On a Quasitoric Virtual Braid Presentation of a Virtual Link

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ABSTRACT. We introduce a quasitoric virtual braid and show that every virtual link can be obtained by the closure of a quasitoric virtual braid. Also, we show that the set of quasitoric virtual braids with n strands forms a group which is a subgroup of the n -virtual braid group.

1. Introduction

The notion of a quasitoric braid is introduced by Manturov[7] as a generalization of a toric braid which can be drawn on the standard torus, and he proved that every link can be presented as the closure of a quasitoric braid. By the virtue of Manturov's theorem, one can define the quasitoric braid index of a link L as the minimum number of strands of a quasitoric braid which presents the given link L . In [1], the authors gave a formula to calculate the quasitoric braid index.

The concept of a virtual link was proposed by Kauffman[4] as a generalization of the notion of a classical link. Kamada[3] proved that any virtual link can be presented by the closure of a virtual braid which is unique up to certain basic moves. The result is analogous to Alexander theorem and Markov theorem for classical braids and links. Kauffman and Lambropoulou[6] proved the same result by using L -moves.

In this paper, we introduce the notion of a quasitoric virtual braid, and prove that every virtual link can be presented as the closure of a quasitoric virtual braid.

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2. Virtual Links and Virtual Braids

A *virtual link diagram* is a link diagram in $\mathbb{R}^2$ possibly with encircled crossings without over/under information, see Figure 1. Such an encircled crossing is called a *virtual crossing*.

Figure 1: Virtual knot

Two virtual link diagrams are said to be *equivalent* if one can be transformed into another by applying finite times of the *generalized Reidemeister moves* which consist of classical Reidemeister moves R_1 , R_2 and R_3 and virtual Reidemeister moves V_{R_1} , V_{R_2} , V_{R_3} and the semi-virtual move V_{R_4} as shown in Figure 2. An equivalence class of a virtual link diagram is called a *virtual link*.

Figure 2: Generalized Reidemeister moves

A succinct depiction of virtual equivalence is that it is generated by classical Reidemeister moves and the detour move which is shown in Figure 3.

Figure 3: The detour move

One can define the *virtual braid group* $\mathbf{vB}_n$ as an extension of the classical braid group $\mathbf{B}_n$.

Definition 2.1.([3]) Let n be a positive integer and $P_n = \{x_i \mid x_1 < x_2 < \cdots < x_n\}$ a set of n interior points of the interval $[0, 1]$. Define E in $\mathbb{R}^2$ by $[0, 1] \times [0, 1]$ and $\pi_2 : E \rightarrow [0, 1]$ by the second factor projection. A *virtual braid diagram* with n strands is an immersed 1-manifold $b = a_1 \cup \cdots \cup a_n$ in E such that

- (1) $\partial b = P_n \times \{0, 1\} \subset E$,
- (2) for each $i = 1, \dots, n$, $\pi_2|_{a_i} : a_i \rightarrow [0, 1]$ is a homeomorphism,
- (3) the multiple point set $V(b)$ consists of transverse double points,
- (4) $\pi_2|_{V(b)} : V(b) \rightarrow [0, 1]$ is injective,
- (5) each point of $V(b)$ is assigned an information of classical crossing or virtual crossing as shown in Figure 4.

Figure 4: Crossings

Two virtual braid diagrams said to be *equivalent* if one is transformed to the other continuously keeping the above conditions. An equivalence class of a virtual braid diagram with n strands is called a *n -virtual braid*. The set of n -virtual braid, with the concatenation product, forms a monoid which is generated by σ_i, σ_i^{-1} and v_i , in Figure 5, for $i = 1, 2, \dots, n - 1$.

Figure 5: Generators

Definition 2.2.([3]) The virtual braid group $\mathbf{vB}_n$ is the group obtained from the monoid of n -virtual braids by introducing the following relations:

$$\begin{aligned}
 \sigma_i \sigma_i^{-1} &= \sigma_i^{-1} \sigma_i = 1, \quad v_i^2 = 1 \\
 \sigma_i \sigma_j &= \sigma_i \sigma_j, \quad v_i v_j = v_j v_i, \quad |i - j| \geq 2 \\
 \sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1} \\
 v_i v_{i+1} v_i &= v_{i+1} v_i v_{i+1} \\
 \sigma_i v_j &= v_j \sigma_i, \quad |i - j| \geq 2 \\
 v_i \sigma_{i+1} v_i &= v_{i+1} \sigma_i v_{i+1}
 \end{aligned}$$

In [5], Kauffman and Lambropoulou introduced the following relation,

$$\sigma_{i+1} = (v_i \cdots v_2 v_1)(v_{i+1} \cdots v_3 v_2) \sigma_1 (v_2 v_3 \cdots v_{i+1})(v_1 v_2 \cdots v_i)$$

for each $i = 1, \dots, n-2$. In terms of n -virtual braids, this relation is a consequence of the above defining relations and it called the *braid detour moves* of the strands $1, 2, \dots, i$, around the crossing σ_{i+1} . See Figure 6.

