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A Note on the Fitting Ideals in Free Resolutions

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In this paper, we prove the following theorems. If R is ring such that $charK = 0$, then there exist an integer k such that :
(a) $J^k ExtR^{d+1}(M,N) = 0$ for every pair of finitely generated R-modules M and N ; and
(b) if M is a finitely generated R-module having a well-defined rank and $(F.,\phi.)$ is any finitely generated free resolution of M, then $J^k I_i(\phi_j) \subseteq I_{i+1}(\phi_j) \forall i = 0,\cdots,t_j - 1$ and $\forall j \geq d+1$ where $t_j = \text{rank}\phi_j$

1. Introduction

Throughout this paper, all rings are commutative with identity. If R is a ring and if $\phi : F \to G$ is a map of finitely generated free R-modules, then we define $I_i(\phi)(i \geq 0)$ to be the ideal of R generated by the $i \times i$ minors of a matrix representing ϕ and the rank of ϕ, to be the largest number t such that $I_t(\phi) \neq 0$. The ideals $I_i(\phi)$ are called the Fitting ideals of ϕ.
Let (R, m, K) be a d-dimensional complete Noetherian local ring containing a field with maximal ideal m and residue class field $K = \frac{R}{m}$. The purpose of this paper is to study a conjecture of C. Huneke concerning the behavier of Fitting ideals in free resolutions of finitely generated modules over R. In order to present these questions, more definitions are needed.

Let R be as above. Then, by Cohen structure theorem,

$$R \cong K[| X_1, \cdots, X_n |]/(f_1, \cdots, f_t)$$

for some indeterminates $X_1, \cdots, X_n$ and some power series

$$f_1, \cdots, f_t \in K[| X_1, \cdots, X_n |].$$

Therefore, from this representation, we may define the Jacobian ideal of R to be $I_h(\partial(f_1,\cdots,f_t)/\partial(X_1,\cdots,X_n))R$, that is, the ideal generated by the image of $h \times h$ minors of the Jacobian matrix $(\partial(f_1,\cdots,f_t)\partial(X_1,\cdots,X_n))$, where h is the height of $(f_1,\cdots,f_t)$. Furthermore, we denote by $I_s(R)$ the ideal defining the singular

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locus of R ; that is, $I_s(R) = \cap_{P \notin Reg(R)} P$. If M is a finitely generated R-module then M is said to have a well-defined rank r, if for any $P \in Ass(R)$, M_p is free and $\mu_p(M) = r$. Finally, we denote by $(F., \phi.)$ an acyclic complex of finitely generated free R-modules :

$$\cdots F_d \xrightarrow{\phi_d} F_{d-1} \xrightarrow{\phi_{d-1}} \cdots \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0.$$

Let us state the questions as follows.

Conjecture 1.1. Let (R, m, K) be a d-dimensional complete Noetherian local ring containing a field and let J be the Jacobian ideal of R. Let M be a finitely generated R-module and let $(F., \phi.)$ be any finitely generated free resolution of M. Assume that M has a well-defined rank. Then

$$J \subseteq I_1(\phi_j)$$
$$JI_1(\phi_j) \subseteq I_2(\phi_j)$$
$$\vdots$$
$$JI_{t_j-1}(\phi_j) \subseteq I_{t_j}(\phi_j)$$

for all $j \geq d+1$, where $t_j = \text{rank}(\phi_j)$. In particular, $J^k \subseteq I_k(\phi_j)$ for all $k \leq t_j$.

Question 1.2. Let (R, m, K) be a d-dimensional complete Noetherian local ring containing a field, with J the Jacobian ideal of R. Then does it hold that $JExt_R^{d+1}(-,-) = 0$?

The following theorem [5, Theorem 1.1], due to Eisenbud and Green, was concerned with Fitting ideals and was initially conjectured by C. Huneke.

