Subnormality and Weighted Composition Operators on L^2 Spaces

MOHAMMAD REZA AZIMI

Department of Mathematics, Faculty of Sciences, University of Maragheh, Maragheh, Iran

e-mail: `mhr.azimi@maragheh.ac.ir`

ABSTRACT. Subnormality of bounded weighted composition operators on $L^2(\Sigma)$ of the form $Wf = uf \circ T$, where T is a nonsingular measurable transformation on the underlying space X of a σ -finite measure space (X, Σ, μ) and u is a weight function on X ; is studied. The standard moment sequence characterizations of subnormality of weighted composition operators are given. It is shown that weighted composition operators are subnormal if and only if $\{J_n(x)\}_{n=0}^{+\infty}$ is a moment sequence for almost every $x \in X$, where $J_n = h_n E_n(|u|^2) \circ T^{-n}$, $h_n = d\mu \circ T^{-n}/d\mu$ and E_n is the conditional expectation operator with respect to $T^{-n}\Sigma$.

1. Introduction

Weighted composition operators acting on certain function spaces are operators of the form $Wf = uf \circ T$, where T is a measurable transformation on a σ -finite measure space (X, Σ, μ) and u is a weight function on X . These operators include composition operators of the form $Cf = f \circ T$ and multiplication operators with symbol u . So far most general properties of composition operators and major classes of known normal, quasinormal, hyponormal and weak hyponormal composition operators have been widely studied by numerous mathematicians (See [1],[2],[13],[15],[17],[19]). Conditions for weighted composition operators to belong to some certain specific classes are found in [8]. However general properties of weighted composition operators have not been studied as well as composition operators. Originally subnormal composition operators have been studied by A. Lambert in [12] based on the standard moment sequence characterizations. Recently, in [18] subnormality of composition operators have been characterized in the Laplace density case. In this paper we shall use the standard moment sequence

Received March 00, 2014; revised May 00, 2014; accepted November 00, 2014.

2010 Mathematics Subject Classification: Primary 47B20; Secondary 47B38.

Key words and phrases: Subnormal, Weighted composition operators, Conditional expectation, Moment sequence.

characterizations of subnormality of weighted composition operators.

2. Preliminaries

Let (X, Σ, μ) be a complete σ -finite measure space and T be a nonsingular measurable transformation from X onto X , i.e. $T^{-1}\Sigma \subseteq \Sigma$ and such that $\mu \circ T^{-1}$ is absolutely continuous with respect to μ which is denoted by $\mu \circ T^{-1} \ll \mu$. Given a real or complex valued measurable weight function u on X , the weighted composition operator on $L^2(\Sigma) := L^2(X, \Sigma, \mu)$ induced by T and u is defined by $Wf = uf \circ T$. It is worth reminding the reader that the nonsingularity of T assures that W is well defined as a mapping of equivalence classes of functions on the support of u . Let $h = h_1 := d\mu \circ T^{-1}/d\mu$ be the Radon-Nikodym derivative and for each $n \geq 1$ let $h_n := d\mu \circ T^{-n}/d\mu$. For $f \in L^2(\Sigma)$ or $f \geq 0$ a.e let $E(f) := E(f|T^{-1}\Sigma)$ be the *conditional expectation* of f with respect to σ -subalgebra $T^{-1}\Sigma$. For each $n \geq 2$, let $E_n(f) := E(f|T^{-n}\Sigma)$. Conditional expectation operator E is defined as an orthogonal projection from $L^2(\Sigma)$ onto $L^2(T^{-1}\Sigma)$ by the assignment $f \rightarrow E(f)$. If g is a $T^{-1}\Sigma$ -measurable function then $E(fg) = gE(f)$. One may find more noteworthy properties of conditional expectation operator in [6, 16, 2] and the references therein. In [15] it is shown that C defines a bounded operator on $L^2(\Sigma)$ if and only if $h \in L^\infty(\Sigma) := L^\infty(X, \Sigma, \mu)$. Necessary and sufficient condition for W to be bounded on $L^2(\Sigma)$ is that $J := hE(|u|^2) \circ T^{-1} \in L^\infty(\Sigma)$ which is found in [7]. Conditional expectation is an functional projection and we shall use it in this paper frequently. For example observe the recurrent relationship $h_{n+1} = hE(h_n) \circ T^{-1}$ proved in [11].

