = X$ then the quarter symmetric connection reduces to the semisymmetric connection. Thus the notion of quarter symmetric connection generalizes the notion of the semisymmetric connection. Again if the quarter symmetric connection $\overline{\nabla}$ satisfies the condition

$$(\overline{\nabla}_X g)(Y,Z) = 0 \tag{1.6}$$

for all X, Y, $Z \in \chi(M)$, where $\chi(M)$ is the Lie algebra of vector fields on the manifold M, then $\overline{\nabla}$ is said to be a quarter symmetric metric connection. Quarter symmetric metric connection have been studied by many authors in several ways to a different extent such as [1, 2, 3, 4, 6, 20, 23, 25, 26, 27, 28, 34, 35, 36, 37, 39, 44, 45, 48]. Recently Prakasha [33] studied ϕ-symmetric Kenmotsu manifolds with respect to quarter symmetric metric connection. In this connection Hui [22] studied ϕ-pseudo symmetric and ϕ-pseudo Ricci symmetric Kenmotsu manifolds with respect to quarter symmetric metric connection.

Motivated by the above studies the present paper deals with the study of generalized ϕ-recurrent and generalized ϕ- Ricci recurrent Kenmotsu manifolds with respect to quarter symmetric metric connection. The paper is organized as follows. Section 2 is concerned with preliminaries. Section 3 is devoted to the study of generalized ϕ-recurrent Kenmotsu manifolds with respect to quarter symmetric metric connection and we obtained the necessary and sufficient condition of generalized ϕ-recurrent Kenmotsu manifolds with respect to quarter symmetric metric connection to be generalized Ricci-recurrent Kenmotsu manifolds with respect to quarter symmetric metric connection. We also found the Ricci tensor and scalar curvature of generalized ϕ-recurrent Kenmotsu manifolds with respect to quarter symmetric metric connection. In section 4, we have studied generalized ϕ-Ricci recurrent Kenmotsu manifolds with respect to quarter symmetric metric connection.

2. Preliminaries

A smooth manifold (M^n, g), $(n = 2m+1 > 3)$ is said to be an *almost contact metric manifold* [7] if it admits a (1,1) tensor field ϕ, a vector field ξ, an 1-form η and a

Riemannian metric g which satisfy

$$\phi\xi = 0, \qquad \eta(\phi X) = 0, \qquad \phi^2 X = -X + \eta(X)\xi, \tag{2.1}$$

$$g(\phi X, Y) = -g(X, \phi Y), \qquad \eta(X) = g(X, \xi), \qquad \eta(\xi) = 1, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2.3}$$

for all vector fields X, Y on M.

An almost contact metric manifold $M^n(\phi, \xi, \eta, g)$ is said to be *Kenmotsu manifold* if the following condition holds [24]:

$$\nabla_X \xi = X - \eta(X)\xi, \tag{2.4}$$

$$(\nabla_X \phi)(Y) = g(\phi X, Y)\xi - \eta(Y)\phi X, \tag{2.5}$$

where ∇ denotes the Riemannian connection of g.
In a Kenmotsu manifold, the following relations hold [24]:

$$(\nabla_X \eta)(Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.6}$$

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.7}$$

$$R(\xi, X)Y = \eta(Y)X - g(X, Y)\xi, \tag{2.8}$$

$$\eta(R(X, Y)Z) = \eta(Y)g(X, Z) - \eta(X)g(Y, Z), \tag{2.9}$$

$$S(X, \xi) = -(n-1)\eta(X), \tag{2.10}$$

$$S(\xi, \xi) = -(n-1), \quad \text{i.e.,} \quad Q\xi = -(n-1)\xi, \tag{2.11}$$

$$S(\phi X, \phi Y) = S(X, Y) + (n-1)\eta(X)\eta(Y), \tag{2.12}$$

$$(\nabla_W R)(X, Y)\xi = g(X, W)Y - g(Y, W)X - R(X, Y)W \tag{2.13}$$

for any vector field X, Y, Z on M and R is the Riemannian curvature tensor and S is the Ricci tensor of type (0,2) such that $g(QX, Y) = S(X, Y)$.
Let M be an n-dimensional Kenmotsu manifold and ∇ be the Levi-Civita connection on M. A quarter symmetric metric connection $\overline{\nabla}$ in a Kenmotsu manifold is defined by [18, 33]