Figure 6: Detouring the crossing σ_{i+1}

Since the strands of an n -virtual braid β connect injectively the points in $\{p_i \mid p_i = (x_i, 1), x_i \in P_n\}$ to the points in $\{q_i \mid q_i = (x_i, 0), x_i \in P_n\}$ for an n -virtual braid β , there is an element $\pi(\beta)$ in the permutation group $\mathbf{S}_n$ which is uniquely determined by each strand. Indeed, if the k th strand of β connects p_i to q_{i_k} , ($k = 1, 2, \dots, n$), then the permutation $\pi(\beta)$ corresponding to β is

$\begin{pmatrix} 1 & 2 & \cdots & n \\ i_1 & i_2 & \cdots & i_n \end{pmatrix}$. For example, the permutation corresponding to β in Figure 7 (a) is $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}$.

Figure 7: A virtual braid and its closure

A virtual braid is said to be *pure* if its permutation is the identity permutation. The *closure* $\widehat{\beta}$ of a virtual braid β is the virtual link obtained by joining the points $p_1, p_2, \dots, p_n$ to the points $q_1, q_2, \dots, q_n$ by parallel curves as shown in Figure 7 (b).

Proposition 2.3 and Proposition 2.4 were proved by Kamada which are the analogues of Alexander theorem and Markov theorem respectively.

Proposition 2.3.([3]) *Virtual links can be represented by the closures of virtual braids.*

Proposition 2.4.([3]) *Two virtual braid diagrams have equivalent closures as virtual links if and only if they are related to each other by a finite sequence of the following VM moves (1) - (4).*

- (1) (VM0-move) a virtual braid move.
- (2) (VM1-move) a conjugation in the virtual braid group.
- (3) (VM2-move) a right stabilization of positive, negative or virtual crossing and its inverse operation, see Figure 8.
- (4) (VM3-move) a right/left virtual exchange moves, see Figure 9.

VM0-move, VM1-move and VM2-move are analogous to those in the classical case and the new move has two variants, the right one and the left one. It is remarkable that the right VM3-move is not consequences of the left VM3-move, vice versa. Two virtual braids are *VM-equivalent* if they are related by a finite sequence of VM moves which are mentioned in Proposition 2.4.

Figure 8: Right stabilizations/destabilizations

Figure 9: Right/left exchange moves

3. Quasitoric Virtual Braids

We introduce the definition of a *quasitoric virtual braid*.

Definition 3.1. Let p and q be positive integers with $p \geq 2$. A virtual braid β is said to be *quasitoric* of type (p, q) if it can be expressed as $\beta_1 \beta_2 \cdots \beta_q$ where for each $\beta_j = x_{p-1}^{e_{j,p-1}} \cdots x_1^{e_{j,1}}$ where $x_i \in \{\sigma_i, v_i\}$ and each $e_{j,i}$ is either 1 or -1 if $x_i = \sigma_i$ and $e_{j,i} = 1$ if $x_i = v_i$, see Figure 10. Shortly, a quasitoric braid of type (p, q) is called a *p-quasitoric virtual braid*.

Figure 10: Example of a quasitoric virtual braid

Lemma 3.2. Let β be a quasitoric virtual braid of type (p, q) . If β is a pure virtual braid, then $q = p \cdot k$ for some positive integer k .

Proof. Let $\beta = \beta_1 \beta_2 \cdots \beta_q$ where for each $\beta_j = x_{p-1}^{e_{j,p-1}} \cdots x_1^{e_{j,1}}$ where $x_i \in \{\sigma_i, v_i\}$ and each $e_{j,i}$ is either 1 or -1 if $x_i = \sigma_i$ and $e_{j,i} = 1$ if $x_i = v_i$. Then $\pi(\beta_j) = (12 \cdots p)$ for all j and the order of $(12 \cdots p)$ is p . Since β is a pure virtual braid, $\pi(\beta)$ is the identity permutation. Since $\pi(\beta) = \pi(\beta_1 \beta_2 \cdots \beta_q) = \pi(\beta_1) \pi(\beta_2) \cdots \pi(\beta_q) = (12 \cdots p)^q$, p divides q . Hence $q = p \cdot k$ for some positive integer k . $\square$

Also, we introduce the definition of a (i, j) -quasitoric virtual braid.

Definition 3.3. For positive integers i, j with $1 \leq i < j \leq p$, a p -strand virtual braid β is called a (i, j) -quasitoric virtual braid with p strands if it has a virtual braid diagram of the form in Figure 11, where γ is a $(j - i + 1)$ -quasitoric virtual braid.

