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Extensions of Strongly α-semicommutative Rings

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Abstract. This paper is devoted to the study of strongly α-semicommutative rings, a generalization of strongly semicommutative and α-rigid rings. Although the n-by-n upper triangular matrix ring over any ring with identity is not strongly $\bar{\alpha}$-semicommutative for $n \geq 2$, we show that a special subring of the upper triangular matrix ring over a reduced ring is strongly $\bar{\alpha}$-semicommutative under some additional conditions. Moreover, it is shown that if R is strongly α-semicommutative with $\alpha(1) = 1$ and S is a domain, then the Dorroh extension D of R by S is strongly $\bar{\alpha}$-semicommutative.

1. Introduction

Throughout this paper, R denotes an associative ring with identity and α denotes a nonzero and non-identity endomorphism, unless specified otherwise. A ring R is called semicommutative, if for all $a, b \in R, ab = 0$ implies $aRb = 0$. This is equivalent to the usual definition by [18, Lemma 1.2] or [8, Lemma 1]. Properties, examples and counterexamples of semicommutative rings were given in Huh, Lee and Smoktunowicz [8], Kim and Lee [10], Liu [13] and Yang [19]. One of general-

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izations of semicommutative rings was investigated by Liu and Zhao in [14].

Recall that an endomorphism α of a ring R is called rigid [11] if for $a \in R$, $a\alpha(a) = 0$ implies $a = 0$, and R is called an α-rigid ring [6] if there exists a rigid endomorphism α of R. Note that any rigid endomorphism of a ring is a monomorphism, and α-rigid rings are reduced rings by [6, Proposition 5]. Due to [1], an endomorphism α of a ring R is called semicommutative if whenever $ab = 0$ for $a, b \in R, aR\alpha(b) = 0$. A ring R is called α-semicommutative if there exists a semicommutative endomorphism α of R. Gang and Ruijuan [5] called a ring R strongly semicommutative, if whenever polynomials $f(x), g(x)$ in $R[x]$ satisfy $f(x)g(x) = 0$, then $f(x)R[x]g(x) = 0$. In general the polynomial rings over α-semicommutative rings need not be α-semicommutative. In this paper, we consider the α-semicommutative rings over which polynomial rings are also α-semicommutative and we call them strongly α-semicommutative rings, i.e., if α is an endomorphism of R, then α is called strongly semicommutative if whenever polynomials $f(x), g(x) \in R[x]$ satisfy $f(x)g(x) = 0$, then $f(x)R[x]\alpha(g(x)) = 0$. A ring R is called strongly α-semicommutative if there exists a strongly semicommutative endomorphism α of R. Clearly strongly α-semicommutative rings are α-semicommutative but not conversely. If R is Armendariz, then these two concepts coincide (see, Proposition 2.11). We characterize α-rigid rings by showing that a ring R is α-rigid if and only if R is a reduced strongly α-semicommutative ring and α is a monomorphism. It is also shown that a ring R is strongly α-semicommutative if and only if the polynomial ring $R[x]$ over R is strongly α-semicommutative. Some extensions of α-semicommutative rings are considered.

2. Strongly α-semicommutative Rings

In this section we introduce the concept of a strongly α-semicommutative ring and study its properties. Observe that the notion of strongly α-semicommutative rings not only generalizes that of α-rigid rings, but also extends that of strongly semicommutative rings. We start by the following definition.

Definition 2.1. An endomorphism α of a ring R is called *strongly semicommutative* if whenever polynomials $f(x), g(x) \in R[x]$ satisfy $f(x)g(x) = 0$, then $f(x)R[x]\alpha(g(x)) = 0$. A ring R is called *strongly α-semicommutative* if there exists a strongly semicommutative endomorphism α of R.

It is clear that a ring R is strongly semicommutative, if R is strongly I_R-semicommutative, where I_R is the identity endomorphism of R. It is easy to see that every subring S with $\alpha(S) \subseteq S$ of a strongly α-semicommutative ring is also strongly α-semicommutative. For any $i \in I$, let R_i be strongly α_i-semicommutative where α_i is an endomorphism of R_i. Set $W = \Pi_{i \in I} R_i$. Define an endomorphism α of W as following:

$$\alpha(a_i)_{i \in I} = (\alpha_i(a_i))_{i \in I}.$$

Then it is easy to see that W is strongly α-semicommutative.

Remark 2.2. Let R be a strongly α-semicommutative ring with $f(x)g(x) = 0$ for $f(x), g(x) \in R[x]$. Then $f(x)R[x]\alpha(g(x)) = 0$ and, in particular, $f(x)\alpha(g(x)) = 0$. Since R is strongly α-semicommutative, we get $f(x)R[x]\alpha^2(g(x)) = 0$. So, by induction hypothesis, we obtain $f(x)R[x]\alpha^k(g(x)) = 0$ and $f(x)\alpha^k(g(x)) = 0$, for any positive integer k.

The following example shows that there exists an endomorphism α of strongly semicommutative ring R such that R is not strongly α-semicommutative.

Example 2.3. Let $\mathbb{Z}_2$ be the ring of integers modulo 2 and consider the ring $R = \mathbb{Z}_2 \bigoplus \mathbb{Z}_2$, with the usual addition and multiplication. Then R is strongly semicommutative, since R is a commutative reduced ring. Now, let $\alpha : R \to R$ be defined by $\alpha((a,b)) = (b,a)$. Then α is an automorphism of R. For $f(x) = (1,0) + (1,0)x$ and $g(x) = (0,1) + (0,1)x$, it is clear that $f(x)g(x) = 0$. But $(0,0) \neq ((1,0) + (1,0)x)(1,1)x((1,0) + (1,0)x) \in f(x)R[x]\alpha(g(x))$. Thus R is not strongly α-semicommutative.

Lemma 2.4. *R is a reduced ring if and only if so is $R[x]$.*

Lemma 2.5. *A ring R is α-rigid if and only if $R[x]$ is α-rigid.*

Theorem 2.6. *A ring R is α-rigid if and only if R is a reduced strongly α-semicommutative ring and α is a monomorphism.*

Proof. ($\Rightarrow$) Let R be an α-rigid ring. Then R is reduced and α is a monomorphism by [6, p.218]. Assume that $f(x)g(x) = 0$, for $f(x), g(x) \in R[x]$. Let $h(x)$ be an arbitrary polynomial of $R[x]$. Then $g(x)f(x) = 0$ since $R[x]$ is reduced by Lemma 2.4. Thus $f(x)h(x)\alpha(g(x))\alpha(f(x)h(x)\alpha(g(x))) = f(x)h(x)\alpha(g(x)f(x))\alpha(h(x))\alpha^2(g(x)) = 0$. Since R is α-rigid, $f(x)h(x)\alpha(g(x)) = 0$ by Lemma 2.5 so $f(x)R[x]\alpha(g(x)) = 0$. Thus R is strongly α-semicommutative.

($\Leftarrow$) Assume that $f(x)\alpha(f(x)) = 0$ for $f(x) \in R[x]$. Since R is reduced and strongly α-semicommutative, $\alpha(f(x))f(x) = 0$ and so $\alpha(f(x))R[x]\alpha(f(x)) = 0$. Hence $\alpha((f(x))^2) = 0$ and so $f(x) = 0$, since α is a monomorphism and R is reduced. Therefore R is α-rigid. $\square$

The following examples show that the condition "R is reduced ring" and "α is a monomorphism" in Theorem 2.6 cannot be dropped respectively.