$$\overline{\nabla}_X Y = \nabla_X Y + H(X, Y), \tag{2.14}$$

where H is a tensor of type (1,1) such that

$$H(X,Y) = \frac{1}{2}\big[\tau(X,Y) + \tau'(X,Y) + \tau'(Y,X)\big] \tag{2.15}$$

and

$$g(\tau'(X,Y),Z) = g(\tau(Z,X),Y). \tag{2.16}$$

From (1.5) and (2.18), we get

$$\tau'(X,Y) = g(\phi Y,X)\xi - \eta(X)\phi Y. \tag{2.17}$$

Using (1.5) and (2.19) in (2.17), we obtain

$$H(X,Y) = -\eta(X)\phi Y. \tag{2.18}$$

Hence a quarter symmetric metric connection $\overline{\nabla}$ in a Kenmotsu manifold is given by

$$\overline{\nabla}_X Y = \nabla_X Y - \eta(X)\phi Y. \tag{2.19}$$

If R and $\overline{R}$ are respectively the curvature tensor of Levi-Civita connection ∇ and the quarter symmetric metric connection $\overline{\nabla}$ in a Kenmotsu manifold then we have [33]

$$\begin{aligned} \overline{R}(X,Y)Z &= R(X,Y)Z - 2d\eta(X,Y)\phi Z \\ &+ \big[\eta(X)g(\phi Y,Z) - \eta(Y)g(\phi X,Z)\big]\xi \\ &+ \big[\eta(Y)\phi X - \eta(X)\phi Y\big]\eta(Z). \end{aligned} \tag{2.20}$$

From (2.21) we have

$$\overline{S}(Y,Z) = S(Y,Z) - 2d\eta(\phi Z,Y) + g(\phi Y,Z) + \psi\eta(Y)\eta(Z), \tag{2.21}$$

where $\overline{S}$ and S are respectively the Ricci tensor of a Kenmotsu manifold with respect to the quarter symmetric metric connection and Levi-Civita connection and $\psi = tr.\omega$, where $\omega(X,Y) = g(\phi X,Y)$. From (2.23) it follows that the Ricci tensor with respect to quarter symmetric metric connection is not symmetric.
Also from (2.23), we have

$$\overline{r} = r + 2(n-1), \tag{2.22}$$

where $\overline{r}$ and r are the scalar curvatures with respect to quarter symmetric metric connection and Levi-Civita connection respectively.
From (2.1), (2.2), (2.5), (2.13), (2.21) and (2.22), we get

$$\begin{aligned} (\overline{\nabla}_W \overline{R})(X,Y)\xi &= g(X,W)Y - g(Y,W)X - R(X,Y)W \\ &+ \big[\eta(Y)g(\phi W,X) - \eta(X)g(\phi W,Y)\big]\xi \\ &- \eta(W)\big[\eta(X)Y - \eta(Y)X \\ &+ \eta(X)\phi Y - \eta(Y)\phi X\big]. \end{aligned} \tag{2.23}$$

Again from (2.21) and (2.22), we have

$$g((\overline{\nabla}_W\overline{R})(X,Y)Z,U) = -g((\overline{\nabla}_W\overline{R})(X,Y)U,Z). \tag{2.24}$$

Definition 3. A Kenmotsu manifold M is said to be η-*Einstein* if its Ricci tensor S of type (0,2) is of the form

$$S = ag + b\eta\otimes\eta, \tag{2.25}$$

where a, b are smooth functions on M.