A measurable transformation $T : X \rightarrow X$ is said to be *essentially surjective* if $\mu(X - T(X)) = 0$. It is shown in [9] that a bounded weighted composition operator W is injective if and only if T is essentially bounded. Throughout this paper it is assumed that T is essentially bounded.

Recall that a bounded operator A defined on a Hilbert space is *normal* if and only if $A^*A = AA^*$ (where A^* is a adjoint of A) and A is *subnormal* if A has an extension to a normal operator on a larger Hilbert space. Let $A = U|A|$ be the canonical polar decomposition for A and let $p \in (0, \infty)$. An operator A is *p-hyponormal* if $(A^*A)^p \geq (AA^*)^p$ and A is *p-quasihyponormal* if $A^*(A^*A)^pA \geq A^*(AA^*)^pA$. For all unit vectors $x \in \mathcal{H}$, if $\| |A|^p U |A|^p x \| \geq \| |A|^p x \|^2$, then A is called a *p-paranormal* operator (See [5]). For instance consider the known relationships between these classed as follows

$$\text{normal} \implies \text{quasinormal} \implies \text{subnormal} \implies \text{hyponormal},$$

$$p - \text{hyponormal} \implies p - \text{quasihyponormal} \implies p - \text{paranormal}.$$

Further information may be found in [1, 8, 2].

3. Subnormal Weighted Composition Operators

The assumption that T is essentially bounded; plays an important role to make use of the following theorem stated in [14].

Theorem 3.1. *Let A be an operator on a Hilbert space H with $\ker\{A\} = 0$. Then A is subnormal if and only if for every x in H the sequence $\{\|A^n x\|\}_{n=0}^{+\infty}$ is a moment sequence, i.e., there is an interval $I = [0, r]$ such that for each x in H one can find a Borel measure m_x such that for each $n \geq 0$,*

$$\|A^n x\|^2 = \int_I t^n dm_x(t).$$

Remark 3.2. For each $n \in \mathbb{N}$ and $F \in \Sigma$, define the measure λ_u^n by

$$\lambda_u^n(F) = \int_{T^{-n}(F)} |u|^2 d\mu.$$

According to the following chain

$$\lambda_u^n \ll \mu \circ T^{-n} \ll \dots \ll \mu \circ T^{-2} \ll \mu \circ T^{-1} \ll \mu,$$

the assumption $\mu \circ T^{-1} \ll \mu$ implies that $\lambda_u^n \ll \mu$. Hence there exists the Radon-Nikodym derivative $J_n := \frac{d\lambda_u^n}{d\mu}$. For $F \in \Sigma$, we have

$$\begin{aligned} \lambda_u^n(F) &= \int_{T^{-n}(F)} |u|^2 d\mu \\ &= \int_{T^{-n}(F)} E_n(|u|^2) d\mu \\ &= \int_F h_n E_n(|u|^2) \circ T^{-n} d\mu. \end{aligned}$$

Then $d\lambda_u^n = h_n[E_n(|u|^2)] \circ T^{-n} d\mu$, since F was chosen arbitrarily. It follows that $J_n = h_n[E_n(|u|^2)] \circ T^{-n}$. Note that $J_1 = J$ and let $J_0 = h_0 = 1$. Moreover, for an arbitrary $F \in \Sigma$,

$$\begin{aligned} \lambda_u^n(F) &= \int_{T^{-n}(T^{-1}F)} |u|^2 d\mu \\ &= \int_{T^{-n}(T^{-1}F)} E_n(|u|^2) d\mu \\ &= \int_{T^{-1}(F)} h_n E_n(|u|^2) \circ T^{-n} d\mu \\ &= \int_{T^{-1}(F)} E(J_n) d\mu \\ &= \int_F h E(J_n) \circ T^{-1} d\mu. \end{aligned}$$

Then $J_{n+1} = hE(J_n) \circ T^{-1}$. If Theorem 3.1 is now applied to a bounded weighted composition operator we have the following.