Example 2.7. Let $\mathbb{Z}$ be the ring of integers. Consider $R = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} | a, b \in \mathbb{Z} \right\}$. Let $\alpha : R \to R$ be an endomorphism defined by $\alpha\left(\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}\right) = \begin{pmatrix} a & -b \\ 0 & a \end{pmatrix}$. Note that α is an automorphism. By [1, Example 2.5(1)] R is not reduced and hence R is not α-rigid. Thus $R[x]$ is not α-rigid by Lemma 2.5.

Let $f(x)g(x) = 0$ for $f(x) = \begin{pmatrix} f_0(x) & f_1(x) \\ 0 & f_0(x) \end{pmatrix}, g(x) = \begin{pmatrix} g_0(x) & g_1(x) \\ 0 & g_0(x) \end{pmatrix} \in R[x]$. Then $f_0(x)g_0(x) = 0$ and $f_0(x)g_1(x) + f_1(x)g_0(x) = 0$. For $h(x) =$

$\begin{pmatrix} h_0(x) & h_1(x) \\ 0 & h_0(x) \end{pmatrix} \in R[x]$, we have

$$\begin{pmatrix} f_0(x) & f_1(x) \\ 0 & f_0(x) \end{pmatrix} \begin{pmatrix} h_0(x) & h_1(x) \\ 0 & h_0(x) \end{pmatrix} \alpha\left(\begin{pmatrix} g_0(x) & g_1(x) \\ 0 & g_0(x) \end{pmatrix}\right)$$
$$= \begin{pmatrix} f_0(x)h_0(x)g_0(x) & -f_0(x)h_0(x)g_1(x) + f_0(x)h_1(x)g_0(x) + f_1(x)h_0(x)g_0(x) \\ 0 & f_0(x)h_0(x)g_0(x) \end{pmatrix}.$$

Since $f_0(x)g_0(x) = 0, f_0(x) = 0$ or $g_0(x) = 0$. If $f_0(x) = 0$ then $f_1(x)g_0(x) = 0$. So $f(x)R[x]\alpha(g(x)) = 0$. If $g_0(x) = 0$ then $f_0(x)g_1(x) = 0$. Again $f(x)R[x]\alpha(g(x)) = 0$. Thus R is strongly α-semicommutative.

Example 2.8. Let F be a field and $R = F[x]$ the polynomial ring over F. Define $\alpha : R[x] \to R[x]$ by $\alpha(f(x)) = f(0)$ where $f(x) \in R[x]$. Then $R[x]$ is a commutative domain (and so reduced) and α is not a monomorphism. If $f(x)g(x) = 0$ for $f(x), g(x) \in R[x]$ then $f(x) = 0$ or $g(x) = 0$, and so $f(x) = 0$ or $\alpha(g(x)) = 0$. Hence $f(x)R[x]\alpha(g(x)) = 0$, and thus R is strongly α-semicommutative. Note that R is not α-rigid, since $x\alpha(x) = 0$ for $0 \neq x \in R$.

Observe that if R is a domain then R is both strongly semicommutative and strongly α-semicommutative for any endomorphism α of R. Example 2.7 also shows that there exists a strongly α-semicommutative ring R which is not a domain. According to Cohn [4], a ring R is called *reversible* if $ab = 0$ implies $ba = 0$ for $a, b \in R$. Baser and et al. [2] called a ring R *right* (respectively, *left*) *α-reversible* if there exists a right (respectively, left) reversible endomorphism α of R. A ring is *α-reversible* if it is both left and right α-reversible.

Lemma 2.9.([16, Proposition 3]) *A reduced α-reversible ring is α-semicommutative.*

Proposition 2.10. *Let R be a reduced and α-reversible ring. Then R is strongly α-semicommutative.*

Proof. Let $f(x) = \Sigma_{i=0}^{n} a_i x^i, g(x) = \Sigma_{j=0}^{m} b_j x^j \in R[x]$ be such that $f(x)g(x) = 0 = \Sigma_{s=0}^{n+m} \Sigma_{i+j=s} a_i b_j x^s$. Since every reduced ring is an Armendariz ring, we obtain $a_i b_j = 0$. Then $\alpha(b_j)a_i = 0$ (by α-reversibility). Now for arbitrary element $h(x) = \Sigma_{k=0}^{r} c_k x^k \in R[x]$, we have $\alpha(b_j)a_i c_k = 0$ for each i, j, k, so $a_i c_k \alpha(b_j) = 0$ (by reducibility). Hence, $f(x)h(x)\alpha(g(x)) = 0$. Therefore R is strongly α-semicommutative. □

Rege and Chhawchharia [17] called a ring R an Armendariz ring if whenever polynomials $f(x) = a_0 + a_1 x + \cdots + a_m x^m, g(x) = b_0 + b_1 x + \cdots + b_n x^n \in R[x]$ satisfy $f(x)g(x) = 0$, then $a_i b_j = 0$ for each i and j. Hong et al. [7] called a ring R α-Armendariz if whenever $f(x) = a_0 + a_1 x + \cdots + a_m x^m, g(x) = b_0 + b_1 x + \cdots + b_n x^n \in R[x; \alpha]$ satisfy $f(x)g(x) = 0$, then $a_i b_j = 0$ for each i and j.

Proposition 2.11. *Let R be an Armendariz ring. If R is α-semicommutative, then R is strongly α-semicommutative.*

Proof. Suppose that $f(x) = \Sigma_{i=0}^{n} a_i x^i, g(x) = \Sigma_{j=0}^{m} b_j x^j \in R[x]$ satisfy $f(x)g(x) = 0$.

Then, since R is Armendariz, each a_ib_j is zero, additionally R is α-semicommutative, therefore $a_ic_k\alpha(b_j) = 0$ for any element c_k in R for all i, j, k. Now it is easy to check that $f(x)h(x)\alpha(g(x)) = 0$ for any $h(x) = \Sigma_{k=0}^{r} c_kx^k \in R[x]$. □

Lemma 2.12.([10, Proposition 3.1(2)]) *If R is a reversible α-Armendariz ring, then R is α-semicommutative.*

Liu and Yang [20] called a ring R *strongly reversible*, if whenever polynomials $f(x), g(x) \in R[x]$ satisfy $f(x)g(x) = 0$, then $g(x)f(x) = 0$.

Proposition 2.13. *If R is a strongly reversible α-Armendariz ring, then R is strongly α-semicommutative.*

Proof. Let $f(x)g(x) = 0$, for $f(x), g(x) \in R[x]$. Then $g(x)f(x) = 0$ since R is strongly reversible. By [7, Proposition 1.3(1)], we obtain $\alpha(g(x))f(x) = 0$, and so $\alpha(g(x))f(x)h(x) = 0$ for all $h(x) \in R[x]$. Hence, $f(x)h(x)\alpha(g(x)) = 0$ for all $h(x) \in R[x]$ since R is strongly reversible and $f(x)R[x]\alpha(g(x)) = 0$. Therefore, R is strongly α-semicommutative. □

Recall that an element u of a ring R is right regular if $ur = 0$ implies $r = 0$ for $r \in R$. Similarly, left regular elements can be defined. An element is regular if it is both left and right regular (and hence not a zero divisor).