Figure 11: (i, j) -quasitoric virtual braid with p strands

Note that if a (i, j) -quasitoric virtual braid β is pure, then the corresponding $(j - i + 1)$ -quasitoric virtual braid γ is a quasitoric pure virtual braid of type $(j - i + 1, (j - i + 1)k)$ for some positive integer k .

Theorem 3.4. For integer i, j with $1 \leq i < j \leq p$, every (i, j) -quasitoric pure virtual braid with p strands is quasitoric.

Proof. For a positive integer p with $p \geq 3$, it suffice to show that $(1, p - 1)$ -quasitoric pure virtual braid and $(2, p)$ -quasitoric pure virtual braid with p strands are quasitoric. Consider a $(p - 1)$ -quasitoric pure virtual braid diagram γ which lies in a $(1, p - 1)$ -quasitoric pure virtual braid diagram D with p strands. We slide the p th vertical strand into γ such that the p th strands is parallel with the $(p - 1)$ th strand by using detour moves as shown in Figure 12. No sooner than the sliding of the p th strand, we can pass the section of the p th strand with positive slope over the $(p - 1)$ th strand by using detour moves as shown in Figure 13. Then we obtain a p -quasitoric pure virtual braid diagram which is equivalent to the $(1, p - 1)$ -quasitoric pure virtual braid diagram with p strands. Similarly, we can make a p -quasitoric pure virtual braid diagram which is equivalent to the $(2, p)$ -quasitoric pure virtual

Figure 12: Slide the p th strand into γ Figure 13: the p th strand pass over the $(p-1)$ th strand

braid diagram with p strands. By using this process inductively, we have a quasitoric pure virtual braid diagram of a (i, j) -quasitoric pure virtual braid. $\square$

For example, we can make a quasitoric pure virtual braid diagram of a $(1, 4)$ -quasitoric pure virtual braid with 5 strands as shown in Figure 14.

Figure 14: The resulting virtual braid

In [2], Bardakov introduced a presentation of the pure virtual braid group $\mathbf{PvB}_n$ with n -strands as the following.

$$PvB_n = \langle R_{ij} \mid R_{ij}R_{ik}R_{jk} = R_{jk}R_{ik}R_{ij}, R_{ij}R_{kl} = R_{kl}R_{ij} \rangle,$$

where $1 \leq i \neq j \leq n$, the generators R_{ij} and R_{ji} are shown in Figure 15.

Figure 15: Generators R_{ij} and R_{ji}

Theorem 3.5. *Every pure virtual braid is quasitoric.*

Proof. It suffices to show that the generators R_{ij} and R_{ji} can be presented as a quasitoric virtual braid. R_{ij} and R_{ji} can be deformed to the diagrams in the right of Figure 15. Note that r_1 and t_1 in Figure 15 are quasitoric. Firstly, we will show that R_{ij} is quasitoric.

Figure 16: Make r_2 as a quasitoric virtual braid

Note that the braid word of r_2 is $\sigma_i v_{i+1} v_{i+2} \cdots v_{j-1}$. By using the following equivalence:

$$\begin{aligned} & \sigma_i v_{i+1} v_{i+2} \cdots v_{j-1} \\ \simeq & \sigma_i v_i v_{i+1} v_{i+2} \cdots v_{j-1} \\ \simeq & \sigma_i v_i (v_{j-1} \cdots v_{i-1} v_i)^{j-i}, \end{aligned}$$

as shown in Figure 16, we can make r_2 as the product of a $(i, i+1)$ -quasitoric pure virtual braid with $(j-i+1)$ strands and a quasitoric virtual braid. Since we already show that a (i, j) -quasitoric pure virtual braid is quasitoric in Theorem 3.4, we can present r_2 as a quasitoric virtual braid.

R_{ij}^{-1} is depicted in Figure 17 (a), and we can deform R_{ij}^{-1} to the diagram in the right of Figure 17 (b).

Figure 17: $R_{i,j}^{-1}$

Note that the braid word of s_2 is $v_i v_{i+1} v_{i+2} \cdots v_{j-1}$. In a similar manner of r_2 , we can make s_2 as a quasitoric virtual braid by using the following equivalence:

$$v_i v_{i+1} v_{i+2} \cdots v_{j-1} \simeq (v_{j-1} \cdots v_{i-1} v_i)^{j-i}.$$

Since s_2 is a quasitoric virtual braid, R_{ij}^{-1} can be presented as a (i, j) -quasitoric pure virtual braid. By Theorem 3.4, R_{ij}^{-1} is quasitoric. Similarly, we can show that R_{ji} and R_{ji}^{-1} are quasitoric also. Therefore every pure virtual braid is quasitoric. $\square$

In [3], Kamada showed that any virtual link can be presented by the closure of a virtual braid β . In order to obtain a quasitoric virtual braid presentation of β , we must transform β in a proper way.