Theorem 3.3. *W is subnormal if and only if for each $f \in L^2(\Sigma)$, $\{\int_x J_n |f|^2 d\mu\}_{n=0}^{+\infty}$ is a moment sequence.*

Proof. Let $f \in L^2(\Sigma)$, by calculating the n -th iteration of W we will have

$$W^n f = \prod_{i=0}^{n-1} (u \circ T^i)(f \circ T^n).$$

Thus,

$$\begin{aligned} \|W^n f\|^2 &= \int_X \left| \prod_{i=0}^{n-1} (u \circ T^i)(f \circ T^n) \right|^2 d\mu \\ &= \int_X \prod_{i=0}^{n-2} |u \circ T^i|^2 |f \circ T^{n-1}|^2 d\lambda_u^1 \\ &= \int_X \prod_{i=0}^{n-3} |u \circ T^i|^2 |f \circ T^{n-2}|^2 d\lambda_u^2 \\ &\vdots \\ &= \int_X |f|^2 d\lambda_u^n \\ &= \int_X J_n |f|^2 d\mu. \end{aligned}$$

□

In what follows we are going to determine the subnormality of the weighted composition operators just by the sequence $\{J_n\}_{n=0}^{+\infty}$ based on the following standard moment sequence characterization.

Theorem 3.4. *Let $\{\lambda_n\}_{n=0}^{+\infty}$ be a sequence of positive real numbers. Then $\{\lambda_n\}_{n=0}^{+\infty}$ is moment sequence if and only if for some interval I the linear functional φ defined on $P(I)$ - the set of polynomials over I - by $\varphi(\sum_{n=0}^{+\infty} a_n t^n) = \sum_{n=0}^{+\infty} a_n \lambda_n$ is positive (i.e., $\varphi(p) \geq 0$ whenever $P(t) \geq 0$ on I).*

Proof. See [20].

□

The application of above theorem characterizes subnormality of weighted composition operators initially.

Theorem 3.5. *Let $W \in \mathcal{B}(L^2(\Sigma))$ and $I = [0, \|J\|_\infty]$. Define the linear transformation $\mathcal{L} : P(I) \rightarrow L^\infty(\Sigma)$ by $\mathcal{L}(\sum_{n=0}^{+\infty} a_n t^n) = \sum_{n=0}^{+\infty} a_n J_n$. Then W is subnormal if and only if $\mathcal{L}$ is positive, in a sense that $\mathcal{L}(p) \geq 0$ a.e $[\mu]$ whenever $p \geq 0$ on I .*

Proof. Let W be a subnormal and p be an arbitrary nonnegative polynomial on I . Let $K \subseteq \{x \in X : \mathcal{L}(p)(x) < 0\}$ with $\mu(K) < +\infty$. By Theorem 3.3 there is a Borel measure m on I such that for every $n \geq 0$,

$$\int_K J_n d\mu = \int_I t^n dm(t).$$

Then

$$\begin{aligned} \int_K \mathcal{L}(p) d\mu &= \sum_{n=0}^{+\infty} a_n \int_K J_n d\mu \\ &= \sum_{n=0}^{+\infty} a_n \int_I t^n dm(t) \\ &= \int_I p(t) dm(t) \geq 0. \end{aligned}$$

It follows that $\mathcal{L}(p) \geq 0$ a.e, hence $\mathcal{L}$ is positive.