Proposition 2.14. *Let Δ be a multiplicatively closed subset of a ring R consisting of central regular elements. Then R is strongly α-semicommutative if and only if so is $\Delta^{-1}R$.*

Proof. It is enough to show that the necessity. Suppose that R is strongly α-semicommutative. Let $F(x)G(x) = 0$, for $F(x) = u^{-1}f(x)$ and $G(x) = v^{-1}g(x) \in (\Delta^{-1}R)[x]$ where u, v are regular and $f(x), g(x) \in R[x]$. Since Δ is contained in the center of R we have $0 = F(x)G(x) = u^{-1}f(x)v^{-1}g(x) = (u^{-1}v^{-1})f(x)g(x) = (uv)^{-1}f(x)g(x)$ and so $f(x)g(x) = 0$. Since R is strongly α-semicommutative, $f(x)R[x]\alpha(g(x)) = 0$ and $f(x)(s^{-1}R)[x]\alpha(g(x)) = 0$ for any regular element s. This implies $F(x)(\Delta^{-1}R)[x]\alpha(G(x)) = 0$. Therefore $\Delta^{-1}R$ is strongly α-semicommutative. □

The ring of Laurent polynomials in x with coefficients in a ring R, denoted by $R[x; x^{-1}]$, consists of all formal sums $\sum_{i=k}^{n} m_ix^i$ with obvious addition and multiplication, where $m_i \in R$ and k, n are (possibly negative) integers.

Corollary 2.15. *Let R be a ring with $\alpha(1) = 1$. Then $R[x]$ is strongly α-semicommutative if and only if $R[x; x^{-1}]$ is strongly α-semicommutative.*

Corollary 2.16. *Let R be an Armendariz ring. Then the following are equivalent:*

(1) *R is α-semicommutative.*

(2) *R is strongly α-semicommutative.*

(3) *$R[x; x^{-1}]$ is strongly α-semicommutative.*

Proposition 2.17. *Let R be a ring, e a central idempotent of R, with $\alpha(e) = e$. Then the following statements are equivalent:*

(1) *R is strongly α-semicommutative rings.*

(2) *eR and $(1-e)R$ are strongly α-semicommutative rings.*

Proof. (1)⇔(2) This is straightforward since subrings and finite direct products of strongly α-semicommutative rings are strongly α-semicommutative. □

We denote by $M_n(R)$ and $T_n(R)$ the $n \times n$ matrix ring and $n \times n$ upper triangular matrix ring over R, respectively.

Given a ring R and a bimodule ${}_RM_R$, the trivial extension of R by M is the ring $T(R,M) = R \bigoplus M$ with the usual addition and the following multiplication $(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + m_1r_2)$. This is isomorphic to the ring of all matrices $\begin{pmatrix} r & m \\ 0 & r \end{pmatrix}$, where $r \in R, m \in M$ and the usual matrix operations are used.

For an endomorphism α of a ring R and the trivial extension $T(R,R)$ of R, $\bar{\alpha} : T(R,R) \to T(R,R)$ defined by $\bar{\alpha}\left(\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}\right) = \begin{pmatrix} \alpha(a) & \alpha(b) \\ 0 & \alpha(a) \end{pmatrix}$ is an endomorphism of $T(R,R)$. Since $T(R,0)$ is isomorphic to R, we can identify the restriction of $\bar{\alpha}$ by $T(R,0)$ to α. Notice that the trivial extension of a α-semicommutative ring is not $\bar{\alpha}$-semicommutative by [1, Example 2.9]. Now, we may ask whether the trivial extension $T(R,R)$ is strongly $\bar{\alpha}$-semicommutative if R is strongly α-semicommutative. But the following example erases the possibility.

Example 2.18. Consider the strongly α-semicommutative ring $R = \left\{\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in \mathbb{Z}\right\}$ with an endomorphism α defined by $\alpha\left(\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}\right) = \begin{pmatrix} a & -b \\ 0 & a \end{pmatrix}$ in Example 2.7. For

$$A = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix}, B = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \in T(R,R)$$

we have $AB = 0$. However, for

$$C = \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix} \in T(R,R),$$

we obtain

$$0 \neq \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix} = AC\bar{\alpha}(B) \in AT(R,R)\bar{\alpha}(B).$$

Thus, $T(R,R)$ is not strongly $\bar{\alpha}$-semicommutative.

It was shown in [1, Proposition 2.10], that if R is a reduced α-semicommutative ring, then $T(R,R)$ is an $\bar{\alpha}$-semicommutative. Here we have the following results.

Proposition 2.19. *Let R be a reduced ring. If R is α-semicommutative, then $T(R,R)$ is strongly $\bar{\alpha}$-semicommutative.*

Proof. Let $f(x) = (f_0(x), f_1(x)), g(x) = (g_0(x), g_1(x)) \in T(R,R)[x]$ with $f(x)g(x) = 0$. We shall prove $f(x)T(R,R)[x]\alpha(g(x)) = 0$. Now we have

$$f_0(x)g_0(x) = 0, \tag{2.1}$$

$$f_0(x)g_1(x) + f_1(x)g_0(x) = 0. \tag{2.2}$$

Since R is reduced, $R[x]$ is reduced. Therefore, (2.1) implies $g_0(x)f_0(x) = 0$. Multiplying (2.2) on the left side by $g_0(x)$ we get $f_1(x)g_0(x) = 0$, and so $f_0(x)g_1(x) = 0$. Let $f(x) = \Sigma_{i=0}^{n}(a_i, b_i)x^i, g(x) = \Sigma_{j=0}^{m}(a'_j, b'_j)x^j$, where $f_0(x) = \Sigma_{i=0}^{n}a_ix^i, f_1(x) = \Sigma_{i=0}^{n}b_ix^i, g_0(x) = \Sigma_{j=0}^{m}a'_jx^j$ and $g_1(x) = \Sigma_{j=0}^{m}b'_jx^j$. Since every reduced ring is an Armendariz ring, we obtain that $a_ia'_j = 0, a_ib'_j = 0, b_ia'_j = 0$ for all i, j by the preceding results. With these facts and the fact that R is α-semicommutative, we have $a_ic_k\alpha(a'_j) = 0, a_ic_k\alpha(b'_j) = 0, a_id_k\alpha(b'_j) = 0, b_ic_k\alpha(a'_j) = 0$, for any elements c_k, d_k. Thus, $f(x)h(x)\alpha(g(x)) = 0$, for any arbitrary $h(x) = \Sigma_{k=0}^{r}(c_k, d_k)x^k \in R[x]$. This implies that $T(R,R)$ is strongly $\bar{\alpha}$-semicommutative. $\square$

The trivial extension $T(R,R)$ of a ring R is extended to

$$S_3(R) = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} | a, b, c, d \in R \right\}$$

and an endomorphism α of a ring R is also extended to the endomorphism $\bar{\alpha}$ of $S_3(R)$ defined by $\bar{\alpha}((a_{ij})) = (\bar{\alpha}(a_{ij}))$. There exists a reduced ring R such that $S_3(R)$ is not strongly $\bar{\alpha}$-semicommutative by the following example.