Theorem 1.3. *Let R be a Noetherian local ring containing a field and let M be a finitely generated R-module. Let $I = ann_R M$ and let $(F., \phi.)$ be a finitely generated free resolution of M. Assume that I contains a non-zero-divisor. Then*

$$II_i(\phi_j) \subset I_{i+1}(\phi_j) \forall i = 0, \cdots, t_j - 1 \text{ and } {}^\forall j \geq 1,$$

where $t_j = \text{rank}\phi_j$

The main results of this paper are as follows.
If R is a ring such that $charK = 0$. then there exists an integer k such that:

(a) $J^k Ext_R^{d+1}(M, N) = 0$ for every pair of finitely generated R-modules M and N ; and

(b) if M is a finitely generated R-module having a well-defined rank and $(F., \phi.)$ is any finitely generated free resolution of M, then

$$J^k I_i(\phi_j) \subseteq I_{i+1}(\phi_j) \ {}^\forall i = 0, \cdots, t_j - 1 \ \textit{and} \ {}^\forall j \geq d+1$$

where $t_j = \text{rank}\phi_j$

2. Preliminaries

First we give some definitions and preliminaries.

Definition 2.1. Let (R, m, K) be a d-dimensional complete Noetherian local ring containing a field. A regular local ring A of the form $K[| X_1, \cdots, X_d |]$ is called a (Noether) normalization of R if $A \subseteq R$ and R is finite over A.

By the Cohen structure theorem , if $x_1, \cdots, x_d$ is a system of parameters (s.o.p.)of R then $K[| X_1, \cdots, X_d |]$ is a normalization of R; in fact, every normalization of R can be constructed in this way.

Definition 2.2. Let A be a Noetherian ring and R a finitely generated A-algebra. Let $R = A[X_1, \cdots, X_n]/(f_1, \cdots, f_t)$ be a presentation of R over A. Then the ideal in R generated by $n \times n$ minors of the jacobian matrix $\frac{\partial f_i}{\partial X_j}$ is called the Jacobian ideal of R over A, denote $J_{R/A}$.

Lemma 2.3. *Let R be Noetherian ring and M a finitely generated R-module. Then if M_1 is a first syzygy of M, then*

$$ann_R Ext^1_R(M, -) = ann_R Ext^1_R(M, M_1).$$

Proof. See [1, Corollary 1.6]

Proposition 2.4. *Let R be a Noetherian ring, M a finitely generated R-module, and $x \in R$. Suppose that M has a well-defined rank and that $xExt^1_R(M, -) = 0$. Then for any finitely generated free resolution $(F., \phi.)$ of M, we have*

$$(t_j - x)I_i(\phi_j) \subseteq I_{i+1}(\phi_j) \forall i = 0, \cdots, t_j - 1 and \forall j \geq 1$$

where $t_j = \text{rank}\phi_j$

Proof. See [1, Proposition 2.4]

Lemma 2.5. *Let (R, m, K) be a d-dimensional complete Noetherian local ring containing a field and let J be the Jacobian ideal of R. Then $J = \sum_A J_{R/A}$, where the sum is over all normalization of R.*

Proof. See [1, Lemma 4.3]

Proposition 2.6. *Let (R, m, K) be a d-dimensional complete Noetherian local ring containing a field and let J be the Jacobian ideal of R. Let $P \in Spec(R) - \{m\}$ be such that R_P is regular. Then there exists a normalization A of R such that*

1. $J_{R/A} \nsubseteq P$
2. *$R_{P \cap A}$ is Cohen Macaulay ring (CM).*

Proof. See [1, Proposition 4.4]

Definition 2.7. Let R be a commutative ring and R an A-algebra. Let R^e denote the envelope algebra $R \bigotimes_A R$ and let $\mu : R \bigotimes_A R \to R$ be the augmented map defined by $\mu(x \bigotimes y) = x.y$ for $x, y \in R$. Let I be the kernel of μ. Then the Noetherian different ideal of R over A, denoted N_R^A, is the ideal $\mu(ann_{R^e} I)$.

Lemma 2.8. *Let A be a Noetherian ring and R a finitely generated A-algebra. Then $J_{R/A} \subseteq N_R^A$.*

Proof. See [1, Lemma 5.8].

Proposition 2.9. *Let A be a Noetherian ring and R a finitely generated A-algebra. Then, for any generated R-modules M and N,*

$$N_R^A ann_A Ext_A^1(M, N) \subseteq ann_R Ext_R^1(M, N).$$

Proof. See [1, Proposition 5.9].