Theorem 3.6. *Every virtual link can be obtained as the closure of a quasitoric virtual braid.*

Proof. For a given virtual link L , there exist a virtual braid β such that the closure of β is equivalent to L . Suppose that β is a virtual braid with n strands and $\pi(\beta)$ is the permutation of β . Now consider orbits of the action of $\pi(\beta)$ on the set $\{1, 2, \dots, n\}$.

Then these orbits contain different numbers of elements. Let us apply the VM2-move and the VM3-move for transforming these orbits. The VM2-move conjugates the braid. So, the number of elements in orbits does not change, but elements in orbits are permuted. Because, it conjugates the corresponding permutation. The VM3-move increases the number of strands by one, since it adds the element $n+1$ to the orbit which is containing the element n and does not change other orbits. Thus, by using the VM2-move and the VM3-move, one can re-enumerate elements in such a way that the smallest orbit by one. Moreover, repeating this many times, we can obtain a virtual braid τ with the same number of elements for all orbit. Now suppose that the number of strands of τ is km for positive integers k and m . By conjugating τ , we can get a virtual braid α such that the corresponding permutation of α is the r th power of the cyclic permutation $(12 \cdots km)$ for positive integer r . Then the resulting virtual braid α is VM-equivalent to β . Let $\kappa = \sigma_{km-1}\sigma_{km-2} \cdots \sigma_2\sigma_1$, then $\alpha\kappa^{-r}$ is a pure virtual braid. By using Theorem 3.5, $\alpha\kappa^{-r}$ is quasitoric and put $\alpha\kappa^{-r} = \delta$. Then we have

$$\alpha = \delta\kappa^r.$$

Note that the product of two n -quasitoric virtual braids is a n -quasitoric virtual braid. Since κ and δ are quasitoric, α is quasitoric. Therefore there exist a quasitoric virtual braid α such that α is quasitoric and VM-equivalent to β . It means that α is a quasitoric presentation of the given link L . $\square$

Now we will show that the set $\mathbf{QvB}_n$ of n -quasitoric virtual braids forms a group which is a subgroup of the n -virtual braid group.

Theorem 3.7. *Let $\mathbf{QvB}_n$ be the set of n -quasitoric virtual braids. Then $\mathbf{QvB}_n$ is a subgroup of the virtual braid group $\mathbf{vB}_n$.*

Proof. It suffices to show that the inverse of a n -quasitoric virtual braid in $\mathbf{vB}_n$ is quasitoric, for every positive integer n . Let $\beta = x_{n-1}^{e_{n-1}}x_{n-2}^{e_{n-2}} \cdots x_2^{e_2}x_1^{e_1}$ where $x_i \in \{\sigma_i, v_i\}$ and each e_i is either 1 or -1 if $x_i = \sigma_i$ and $e_i = 1$ if $x_i = v_i$ for positive integer n , we have to prove that there exist a quasitoric virtual braid δ such that $\beta \cdot \delta$ and $\delta \cdot \beta$ are the trivial. first put δ as $x_1^{d_1}x_2^{d_2} \cdots x_{n-2}^{d_{n-2}}x_{n-1}^{d_{n-1}}$ each $d_i = -e_i$ if $x_i = \sigma_i$ and $d_i = 1$ if $x_i = v_i$ then by using two relations $\sigma\sigma^{-1} = \sigma^{-1}\sigma = 1$ and $v_iv_i = 1$, we have the following:

$$x_{n-1}^{e_{n-1}}x_{n-2}^{e_{n-2}} \cdots x_2^{e_2}x_1^{e_1}x_1^{d_1}x_2^{d_2} \cdots x_{n-2}^{d_{n-2}}x_{n-1}^{d_{n-1}} = 1$$

and

$$x_1^{e_1}x_1^{d_1}x_2^{d_2} \cdots x_{n-2}^{d_{n-2}}x_{n-1}^{d_{n-1}}x_{n-1}^{e_{n-1}}x_{n-2}^{e_{n-2}} \cdots x_2^{e_2}x_1^{e_1} = 1.$$

Hence δ is the inverse of β , see Figure 18. We will show that δ is quasitoric. To begin with, we will product a trivial quasitoric virtual braid to the given inverse, as shown in Figure 19 (b). Then we can see that the resulting virtual braid is presented by a product of two virtual braids, see Figure 19 (b). Moreover, we can see that one of two braids is a pure virtual braid and the other is a quasitoric virtual braid. In Theorem 3.5, we show that every pure virtual braid is quasitoric, therefore we can

Figure 18: Generator and its inverse

Figure 19: Quasitoric presentation of δ

make δ as a quasitoric virtual braid. Since every quasitoric virtual braid is generated by β , we can show that the inverse of a given quasitoric virtual braid is quasitoric. $\square$

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