Example 2.20. We consider the commutative reduced ring $R = \mathbb{Z}_2 \bigoplus \mathbb{Z}_2$, and the automorphism α of R defined by $\alpha((a,b)) = (b,a)$, in Example 2.3. Then $S_3(R)$ is not strongly $\bar{\alpha}$-semicommutative. For $A = \begin{pmatrix} (1,0) & (0,0) & (0,0) \\ (0,0) & (1,0) & (0,0) \\ (0,0) & (0,0) & (1,0) \end{pmatrix}$, $B = \begin{pmatrix} (0,1) & (0,0) & (0,0) \\ (0,0) & (0,1) & (0,0) \\ (0,0) & (0,0) & (0,1) \end{pmatrix} \in S_3(R)$, then $AB = 0$, but $AA\bar{\alpha}(B) = A \neq 0$. Thus $AS_3(R)\bar{\alpha}(B) \neq 0$, and therefore $S_3(R)$ is not strongly $\bar{\alpha}$-semicommutative.

However, we obtain that $S_3(R)$ is strongly $\bar{\alpha}$-semicommutative for a reduced α-semicommutative ring R by the similar method to the proof of Proposition 2.19 as follows:

Proposition 2.21. *Let R be a reduced ring. If R is α-semicommutative, then*

$$S_3(R) = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \mid a, b, c, d \in R \right\}$$

is strongly $\bar{\alpha}$-semicommutative.

Proof. For

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ 0 & a_1 & d_1 \\ 0 & 0 & a_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 & c_2 \\ 0 & a_2 & d_2 \\ 0 & 0 & a_2 \end{pmatrix} \in S_3(R),$$

we can denote their addition and multiplication by

$$(a_1, b_1, c_1, d_1) + (a_2, b_2, c_2, d_2) = (a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2),$$

$$(a_1, b_1, c_1, d_1)(a_2, b_2, c_2, d_2) = (a_1a_2, a_1b_2 + b_1a_2, a_1c_2 + b_1d_2 + c_1a_2, a_1d_2 + d_1a_2),$$

respectively. So every polynomial in $S_3[x]$ can be expressed in the form of (f_0, f_1, f_2, f_3) for some f_i's in $R[x]$. Let $f(x) = (f_0(x), f_1(x), f_2(x), f_3(x)), g(x) = (g_0(x), g_1(x), g_2(x), g_3(x)) \in S_3[x]$ with $f(x)g(x) = 0$. Then $f(x)g(x) = (f_0(x)g_0(x), f_0(x)g_1(x)+f_1(x)g_0(x), f_0(x)g_2(x)+f_1(x)g_3(x)+f_2(x)g_0(x), f_0(x)g_3(x)+f_3(x)g_0(x))$, we shall prove $f(x)S_3(R)[x]\alpha(g(x)) = 0$. So we have the following system of equations:

$$f_0(x)g_0(x) = 0, \tag{2.3}$$

$$f_0(x)g_1(x) + f_1(x)g_0(x) = 0, \tag{2.4}$$

$$f_0(x)g_2(x) + f_1(x)g_3(x) + f_2(x)g_0(x) = 0, \tag{2.5}$$

$$f_0(x)g_3(x) + f_3(x)g_0(x) = 0. \tag{2.6}$$

Use the fact that $R[x]$ is reduced. From Eq. (2.3), we get $g_0(x)f_0(x) = 0$. If we multiply Eq. (2.4), on the right side by $g_0(x)$, then $0 = (f_0(x)g_1(x) + f_1(x)g_0(x))g_0(x) = f_1(x)g_0^2(x)$, and so $f_1(x)g_0(x) = 0$ and $f_0(x)g_1(x) = 0$. Similarly, from Eq. (2.6), we have $f_3(x)g_0(x) = 0$, and $f_0(x)g_3(x) = 0$. Also, in Eq. (2.5), $0 = (f_0(x)g_2(x)+f_1(x)g_3(x)+f_2(x)g_0(x))g_0(x) = f_2(x)g_0^2(x)$ implies $f_2(x)g_0(x) = 0$ and

$$f_0(x)g_2(x) + f_1(x)g_3(x) = 0. \tag{2.7}$$

Multiplying (2.7) on left side by $f_0(x)$ gives $0 = f_0(x)(f_0(x)g_2(x)+f_1(x)g_3(x)) = f_0^2(x)g_2(x)$, and so $f_0(x)g_2(x) = 0$ hence $f_1(x)g_3(x) = 0$. Let

$$f(x) = \sum_{i=0}^{n} \begin{pmatrix} a_i & b_i & c_i \\ 0 & a_i & d_i \\ 0 & 0 & a_i \end{pmatrix} x^i, g(x) = \sum_{j=0}^{m} \begin{pmatrix} a'_j & b'_j & c'_j \\ 0 & a'_j & d'_j \\ 0 & 0 & a'_j \end{pmatrix} x^j$$

and $h(x) = \sum_{k=0}^{r} \begin{pmatrix} a''_k & b''_k & c''_k \\ 0 & a''_k & d''_k \\ 0 & 0 & a''_k \end{pmatrix} x^k \in S_3(R)$,
where $f_0(x) = \Sigma_{i=0}^{n} a_i x^i, f_1(x) = \Sigma_{i=0}^{n} b_i x^i, f_2(x) = \Sigma_{i=0}^{n} c_i x^i, f_3(x) = \Sigma_{i=0}^{n} d_i x^i, g_0(x) = \Sigma_{j=0}^{m} a'_j x^j, g_1(x) = \Sigma_{j=0}^{m} b'_j x^j, g_2(x) = \Sigma_{j=0}^{m} c'_j x^j, g_3(x) = \Sigma_{j=0}^{m} d'_j x^j$. Since every reduced ring is an Armendariz ring, we obtain that $a_i a'_j = 0, a_i b'_j = 0, b_i a'_j = 0, a_i c'_j = 0, b_i d'_j = 0, c_i a'_j = 0, a_i d'_j = 0, d_i a'_j = 0$, for all i, j by the preceding results. With these facts and the fact that R is α-semicommutative ring, we have $a_i a''_k \alpha(a'_j) = 0, a_i a''_k \alpha(b'_j) = 0, b_i a''_k \alpha(a'_j) = 0, b_i a''_k \alpha(d'_j) = 0, a_i a''_k \alpha(c'_j) = 0, a_i b''_k \alpha(d'_j) = 0, b_i a''_k \alpha(d'_j) = 0, a_i c''_k \alpha(a'_j) = 0, b_i d''_k \alpha(a'_j) = 0, c_i a''_k \alpha(a'_j) = 0, a_i a''_k \alpha(d'_j) = 0, a_i d''_k \alpha(a'_j) = 0, d_i a''_k \alpha(a'_j) = 0$. Consequently, we get the equation:

$$\begin{aligned} f(x)h(x)\alpha(g(x)) &= (f_0(x), f_1(x), f_2(x), f_3(x))S_3(R)[x]\alpha((g_0(x), g_1(x), g_2(x), g_3(x)) \\ &= (f_0(x)S_3(R)[x]\alpha(g_0(x)), f_0(x)S_3(R)[x]\alpha(g_1(x)) + f_1(x)S_3(R)[x]\alpha(g_0(x)), \\ &\quad f_0(x)S_3(R)[x]\alpha(g_2(x)) + f_1(x)S_3(R)[x]\alpha(g_3(x)) + f_2(x)S_3(R)[x]\alpha(g_0(x)), \\ &\quad f_0(x)S_3(R)[x]\alpha(g_3(x)) + f_3(x)S_3(R)[x]\alpha(g_0(x))) = 0. \end{aligned}$$

Therefore $S_3(R)$ is strongly $\bar{\alpha}$-semicommutative. □

Let R be a ring. Define a subring S_n of the n-by-n full matrix ring $M_n(R)$ over R as follows:

$$S_n(R) = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix} \mid a, a_{ij} \in R \right\}.$$

For an α-rigid ring R and $n \geq 2$, by Proposition 2.21, we may suspect that $S_n(R)$ may be strongly $\bar{\alpha}$-semicommutative ring for $n \geq 4$. But the possibility is eliminated by the next example.