Conversely, suppose that $\mathcal{L}$ is positive. Note that $\mathcal{L}$ is continuous with respect to essential sup norm since $\mathcal{L}$ preserves the unit element of both sides and $J_n(x) \in I = [0, \|J\|_\infty]$ a.e for each $n \geq 0$. Then $\mathcal{L}$ can be normally extended to a positive linear mapping from $C(I)$ (the space of all continuous functions on I) into $L^\infty(\Sigma)$. Correspond for each $f \in L^2(\Sigma)$ a linear functional φ_f on $C(I)$ by

$$\varphi_f(g) = \int_X \mathcal{L}(g) |f|^2 d\mu.$$

Since $\mathcal{L}(g) \in L^\infty(\Sigma)$, φ_f is bounded. Actually its norm can not be exceed $\|\mathcal{L}\| \|f\|_2^2$. By the Riesz representation theorem there is a unique Borel measure m_f such that

$$\varphi_f(g) = \int_I g(t) dm_f(t), \quad g \in C(I).$$

Hence $\int_X \mathcal{L}(g) |f|^2 d\mu = \int_I g(t) dm_f(t)$. For each $n \geq 0$, putting $g(t) = t^n$ in the last equation leads that

$$\int_X J_n |f|^2 d\mu = \int_I t^n dm_f(t).$$

It follows from Theorem 3.3 that W is subnormal. □

Corollary 3.6. $W \in \mathcal{B}(L^2(\Sigma))$ is subnormal if and only if for every Σ -measurable set A of finite measure the sequence $\{\int_{T^{-n}A} |u|^2 d\mu\}_{n=0}^{+\infty}$ is a moment sequence.

Proof. First suppose that W is subnormal. Then by Theorem 3.3 we have

$$\begin{aligned} \int_X J_n \chi_A d\mu &= \int_A h_n[E_n(|u|^2)] \circ T^{-n} d\mu \\ &= \int_{T^{-n}A} E_n(|u|^2) d\mu \\ &= \int_{T^{-n}A} |u|^2 d\mu. \end{aligned}$$

Conversely, let A be an arbitrary Σ -measurable set of finite measure on which $\{\int_{T^{-n}A} |u|^2 d\mu\}_{n=0}^{+\infty}$ is a moment sequence. Let $p(t) \geq 0$ on $I = [0, \|J\|_\infty]$. Then there is a Borel measure m such that for each n , $\int_{T^{-n}A} |u|^2 d\mu = \int_I t^n dm(t)$. Then

$$\begin{aligned} \int_A \mathcal{L}(p) d\mu &= \sum_{n=0}^{+\infty} a_n \int_A J_n d\mu \\ &= \sum_{n=0}^{+\infty} a_n \int_{T^{-n}A} |u|^2 d\mu \\ &= \sum_{n=0}^{+\infty} a_n \int_I t^n dm(t) \\ &= \int_I p(t) dm(t) \\ &\geq 0. \end{aligned}$$

Hence $\mathcal{L}(p) \geq 0$ a.e, since A was chosen arbitrarily. This fact yields that W is subnormal by Theorem 3.5. $\square$

Proposition 3.7. $W \in \mathcal{B}(L^2(\Sigma))$ is subnormal if and only if $\{J_n(x)\}_{n=0}^{+\infty}$ is a moment sequence for almost every $x \in X$.

Proof. The proof follows very closely the proof of Corollary 4 in [12]. Suppose first that W is subnormal. Let $P^+(I)$ be the set of all nonnegative polynomials on I . Let $P_r^+(I)$ be the set of all polynomials with rational coefficients which are nonnegative on I . By Theorem 3.5 for each $p \in P_r^+(I)$, $\mathcal{L}(p) \geq 0$ a.e. Let $X_p := \{x \in X : \mathcal{L}(p) \geq 0\}$. Then $\mu(X - X_p) = 0$. Since $P_r^+(I)$ is a countable dense subset of $P^+(I)$ we have $\mu(X - X^*) = 0$ where $X^* = \bigcup \{X_p : p \in P_r^+(I)\}$. It means that for each $p \in P^+(I)$, $\mathcal{L}(p) \geq 0$ a.e. Therefore by Theorem 3.5 it is inferred that $\{J_n(x)\}_{n=0}^{+\infty}$ is a moment sequence for every $x \in X^*$.