3. Generalized ϕ-recurrent Kenmotsu Manifolds with respect to Quarter Symmetric Metric Connection

Definition 4. A Kenmotsu manifold $M^n(\phi,\xi,\eta,g),(n=2m+1>3)$ is said to be *generalized ϕ-recurrent with respect to quarter symmetric metric connection* if the curvature tensor R with respect to quarter symmetric metric connection satisfies

$$\phi^2((\overline{\nabla}_W\overline{R})(X,Y)Z) = A(W)\overline{R}(X,Y)Z + B(W)G(X,Y)Z \tag{3.1}$$

for any vector field X, Y,Z and W, where A and B are non-vanishing 1-form.

In particular if $A=0=B$ then the manifold is said to be ϕ-symmetric Kenmotsu manifold with respect to quarter symmetric metric connection [33].

We now consider a Kenmotsu manifold $M^n(\phi,\xi,\eta,g),(n=2m+1>3)$, which is generlized ϕ-recurrent with respect to quarter symmetric metric connection. Then by virtue of (2.1) it follows from (3.1) that

$$\begin{aligned} -(\overline{\nabla}_W\overline{R})(X,Y)Z &+ \eta((\overline{\nabla}_W\overline{R})(X,Y)Z)\xi \\ = A(W)\overline{R}(X,Y)Z &+ B(W)[g(Y,Z)X - g(X,Z)Y] \end{aligned} \tag{3.2}$$

from which it follows that

$$\begin{aligned} -g((\overline{\nabla}_W\overline{R})(X,Y)Z,U) &+ \eta((\overline{\nabla}_W\overline{R})(X,Y)Z)\eta(U) \\ = A(W)g(\overline{R}(X,Y)Z,U) &+ B(W)[g(Y,Z)g(X,U) - g(X,Z)g(Y,U)] \end{aligned} \tag{3.3}$$

Taking an orthonormal frame field and then contracting (3.3) over X and U and then using (2.1) and (2.2), we get

$$\begin{aligned} -(\overline{\nabla}_W\overline{S})(Y,Z) &+ g((\overline{\nabla}_W\overline{R})(\xi,Y)Z,\xi) \\ = A(W)\overline{S}(Y,Z) &+ (n-1)B(W)g(Y,Z). \end{aligned} \tag{3.4}$$

Using (2.8), (2.23) and (2.24), we have

$$\begin{aligned} g((\overline{\nabla}_W\overline{R})(\xi,Y)Z,\xi) &= -g((\overline{\nabla}_W\overline{R})(\xi,Y)\xi,Z) \\ &= g(\phi W,Y)\eta(Z) + g(\phi Y,Z)\eta(W) \\ &+ \big[g(Y,Z) - \eta(Y)\eta(Z)\big]\eta(W). \end{aligned} \tag{3.5}$$

By virtue of (3.5) it follows from (3.4) that

$$(3.6)\qquad \begin{aligned}(\overline{\nabla}_W\overline{S})(Y,Z) &= -A(W)\overline{S}(Y,Z)-(n-1)B(W)g(Y,Z)\\ &+ g(\phi W,Y)\eta(Z)+g(\phi Y,Z)\eta(W)\\ &+ \big[g(Y,Z)-\eta(Y)\eta(Z)\big]\eta(W)\end{aligned}$$

This leads to the following:

Theorem 1. *A generalized ϕ-recurrent Kenmotsu manifold with respect to quarter symmetric metric connection is generalized Ricci recurrent with respect to quarter symmetric metric connection if and only if*

$$g(\phi W,Y)\eta(Z)+g(\phi Y,Z)\eta(W)+\big[g(Y,Z)-\eta(Y)\eta(Z)\big]\eta(W)=0.$$

Setting $Z=\xi$ in (3.4) and using (3.5), we get

$$(3.7)\qquad \begin{aligned}&- (\overline{\nabla}_W\overline{S})(Y,\xi)+g(\phi W,Y)\\ &= A(W)\overline{S}(Y,\xi)+(n-1)B(W)\eta(Y).\end{aligned}$$