Example 2.22. Let R be an α-rigid and

$$S_4 = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & a_{14} \\ 0 & a & a_{23} & a_{24} \\ 0 & 0 & a & a_{34} \\ 0 & 0 & 0 & a \end{pmatrix} \mid a, a_{ij} \in R \right\}.$$

Note that if R is an α-rigid ring, then $\alpha(e) = e$, for $e^2 = e \in R$ by [6, Proposition 5]. In particular $\alpha(1) = 1$. For $A = \begin{pmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in$

$S_4(R)$, we obtain $AB = 0$. But we have $0 \neq \begin{pmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = AC\bar{\alpha}(B) \in$ $S_4(R)$, for $C = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in S_4(R)$. Thus $AC\bar{\alpha}(B) \neq 0$ and so $S_4(R)$ is not strongly $\bar{\alpha}$-semicommutative. Similarly, it can be proved that $S_n(R)$ is not strongly $\bar{\alpha}$-semicommutative for $n \geq 5$.

Let R be a ring and let

$$V_n(R) = \left\{ S = \begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_{n-2} & a & b \\ 0 & a_1 & a_2 & \cdots & a_{n-3} & a_{n-2} & c \\ 0 & 0 & a_1 & \cdots & a_{n-4} & a_{n-3} & a_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_1 & a_2 & a_3 \\ 0 & 0 & 0 & \cdots & 0 & a_1 & a_2 \\ 0 & 0 & 0 & \cdots & 0 & 0 & a_1 \end{pmatrix} \mid a_i, a, b, c \in R \right\}.$$

Note that if $a = c$, then the matrix S is called an upper triangular Toeplitz matrix over R, see [15].

We proved in Proposition 2.21 and Example 2.22 that when R is a reduced ring and R is an α-semicommutative ring, then $S_3(R)$ is strongly $\bar{\alpha}$-semicommutative, but $S_n(R)$ is not strongly $\bar{\alpha}$-semicommutative for $n \geq 4$. In the next theorem we will show that a special subring $V_n(R)$ of $T_n(R)$ for any positive integer $n \geq 2$ is strongly $\bar{\alpha}$-semicommutative, where R is a reduced and α-semicommutativethe ring.

Theorem 2.23. *Let R be a reduced ring. If R is α-semicommutative, then $V_n(R)$ is strongly $\bar{\alpha}$-semicommutative.*

Proof. Suppose that

$$\begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_{n-2} & a_{1,n-1} & a_{1n} \\ 0 & a_1 & a_2 & \cdots & a_{n-3} & a_{n-2} & a_{2n} \\ 0 & 0 & a_1 & \cdots & a_{n-4} & a_{n-3} & a_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_1 & a_2 & a_3 \\ 0 & 0 & 0 & \cdots & 0 & a_1 & a_2 \\ 0 & 0 & 0 & \cdots & 0 & 0 & a_1 \end{pmatrix}, \begin{pmatrix} b_1 & b_2 & b_3 & \cdots & b_{n-2} & b_{1,n-1} & b_{1n} \\ 0 & b_1 & b_2 & \cdots & b_{n-3} & b_{n-2} & b_{2n} \\ 0 & 0 & b_1 & \cdots & b_{n-4} & b_{n-3} & b_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & b_1 & b_2 & b_3 \\ 0 & 0 & 0 & \cdots & 0 & b_1 & b_2 \\ 0 & 0 & 0 & \cdots & 0 & 0 & b_1 \end{pmatrix}$$

are in $V_n(R)$. So every polynomial in $V_n(R)[x]$ can be expressed in the form of $(f_1, f_2, \cdots, f_{n-2}, f_{1,n-1}, f_{1n}, f_{2n})$ for some f_i's in $R[x]$. Let $f(x) = (f_0(x), f_1(x), \cdots, f_{2n}(x)), g(x) = (g_0(x), g_1(x), \cdots, g_{2n}(x)) \in V_n(R)[x]$ with $f(x)g(x) = 0$. We shall prove $f(x)V_n(R)[x]\alpha(g(x)) = 0$. Now we have the following system of equations:

$$f_1(x)g_1(x) = 0, \tag{2.8}$$
$$f_1(x)g_2(x) + f_2(x)g_1(x) = 0, \tag{2.9}$$
$$f_1(x)g_3(x) + f_2(x)g_2(x) + f_3(x)g_1(x) = 0,$$
$$\vdots$$
$$f_1(x)g_{n-2}(x) + f_2(x)g_{n-3}(x) + \cdots + f_{n-2}(x)g_1(x) = 0,$$
$$f_1(x)g_{1,n-1}(x) + f_2(x)g_{n-2}(x) + \cdots + f_{n-2}(x)g_2(x) + f_{1,n-1}(x)g_1(x) = 0, \tag{2.10}$$
$$f_1(x)g_{1n}(x) + f_2(x)g_{2n}(x) + \cdots + f_{1,n-1}(x)g_2(x) + f_{1n}(x)g_1(x) = 0, \tag{2.11}$$
$$f_1(x)g_{2n}(x) + f_2(x)g_{n-2}(x) + \cdots + f_{n-2}(x)g_2(x) + f_{2n}(x)g_1(x) = 0. \tag{2.12}$$

Use the fact that $R[x]$ is reduced. From Eq. (2.8), we get $g_1(x)f_1(x) = 0$. If we multiply Eq. (2.9) on the right side by $f_1(x)$, then $f_1(x)g_2(x)f_1(x) + f_2(x)g_1(x)f_1(x) = 0$. Thus $f_1(x)g_2(x)f_1(x) = 0$ and hence $f_1(x)g_2(x) = 0$. From Eq. (2.9) it follows that $f_2(x)g_1(x) = 0$. Continuing in this manner, we can show that $f_i(x)g_j(x) = 0$ when $i + j = 2, \ldots, n-1$. Hence $g_j(x)f_i(x) = 0$. Multiplying Eq. (2.10) on the right side by $f_1(x)$, we obtain $0 = f_1(x)g_{1,n-1}(x)f_1(x) + f_2(x)g_{n-2}(x)f_1(x) + \cdots + f_{n-2}(x)g_2(x)f_1(x) + f_{1,n-1}(x)g_1(x)f_1(x) = f_1(x)g_{1,n-1}(x)f_1(x)$. Thus $f_1(x)g_{1,n-1}(x) = 0$. Hence

$$f_2(x)g_{n-2}(x) + \cdots + f_{n-2}(x)g_2(x) + f_{1,n-1}(x)g_1(x) = 0, \tag{2.13}$$

Multiplying Eq. (2.13) on the right side by $f_2(x)$, we obtain

$$\begin{aligned} 0 = {} & f_2(x)g_{n-2}(x)f_2(x) + \cdots + f_{n-2}(x)g_2(x)f_2(x) + f_{1,n-1}(x)g_1(x)f_2(x) \\ = {} & f_2(x)g_{n-2}(x)f_2(x). \end{aligned}$$