Conversely, suppose that $\{J_n(x)\}_{n=0}^{+\infty}$ is a moment sequence for almost every $x \in X$. Let X_0 be a measurable subset of X with $\mu(X - X_0) = 0$ in a way that $\{J_n(x)\}_{n=0}^{+\infty}$ is a moment sequence on X_0 . If p is an arbitrary nonnegative polynomial on I , then by Theorem 3.4, $\mathcal{L}(p) \geq 0$ exactly on X_0 . Thus the positivity of $\mathcal{L}$ yields that W is subnormal by Theorem 3.5. $\square$

Example 3.8. Consider the case that $h \circ T = h$ and $E(|u|^2) \circ T = E(|u|^2)$. This case in turn characterizes p -quasihyponormal and p -paranormal weighted composition

operators; see [8]. Now the relationship $J_{n+1} = hE(J_n) \circ T^{-1}$ follows that $J_n = h^{n-1}J$ for each $n \geq 0$. Thus $\mathcal{L}(\sum_{n=0}^{+\infty} a_n t^n) = a_0 + J \sum_{n=1}^{+\infty} a_n h^{n-1}$ is positive since $J(x) \in [0, \|J\|_\infty]$ for almost every $x \in X$. Therefore W is subnormal by Theorem 3.5.

Example 3.9. Let X be the set of nonnegative integers, let Σ be the σ -algebra of all subsets of X , and take μ to be the point mass measure determined by the $\mu(k) := m_k$, $k \in X$. Define the measurable transformation T by

$$T(k) = \begin{cases} 0 & k = 0, 1, \\ k-1 & k \geq 2 \end{cases}$$

and the weight function u by

$$u(k) = \begin{cases} 1 & k = 0, 1, \\ k & k \geq 2. \end{cases}$$

By routine computations one may verify that

$$\begin{aligned} h(k) &= \frac{1}{m_k} \sum_{j \in T^{-1}(k)} m_j \\ h_n(k) &= \frac{1}{m_k} \sum_{j \in T^{-1}(k)} h_{n-1}(j) m_j \\ J_n(k) &= \frac{1}{m_k} \sum_{j \in T^{-n}(k)} (u(j))^2 m_j. \end{aligned}$$

So by inserting T and u in the above formulas we obtain

$$\begin{aligned} J_1(0) &= \frac{1}{m_0} (m_0 + m_1); \\ J_n(0) &= \frac{1}{m_0} (m_0 + m_1 + 2^2 m_2 + 3^2 m_3 + \dots + n^2 m_n); \\ J_n(k) &= \frac{m_{k+n}}{m_k}, \quad k \geq 1. \end{aligned}$$

In particular, fix $a > 0$ and let $\epsilon > 0$ be an arbitrary. Let $m_0 = \epsilon$ and for every $k \geq 1$, let

$$m_k = \frac{1}{k^2} \int_a^{a+\epsilon} (t^k - t^{k-1}) dt.$$

In Proposition 3.7 it is proved that W is subnormal if and only if for every $k \geq 1$ the sequence $\{J_n(k)\}_{n=0}^{+\infty}$ is a moment sequence. Namely the sequences $\{m_0 + \sum_{i=1}^n i^2 m_i\}_{n=0}^{+\infty}$ and $\{\frac{m_{k+n}}{m_k}\}_{n=0}^{+\infty}$ are moment sequences. But these assertions are

true since for every $k \geq 1$ and $n \geq 0$, we have

$$\begin{aligned} m_0 + \sum_{i=1}^n i^2 m_i &= m_0 + \sum_{i=1}^n \int_a^{a+\epsilon} (t^i - t^{i-1}) dt \\ &= \int_a^{a+\epsilon} t^n dt \end{aligned}$$