In view of (2.10) we get from (2.21) that

$$(3.8)\qquad \overline{S}(Y,\xi)=\big[\psi-(n-1)\big]\eta(Y).$$

We know that

$$(3.9)\qquad (\overline{\nabla}_W\overline{S})(Y,\xi)=\overline{\nabla}_W\overline{S}(Y,\xi)-\overline{S}(\overline{\nabla}_WY,\xi)-\overline{S}(Y,\overline{\nabla}_W\xi).$$

Using (2.4), (2.10), (2.19), (2.21) in (3.9) we get

$$(3.10)\qquad \begin{aligned}(\overline{\nabla}_W\overline{S})(Y,\xi) &= -S(Y,W)+2d\eta(\phi Y,W)-g(\phi Y,W)\\ &+ \big[\psi-(n-1)\big]g(Y,W)-\psi\eta(Y)\eta(W).\end{aligned}$$

By virtue of (3.8), (3.10) and (2.1) it follows from (3.7) that

$$(3.11)\qquad \begin{aligned}S(Y,W) &= \big[\psi-(n-1)\big]g(Y,W)-\psi\eta(Y)\eta(W)\\ &+ \big[\{\psi-(n-1)\}A(W)+(n-1)B(W)\big]\eta(Y).\end{aligned}$$

Contracting (3.11) over Y and W, we get

$$(3.12)\qquad r=(n-1)(\psi-n)+\big[\psi-(n-1)\big]A(\xi)+(n-1)B(\xi).$$

This leads to the following:

Theorem 2. *In a generalized ϕ-recurrent Kenmotsu manifold with respect to quarter symmetric metric connection the Ricci tensor and the scalar curvature are respectively given by* (3.11) *and* (3.12).

If, in particular, $A = B = 0$ then (3.11) reduces to

$$S(Y,W) = \big[\psi - (n-1)\big]g(Y,W) - \psi\eta(Y)\eta(W),$$

which implies that the manifold under consideration is η-Einstein.
This leads to the following:

Corollary 1.([22]) *A ϕ-symmetric Kenmotsu manifold with respect to quarter symmetric metric connection is an η-Einstein manifold.*

In view of (2.24) we get from (3.2) that

$$(3.13) \quad (\overline{\nabla}_W\overline{R})(X,Y)Z = - g(\overline{\nabla}_W\overline{R})(X,Y)\xi,Z)\xi - A(W)(\overline{R}(X,Y)Z - B(W)\big[g(Y,Z)X - g(X,Z)Y\big].$$

Using (2.20), (2.23) and (3.13) we obtain

$$\begin{aligned}(3.14) \quad (\overline{\nabla}_W\overline{R})(X,Y)Z &= \big[R(X,Y,W,Z)\\ &+ g(X,Z)g(Y,W) - g(X,W)g(Y,Z)\\ &+ \{\eta(X)g(\phi W,Y) - \eta(Y)g(\phi W,X)\}\eta(Z)\\ &+ \eta(W)\eta(X)\{g(\phi Y,Z) + g(Y,Z)\}\\ &- \eta(W)\eta(Y)\{g(\phi X,Z) + g(X,Z)\}\\ &- A(W)\{\eta(X)g(\phi Y,Z) - \eta(Y)g(\phi X,Z)\}\big]\xi\\ &- A(W)\big[R(X,Y)Z - 2d\eta(X,Y)\phi Z\\ &+ \{\eta(Y)\phi X - \eta(X)\phi Y\}\eta(Z)\big]\\ &- B(W)\big[g(Y,Z)X - g(X,Z)Y\big]\end{aligned}$$

for arbitrary vector fields X, Y, Z and W.
This leads to the following:

Theorem 3. *A Kenmotsu manifold is generalized ϕ-recurrent with respect to quarter symmetric metric connection if and only if the relation (3.14) holds.*