Thus $f_2(x)g_{n-2}(x) = 0$. Continuing in this manner, we can show that $f_i(x)g_j(x) = 0$ when $i + j = n$ and $f_1(x)g_{1,n-1}(x) = 0, f_{1,n-1}(x)g_1(x) = 0$. Similarly, from Eq. (2.12), it follows that $f_1(x)g_{2n}(x) = 0$ and $f_{2n}(x)g_1(x) = 0$. Now multiplying Eq. (2.11) on the right side by $f_1(x)$, we have

$0 = f_1(x)g_{1n}(x)f_1(x)+f_2(x)g_{2n}(x)f_1(x)+f_3(x)g_{n-2}(x)f_1(x)+\cdots+f_{n-2}(x)g_3(x)$ $f_1(x)+f_{1,n-1}(x)g_2(x)f_1(x)+f_{1n}(x)g_1(x)f_1(x) = f_1(x)g_{1n}(x)f_1(x)$. Thus $f_1(x)g_{1n}(x)$ $= 0$. Hence

$$f_2(x)g_{2n}(x) + f_3(x)g_{n-2}(x) + \cdots + f_{1,n-1}(x)g_2(x) + f_{1n}(x)g_1(x) = 0, \tag{2.14}$$

If we multiply Eq. (2.14) on the right side by $f_2(x)$, then $0 = f_2(x)g_{2n}(x)f_2(x) + f_3(x)g_{n-2}(x)f_2(x)+\cdots+f_{1,n-1}(x)g_2(x)f_2(x)+f_{1n}(x)g_1(x)f_2(x) = f_2(x)g_{2n}(x)f_2(x)$. Thus $f_2(x)g_{2n}(x) = 0$. Continuing in this manner, we can show that $f_i(x)g_j(x) = 0$

when $i+j=n+1, f_{1,n-1}(x)g_2(x)=0$ and $f_{1n}(x)g_1(x)=0$. Let

$$f(x)=\sum_{i=0}^{n}\begin{pmatrix} a_1^i & a_2^i & a_3^i & \cdots & a_{n-2}^i & a_{1,n-1}^i & a_{1n}^i \\ 0 & a_1^i & a_2^i & \cdots & a_{n-3}^i & a_{n-2}^i & a_{2n}^i \\ 0 & 0 & a_1^i & \cdots & a_{n-4}^i & a_{n-3}^i & a_{n-2}^i \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_1^i & a_2^i & a_3^i \\ 0 & 0 & 0 & \cdots & 0 & a_1^i & a_2^i \\ 0 & 0 & 0 & \cdots & 0 & 0 & a_1^i \end{pmatrix} x^i,$$

$$g(x)=\sum_{j=0}^{m}\begin{pmatrix} b_1^j & b_2^j & b_3^j & \cdots & b_{n-2}^j & b_{1,n-1}^j & b_{1n}^j \\ 0 & b_1^j & b_2^j & \cdots & b_{n-3}^j & b_{n-2}^j & b_{2n}^j \\ 0 & 0 & b_1^j & \cdots & b_{n-4}^j & b_{n-3}^j & b_{n-2}^j \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & b_1^j & b_2^j & b_3^j \\ 0 & 0 & 0 & \cdots & 0 & b_1^j & b_2^j \\ 0 & 0 & 0 & \cdots & 0 & 0 & b_1^j \end{pmatrix} x^j$$

and $h(x)=\sum_{k=0}^{r}\begin{pmatrix} c_1^k & c_2^k & c_3^k & \cdots & c_{n-2}^k & c_{1,n-1}^k & c_{1n}^k \\ 0 & c_1^k & c_2^k & \cdots & c_{n-3}^k & c_{n-2}^k & c_{2n}^k \\ 0 & 0 & c_1^k & \cdots & c_{n-4}^k & c_{n-3}^k & c_{n-2}^k \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & c_1^k & c_2^k & c_3^k \\ 0 & 0 & 0 & \cdots & 0 & c_1^k & c_2^k \\ 0 & 0 & 0 & \cdots & 0 & 0 & c_1^k \end{pmatrix} x^k \in V_n(R)[x]$,

where $f_1(x)=\Sigma_{i=0}^n a_1^i x^i, f_2(x)=\Sigma_{i=0}^n a_2^i x^i, \cdots, f_{n-2}(x)=\Sigma_{i=0}^n a_{n-2}^i x^i$, $f_{1,n-1}(x)=\Sigma_{i=0}^n a_{1,n-1}^i x^i, f_{1n}(x)=\Sigma_{i=0}^n a_{1n}^i x^i, f_{2n}(x)=\Sigma_{i=0}^n a_{2n}^i x^i, g_1(x)=\Sigma_{j=0}^m b_1^j x^j, g_2(x)=\Sigma_{j=0}^m b_2^j x^j, \cdots, g_{n-2}(x)=\Sigma_{j=0}^m b_{n-2}^j x^j, g_{1,n-1}(x)=\Sigma_{j=0}^m b_{1,n-1}^j x^j$, $g_{1n}(x)=\Sigma_{j=0}^m b_{1n}^j x^j, g_{2n}(x)=\Sigma_{j=0}^m b_{2n}^j x^j$. Since every reduced ring is an Armendariz ring, we obtain that $a_1^ib_1^j=0, a_1^ib_2^j=0, a_2^ib_1^j=0, a_1^ib_3^j=0, a_2^ib_2^j=0, a_3^ib_1^j=0, \cdots, a_1^ib_{n-2}^j=0, a_2^ib_{n-3}^j=0, \cdots, a_{n-2}^ib_1^j=0, a_1^ib_{1,n-1}^j=0, a_2^ib_{n-2}^j=0, \cdots, a_{n-2}^ib_2^j=0, a_{1,n-1}^ib_1^j=0, a_1^ib_{1n}^j=0, a_2^ib_{2n}^j=0, a_3^ib_{n-1}^j=0, \cdots, a_{n-2}^ib_3^j=0, a_{1,n-1}^ib_2^j=0, a_{1n}^ib_1^j=0, a_1^ib_{2n}^j=0, a_2^ib_{n-2}^j=0, \cdots, a_{n-2}^ib_2^j=0, a_{2n}^ib_1^j=0$ for all i,j by the preceding results. With these facts and the fact that R is α-semicommutative ring, we have $a_1^ic_1^k\alpha(b_1^j)=0$, $a_1^ic_1^k\alpha(b_2^j)=0$, $a_1^ic_2^k\alpha(b_1^j)=0$, $a_2^ic_1^k\alpha(b_1^j)=0$, $a_1^ic_1^k\alpha(b_3^j)=0, a_1^ic_2^k\alpha(b_2^j)=0, a_2^ic_1^k\alpha(b_2^j)=0$, $a_1^ic_2^k\alpha(b_1^j)=0, a_2^ic_2^k\alpha(b_1^j)=0, a_3^ic_1^k\alpha(b_1^j)=0, \cdots$, $a_1^ic_{2n}^k\alpha(b_1^j)=0$, $a_2^ic_{n-2}^k\alpha(b_1^j)=0, \cdots$, $a_{n-2}^ic_2^k\alpha(b_1^j)=0$, $a_{2n}^ic_1^k\alpha(b_1^j)=0$.

Therefore $V_n(R)$ is strongly $\bar{\alpha}$-semicommutative. □

The next result can be proved by using the technique used in the proof of [3, Proposition 2.6]. A ring is called *Abelian* if every idempotent is central. Reduced rings are clearly Abelian.