and

$$\begin{aligned} \frac{m_{k+n}}{m_k} &= \frac{1}{(k+n)^2 m_k} \int_a^{a+\epsilon} (t^{k+n} - t^{k+n-1}) dt \\ &= \int_a^{a+\epsilon} t^n \left(\frac{t^k - t^{k-1}}{(k+n)^2 m_k} \right) dt \\ &= \int_a^{a+\epsilon} t^n d\mu(t), \end{aligned}$$

where $d\mu = \left(\frac{t^k - t^{k-1}}{(k+n)^2 m_k} \right) dt$ is a nonnegative measure on $[a, a + \epsilon]$. Hence W is subnormal.

References

- [1] C. Burnap, I. Jung, *Composition operators with weak hyponormality*, J. Math. Anal. Appl., **337**(2008), 686-694.
- [2] C. Burnap, I. Jung and A. Lambert, *Separating partial normality classes with composition operators*, J. Operator Theory, **53**(2005), 381-397.
- [3] J. B. Conway, *Subnormal Operators*, Pitman Publ. Co., London, 1981.
- [4] J. B. Conway, *The Theory of Subnormal Operators*, Math. Surveys Monogr., Amer. Math. Soc., Providence, RI, 1991.
- [5] M. Fujii, S. Izumino and R. Nakamoto, *Classes of operators determined by the Heinz-Kato-Furuta inequality and the Hölder- McCarthy inequality*, Nihonkai Math. J., **5**(1994), 61-67.
- [6] J. Herron, *Weighted conditional expectation operators*, Oper. Matrices, **5**(2011), 107-118.
- [7] T. Hoover, A. Lambert and J. Quinn, *The Markov process determined by a weighted composition operator*, Studia Math., **72**(3)(1982), 225-235.
- [8] M. R. Jabbarzadeh, M. R. Azimi, *Some weak hyponormal classes of weighted composition operators*, Bull. Korean Math. Soc., **47**(4)(2010), 793-803.
- [9] R. Kumar, *Ascent and descent weighted composition operators on L^p -spaces*, Math. Vesnik, **60**(2008), 47-51.

- [10] A. Lambert, Thomas G. Lucas, *Nagata's principle of idealization in relation to module homomorphisms and conditional expectations*, Kyungpook Math. J., **40**(2000), 327-337.
- [11] A. Lambert, *Normal extentions of subnormal composition operators*, Michigan Math. J., **35**(1988), 443-450.
- [12] A. Lambert, *Subnormal composition operators*, Proc. Amer. Math. Soc., **103**(1988), 750-754.
- [13] A. Lambert, *Hyponormal composition operators*, Bull. Lond. Math. Soc., **18**(1986), 395-400.
- [14] A. Lambert, *Subnormality and weighted shifts*, Bull. Lond. Math. Soc., **14**(2)(1976), 476-480.
- [15] E. Nordgren, *Composition operators on Hilbert spaces*, in: Lecture Notes in Math., **693**, Springer-Verlag, Berlin, 1978.
- [16] M. M. Rao, *Conditional measure and applications*, Marcel Dekker, New York, 1993.
- [17] R. K. Singh, J. S. Manhas, *Composition Operators on Function Spaces*, Elsevier Science Publishers B. V., North-Holland, Amsterdam, 1993.
- [18] J. Stochel, J. B. Stochel, *Seminormal composition operators on L^2 spaces induced by matrices: The Laplace density case*, J. Math. Anal. Appl., **375**(2011), 1-7.
- [19] R. Whitley, *Normal and quasinormal composition operators*, Proc. Amer. Math. Soc., **70**(1978), 114-118.
- [20] D. Widder, *The Laplace transform*, Princeton Univ. Press, Princeton, N. J., 1946.