We now take a generalized ϕ-symmetric Kenmotsu manifold with respect to Levi-Civita connection. then the relation (1.3) holds. By virtue of (2.1), (2.13) and the relation

$$g((\nabla_W R)(X,Y)Z,U) = -g((\nabla_W R)(X,Y)U,Z)$$

it follows from (1.3) that

$$\begin{aligned}(3.15) \quad (\nabla_W R)(X,Y)Z &= \big[R(X,Y,W,Z) + g(X,Z)g(Y,W)\\ &- g(X,W)g(Y,Z)\big]\xi - A(W)R(X,Y)Z\\ &- B(W)G(X,Y)Z\end{aligned}$$

From (3.14) and (3.15) we can state the following:

Theorem 4. *A generalized ϕ-recurrent Kenmotsu manifold is invariant under quarter symmetric metric connection if and only if the relation*

$$\begin{aligned} & \big[\{\eta(X)g(\phi W,Y)-\eta(Y)g(\phi W,X)\}\eta(Z)+\eta(W)\eta(X)\{g\phi Y,Z+g(Y,Z)\} \\ - & \ \eta(W)\eta(Y)\{g(\phi X,Z)+g(X,Z)\}-A(W)\{\eta(X)g(\phi Y,Z)-\eta(Y)g(\phi X,Z)\}\big]\xi \\ + & \ \big[2d\eta(X,Y)\phi Z-\{\eta(Y)\phi X-\eta(X)\phi Y\}\eta(Z)\big]=0. \end{aligned}$$

holds for arbitrary vector fields X, Y, Z and W.

4. Generalized ϕ-Ricci Recurrent Kenmotsu Manifolds with respect to Quarter Symmetric Metric Connection

Definition 5. A Kenmotsu manifold $M^n(\phi,\xi,\eta,g),(n=2m+1>3)$ is said to be generalized ϕ-Ricci recurrent with respect to quarter symmetric metric connection if the Ricci operator Q satisfies

$$\phi^2((\overline{\nabla}_X\overline{Q})(Y))=A(X)\overline{Q}Y+B(X)Y \tag{4.1}$$

for any vector field X, Y where A, B are non-zero 1-forms.

In particular, if $A=0=B$ then (4.1) turns into the notion of ϕ-Ricci symmetric Kenmotsu manifold with respect to quarter symmetric metric connection.

let us take a Kenmotsu manifold $M^n(\phi,\xi,\eta,g),(n=2m+1>3)$ which is generalized ϕ-Ricci recurrent with respect to quarter symmetric metric connection. Then by virtue of (2.1) it follows from (4.1) that

$$-(\overline{\nabla}_X\overline{Q})(Y)+\eta((\overline{\nabla}_X\overline{Q})(Y))\xi=A(X)\overline{\phi}Y+B(X)Y$$

from which it follows that

$$\begin{aligned} - & \ g(\overline{\nabla}_X\overline{Q}(Y),Z)+\overline{S}(\overline{\nabla}_XY,Z)+\eta((\overline{\nabla}_X\overline{Q})(Y))\eta(Z) \\ = & \ A(X)\overline{S}(Y,Z)+B(X)g(Y,Z). \end{aligned} \tag{4.2}$$

Putting $X=\xi$ in (4.2) and using (2.4), (2.10), (2.19), (2.21) and (3.10), we get

$$\begin{aligned} S(X,Z) &= \big[\psi-(n-1)\big]g(X,Z)-\psi\eta(X)\eta(Z) \\ &+ \big[\{\psi-(n-1)\}A(X)+B(X)\big]\eta(Z). \end{aligned} \tag{4.3}$$

This leads to the following:

Theorem 5. *In a generalized ϕ-Ricci recurrent Kenmotsu manifold with respect to quarter symmetric metric connection, the Ricci tensor is of the form (4.3).*

In particular if $A=0=B$ then from (4.3). we get

$$S(X,Z)=\big[\psi-(n-1)\big]g(X,Z)-\psi\eta(X)\eta(Z), \tag{4.4}$$

which implies that the manifold under consideration is η-Einstein.
This leads to the following:

Corollary 2. *A generalized ϕ-Ricci symmetric Kenmotsu manifold with respect to quarter symmetric metric connection is an η-Einstein manifold.*

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