Proposition 2.24. *Let R be a strongly α-semicommutative ring. Then*

(1) $\alpha(1) = 1$, *where* 1 *is the identity of* R, *if and only if* $\alpha(e) = e$ *for any* $e^2 = e \in R$.

(2) *If* $\alpha(1) = 1$, *then* R *is Abelian.*

Let R be an algebra over a commutative ring S. Recall that the Dorroh extension of R by S is the ring $D = R\times S$ with operations $(r_1, s_1)+(r_2, s_2) = (r_1+r_2, s_1+s_2)$ and $(r_1, s_1)(r_2, s_2) = (r_1r_2 + s_1r_2 + s_2r_1, s_1s_2)$, where $r_i \in R$ and $s_i \in S$. For an endomorphism α of R, the S-endomorphism $\bar{\alpha}$ of D defined by $\bar{\alpha}(r, s) = (\alpha(r), s)$ is an S-algebra homomorphism.

Proposition 2.25. *If R is a strongly α-semicommutative ring with $\alpha(1) = 1$ and S is a domain, then the Dorroh extension D of R by S is strongly $\bar{\alpha}$-semicommutative.*

Proof. We apply the method in the proof of [3, Proposition 2.8.] Let $f(x) = (f_1(x), f_2(x)), g(x) = (g_1(x), g_2(x)) \in D(x)$ with $(f_1(x), f_2(x))(g_1(x), g_2(x)) = 0$. Then $f_1(x)g_1(x) + f_2(x)g_2(x) + g_2(x)f_1(x) = 0$ and $f_2(x)g_2(x) = 0$. Since S is a domain, we have $f_2(x) = 0$ or $g_2(x) = 0$. If $f_2(x) = 0$, then $0 = f_1(x)g_1(x) + f_2(x)g_2(x) + g_2(x)f_1(x) = f_1(x)g_1(x) + g_2(x)f_1(x)$ and so $f_1(x)(g_1(x) + g_2(x)) = 0$. Since R is strongly α-semicommutative with $\alpha(1) = 1$, $0 = f_1(x)t\alpha(g_1(x) + g_2(x)) = f_1(x)t\alpha(g_1(x)) + f_1(x)tg_2(x))$, for all $t \in R$. This yields $(f_1(x), f_2(x))(r, s)\bar{\alpha}(g_1(x), g_1(x)) = (f_1(x)r + sf_1(x))\alpha(g_1(x)) + (f_1(x)r + sf_1(x)g_2(x), 0) = 0$ for any $(r, s) \in D$, and hence $(f_1(x), f_2(x))D\bar{\alpha}(g_1(x), g_2(x)) = 0$. Now let $g_2(x) = 0$. Then $(f_1(x) + f_2(x))g_1(x) = 0$, and so $0 = (f_1(x) + f_2(x))R\alpha(g_1(x)) = 0$. We similarly obtain $(f_1(x), f_2(x))D\bar{\alpha}(g_1(x), g_2(x)) = 0$, and thus the Dorroh extension D is strongly $\bar{\alpha}$-semicommutative. $\square$

Corollary 2.26.([17, Proposition 3.17(2)]) *Let R be an algebra over a commutative domain S, and D be the Dorroh extension of R by S. Then R is strongly semicommutative if and only if D is strongly semicommutative.*

Note that the condition $\alpha(1) = 1$ in Proposition 2.25 cannot be dropped by the next example.

Example 2.27. Let $R = \mathbb{Z}_2 \bigoplus \mathbb{Z}_2$, and let $\alpha : R \to R$ defined by $\alpha((a, b)) = (0, b)$. Consider the Dorroh extension D of R by the ring of integers $\mathbb{Z}_2$. We clearly have $((1, 0), 0)((1, 0), -1) = 0$, but $((1, 0), 0)((1, 0), 0)\bar{\alpha}((1, 0), -1) = ((1, 0), -1) \neq 0$ in D. Thus D is not strongly $\bar{\alpha}$-semicommutative.

For an ideal I of R, if $\alpha(I) \subseteq I$, then $\bar{\alpha} : R/I \to R/I$ defined by $\bar{\alpha}(a + I) = \alpha(a) + I$ is an endomorphism of the factor ring R/I.

There exists a non-identity automorphism α of a ring R such that R/I is strongly $\bar{\alpha}$-semicommutative and I is strongly α-semicommutative for any nonzero proper ideal I of R, but R is not strongly α-semicommutative by the next example.

Example 2.28. Let F be a field. Consider the ring $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$ and an endomorphism α of R defined by $\alpha\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} a & -b \\ 0 & c \end{pmatrix}$. Then R is not strongly α-semicommutative. In fact, for $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix} \in R$, we have $AB = 0$, but $0 \neq A \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \alpha(B) \in AR\alpha(B)$. Note that for the only nonzero proper ideals of R

$$I = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}, J = \begin{pmatrix} 0 & F \\ 0 & F \end{pmatrix}, K = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix},$$

it can be easily checked that they are strongly α-semicommutative. Since $R/I \cong F$ and $R/J \cong F, R/I$ and R/J are also strongly $\bar{\alpha}$-semicommutative. Finally, the factor ring R/K is reduced and $\bar{\alpha}$ is an identity map on R/K. Thus, R/K is also strongly $\bar{\alpha}$-semicommutative.

Proposition 2.29. *Let R be a ring with an endomorphism α, and I an ideal of R with $\alpha(I) \subseteq I$. Suppose that R/I is a strongly $\bar{\alpha}$-semicommutative ring. If I is α-rigid as a ring without identity, then R is strongly α-semicommutative.*

Proof. Let $f(x)g(x) = 0$ with $f(x), g(x) \in R[x]$. Then we have $f(x)R\alpha(g(x)) \subseteq I[x]$ and $\alpha(g(x))I\alpha(f(x)) = 0$, since $\alpha(g(x))I\alpha(f(x)) \subseteq I[x], (\alpha((g(x)I\alpha(f(x)))^2 = 0$ and $I[x]$ is reduced. Thus, $(f(x)R\alpha(g(x))I)^2 = f(x)R\alpha(g(x))If(x)R\alpha(g(x))I = 0$ and so $f(x)R\alpha(g(x))I = 0$, thus $f(x)R\alpha(g(x))\alpha(f(x)R\alpha(g(x))) \subseteq f(x)R\alpha(g(x))I = 0$ since $f(x)R\alpha(g(x)) \subseteq I[x]$ and $\alpha(I) \subseteq I$. Then $f(x)R\alpha(g(x)) = 0$ as I is α-rigid. Therefore, R is strongly α-semicommutative. □

Theorem 2.30. *Let α be an endomorphism of a ring R. Then R is strongly α-semicommutative if and only if $R[x]$ is strongly α-semicommutative.*

Proof. $(\Leftarrow)$ The converse is obvious since R is a subring of $R[x]$.