Lemma 2.10. *Let R be a d-dimensional complete Noetherian local ring containing a field, and A a normalization of R. Let $x \in A$ be such that $xExt_A^1(R, -) = 0$. Then $x^d Ext_A^1(M_{d,)} = 0$ for any finitely generated R-module M.*

Proof. See [1, Lemma 5.14]

3. Main Theory

We shall prove the main results of this paper.

Theorem 3.1. *Let (R, m, K) be a d-dimensional complete Noetheiran local ring containing a field, with J the Jacobian ideal of R. Assume that $charK = 0$. Then there exists an integer k such that $J^k Ext_R^{d+1}(M, -) = 0$ for any finitely generated R-module M.*

Proof. We first show the following claim : If $P \in Spec(R)$ is such that $J \nsubseteq P$, then there exists an element $x \in J \backslash P$ such that $xExt_R^1(M_d, -) = 0$ for any finitely generated R-module M. To prove this claim, note that by Proposition 2.6 we know there is a normalization A of R such that (1) $J_{R/A} \nsubseteq P$ and (2) $R_{P \cap A}$ is CM. If $q = P \cap A$ then by [2, Corollaly18.17] R_q is a free A_q-module, and so $(Ext_A^1(R, -))_q = 0$ then by Lemma 2.3 we see there exits $y \in A \backslash q$ such that $yExt_A^1(R, -) = 0$, Thus by Lemma 2.10, we have $y^d Ext_A^1(M_d, -) = 0$ for any finitely generated R-module M. Finally by (1) we can choose an element $z \in J_{R/A} \backslash P$ and set $x = y^d z$. Then $x \in J \backslash P$,and by Lemma 2.8 and Proposition 2.9, $xExt_{\frac{1}{R}}(M_d, -) = 0$.

Let $J_0 = \cap_M ann_R Ext_{\frac{1}{R}}(M, -)$, where the intersection is over all finitely generated R-modules M. Then, by the claim, for any prime $P \nsubseteq J$ there is an element $x \notin P$ such that $xExt_R^{d+1}(M, -) = xExt_{\frac{1}{R}}(M_d, -) = 0$ for any finitely generated module M, which means $x \in J_0$ and hence $P \nsubseteq J_0$. It follows that $J^k \subset J_0$ for some integer k, and then $J^k Ext_R^{d+1}(M, -) = 0$ for any finitely generated module M. $\square$

Theorem 3.2. *Let (R, m, K) be a d-dimensional complete Noetherian local ring containing a field, and let J ba the Jacobian ideal of R. Assume that $charK = 0$. Then there exists an integer k such that for any finitely generated R-module M*

having a well-defined rank and for any finitely generated free resolution $(F., \phi.)$ *of* M

$$J^k I_i(\phi_j) \subseteq I_{i+1}(\phi_j) \forall i = 0, \cdots, t_j - 1 and \forall \geq d+1$$

where $t_j = \text{rank}\phi_j$.

Proof. By Theorem 3.1 and Proposition 2.4 proof is clear.

Corollary 3.3. *Let* (R, m, K) *be a* d*-dimensional complete Noetherian local ring containing a field. Assume that* $\sqrt{J} = I_s(R)$. *Then there exists an integer* k *such that*

$$I_s(R)^k Ext_R^{d+1}(M, -) = 0$$

for any finitely generated R*-module* M.

A Noetherian local ring R is called generalized CM if R_P is CM for all $P \in Spec(R) - \{m\}$.

Corollary 3.4. *Let* (R, m, K) *be a* d*-dimensional complete Noetherian local ring containing a field, and let* J *be the Jacobian ideal of* R. *Assume that* R *is a generalized* CM *ring. Then there exists an integer* k *such that*

$$J^k Ext_R^{d+1}(M, -) = 0$$

for any finitely generated R*-module* M.

Proof. Let $P \in Spec(R)$ be such that $J \nsubseteq P$; then $P \neq m$. In view of the proof of Theorem 3.1, it is enough to show that there exists a normalization A of R such that (1) $J_{R/A} \nsubseteq P$ and (2) $R_{P \cap A}$ is CM. But condition (2) is redundant, as $P \neq m$ guarantees it; hence the assertion follows from Lemma 2.5.

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