$(\Rightarrow)$ Assume that R is strongly α-semicommutative. Let $f(y), g(y) \in R[x][y]$ such that $f(y)g(y) = 0$. Let

$$f(y) = f_0 + f_1 y + \cdots + f_m y^m, g(y) = g_0 + g_1 y + \cdots + g_n y^n,$$

and

$$h(y) = h_0 + h_1 y + \cdots + h_r y^r \in R[x][y].$$

We also let $f_i = a_{i_0} + a_{i_1}x + \cdots + a_{i_w}x^{i_w}, g_j = b_{j_0} + b_{j_1}x + \cdots + b_{j_v}x^{j_v}, h_k = c_{k_0} + c_{k_1}x + \cdots + c_{k_u}x^{k_u} \in R[x]$ for each $0 \leq i \leq m, 0 \leq j \leq n$ and $0 \leq k \leq r$, where $a_{i_0}, a_{i_1}, \cdots, a_{i_w}, b_{j_0}, b_{j_1}, \cdots, b_{j_v}, c_{k_0}, c_{k_1}, \cdots, c_{k_u} \in R$. We claim that $p(y)R[x]q(y) = 0$. Take a positive integer k such that $k \geq \max\{deg(f_i), deg(g_j), deg(h_k)\}$, for any $0 \leq i \leq m, 0 \leq j \leq n, 0 \leq k \leq r$, where the degree is as polynomials in $R[x]$ and the degree of the zero polynomial is taken to be 0. Let $f(x^s) = f_0 + f_1 x^s + \cdots + f_n x^{ms}, g(x^s) = g_0 + g_1 x^s + \cdots + g_n x^{ns}, h(x^s) =$

$h_0 + h_1x^s + \cdots + h_rx^{rs} \in R[x]$. Then the set of coefficients of the f_i's, g_j's (respectively, h_k's) is equal to the set of coefficients of $f(x^s), g(x^s)$ (respectively, $h(x^s)$). Since $f(y)g(y) = 0$, x commutes with elements of R in the polynomial ring $R[x]$, we have $f(x^s)g(x^s) = 0$, in $R[x]$. Since R is strongly α-semicommutative, we have $f(x^s)R\alpha(g(x^s)) = 0$. Hence $f(y)R[x]\alpha(g(y)) = 0$, therefore $R[x]$ is strongly α-semicommutative. □

Corollary 2.31. *Let R be a ring. Then R is strongly semicommutative if and only if $R[x]$ is strongly semicommutative.*

Corollary 2.32. *Let α be an endomorphism of a ring R. Then the following are equivalent:*

(1) R is strongly α-semicommutative.

(2) $R[x]$ is strongly α-semicommutative.

(3) $R[x; x^{-1}]$ is strongly α-semicommutative.

Let $A(R, \alpha)$ or A be the subset $\{x^{-i}a_ix^i | a \in R, i \geq 0\}$ of the skew Laurent polynomial ring $R[x, x^{-1}; \alpha]$, where $\alpha : R \to R$ is an injective ring endomorphism of a ring R (see [9] for more details). Elements of $R[x, x^{-1}; \alpha]$ are finite sums of elements of the form $x^{-i}b_jx^j$, where $b \in R$ and i, j are non-negative integers. Multiplication is subject to $xa = \alpha(a)x$ and $ax^{-1} = x^{-1}\alpha(a)$ for all $a \in R$. Note that for each $j \geq 0, x^{-i}a_ix^i = x^{-(i+j)}\alpha^j(a_i)x^{(i+j)}$. It follows that the set $A(R, \alpha)$ of all such elements forms a subring of $R[x, x^{-1}; \alpha]$ with

$$x^{-i}a_ix^i + x^{-j}b_jx^j = x^{-(i+j)}(\alpha^j(a_i) + \alpha^i(b_j))x^{(i+j)}$$

$$(x^{-i}a_ix^i)(x^{-j}b_jx^j) = x^{-(i+j)}(\alpha^j(a_i)\alpha^i(b_j))x^{(i+j)}$$

for $a, b \in R$ and $i, j \geq 0$. Note that α is actually an automorphism of $A(R, \alpha)$. Let $A(R, \alpha)$ be the ring defined above. Then for the endomorphism α in $A(R, \alpha)$, the map $A(R, \alpha)[t] \to A(R, \alpha)[t]$ defined by

$$\Sigma_{i=0}^m (x^{-i}a_ix^i)t^i \to \Sigma_{i=0}^m (x^{-i}\alpha(a_i)x^i)t^i$$

is an endomorphism of the polynomial ring $A(R, \alpha)[t]$.

Proposition 2.33. *Let $A(R, \alpha)$ be an Armendariz ring. If R is α-semicommutative, then $A(R, \alpha)$ is strongly α-semicommutative.*

Proof. Let $f(t) = \Sigma_{i=0}^m (x^{-i}a_ix^i)t^i, g(t) = \Sigma_{j=0}^n (x^{-j}b_jx^j)t^j \in A(R, \alpha)[t]$ with $f(t)g(t) = 0$. Since $A(R, \alpha)$ is Armendariz, we have $(x^{-i}a_ix^i)(x^{-j}b_jx^j) = 0$, and so $x^{-(i+j)}(\alpha^j(a_i)\alpha^i(b_j))x^{(i+j)} = 0$. This implies that $\alpha^j(a_i)\alpha^i(b_j) = 0$, and so $\alpha^{j+k}(a_i)\alpha^{i+k}(b_j) = 0$. Hence $\alpha^{j+k}(a_i)R\alpha^{i+k+1}(b_j) = 0$. Since R is α-semicommutative, for any $h(t) = \Sigma_{k=0}^{p}(x^{-k}c_kx^k)t^k \in A(R,\alpha)[t]$, we have

$$\begin{aligned} f(t)h(t)g(t) = & \ (\Sigma_{i=0}^{m}(x^{-i}a_ix^i)t^i)(\Sigma_{k=0}^{p}(x^{-k}c_kx^k)t^k)\alpha(\Sigma_{j=0}^{n}(x^{-j}b_jx^j)t^j) \\ = & \ (\Sigma_{i+k=0}^{m+p}(x^{-i}a_ix^i)(x^{-k}c_kx^k)t^{i+k})(\Sigma_{j=0}^{n}(x^{-j}\alpha(b_j)x^j)t^j) \\ = & \ (\Sigma_{i+k=0}^{m+p}(x^{-(i+k)}(\alpha^k(a_i)\alpha^i(c_k))x^{i+k})t^{i+k})(\Sigma_{j=0}^{n}(x^{-j}\alpha(b_j)x^j)t^j) \\ = & \ (\Sigma_{i+j+k=0}^{m+n+p}(x^{-(i+k)}(\alpha^k(a_i)\alpha^i(c_k))x^{i+k})(x^{-j}\alpha(b_j)x^j)t^{i+j+k} \\ = & \ (\Sigma_{i+j+k=0}^{m+n+p}(x^{-(i+j+k)}(\alpha^j(\alpha^k(a_i)\alpha^i(c_k))\alpha^{i+k})(\alpha(b_j))(x^{i+j+k})t^{i+j+k} \\ = & \ (\Sigma_{i+j+k=0}^{m+n+p}(x^{-(i+j+k)}(\alpha^{k+j}(a_i)\alpha^{i+j}(c_k)\alpha^{i+k+1}(b_j))(x^{i+j+k})t^{i+j+k}. \end{aligned}$$

As $(\alpha^{k+j}(a_i)\alpha^{i+j}(c_k)\alpha^{i+k+1}(b_j) = 0$, $f(t)h(t)\alpha(g(t)) = 0$. So $A(R,\alpha)$ is strongly α-semicommutative. □

Corollary 2.34. *Let $A(R,\alpha)$ be an Armendariz ring. If R is semicommutative, then $A(R,\alpha)$ is strongly semicommutative.*

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