Some Symmetric Properties on $(LCS)_n$ -manifolds

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ABSTRACT. We analyze the $(LCS)_n$ -manifolds endowed with some symmetric properties, focusing on Ricci tensor and the 1-form γ . We study some properties of special Weakly Ricci-Symmetric $(LCS)_n$ -manifolds and also shown that Weakly ϕ -Ricci Symmetric $(LCS)_n$ -manifold is an η -Einstein manifold.

1. Introduction

The study of Riemannian symmetric manifolds began with the work of E. Cartan [2]. A Riemannian manifold (M^n, g) is said to be locally symmetric due to E. Cartan [2] if its curvature tensor R satisfies the relation $\nabla R = 0$, where ∇ denotes the operator of covariant differentiation with respect to the metric tensor g .

A non-flat Riemannian manifold (M^n, g) is called a weakly symmetric manifold if its curvature tensor R of type $(0, 4)$ satisfies the condition

$$\begin{aligned} (\nabla_W R)(X, Y, Z, U) &= A(W)R(X, Y, Z, U) + B(X)R(W, Y, Z, U) \\ &+ H(Y)R(X, W, Z, U) + D(Z)R(X, Y, W, U) \\ &+ E(U)R(X, Y, Z, W), \end{aligned}$$

for all vector fields $W, X, Y, Z, U \in \chi(M^n)$, where A, B, H, D and E are 1-forms (not simultaneously zero) and ∇ denotes the operator of covariant differentiation with respect to the Riemannian metric g . The 1-forms are called the associated 1-forms of the manifold and an n -dimensional manifold of this kind is denoted by $(WS)_n$.

In 1989, Tamassy and Binh [13, 14] introduced the notion of weakly symmetric and weakly Ricci-symmetric Riemannian manifolds and studied such structures on

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Sasakian manifolds and proved that such a structure does not exist always. Weakly symmetric and weakly Ricci-symmetric structures are also studied by Shaikh and Jana [10, 11]. A Riemannian manifold (M^n, g) is called weakly Ricci symmetric if its Ricci tensor S of type $(0, 2)$ is not identically zero and satisfies the condition:

$$(\nabla_X S)(Y, Z) = A(X)S(Y, Z) + B(Y)S(Z, X) + D(Z)S(Y, X),$$

where A , B and D are 1-forms (not simultaneously zero). Such an n -dimensional manifold is denoted by $(WRS)_n$.

The above relation can be written as

$$(\nabla_X Q)(Y) = A(X)Q(Y) + B(Y)Q(X) + S(Y, X)\sigma,$$

where σ is the vector field associated to the 1-form D such that $D(Z) = g(Z, \sigma)$ and Q is the Ricci operator, i.e., $g(QX, Y) = S(X, Y)$ for all X, Y .

In this paper, we study some symmetric properties of $(LCS)_n$ manifolds by presenting two new sections, preceded by a preliminaries section containing some background on $(LCS)_n$ manifolds.

The first one is devoted to the study of special weakly Ricci-symmetric $(LCS)_n$ manifolds. We begin by studying special Weakly Ricci-Symmetric $(LCS)_n$ -manifolds with a cyclic parallel Ricci tensor, in this case the 1-form γ must vanish and also we show that if a special weakly Ricci-symmetric $(LCS)_n$ -manifolds does not satisfy the condition of Einstein manifold, then the 1-form γ is non zero. Further it is proved that the Ricci tensor is parallel in a special weakly Ricci-symmetric $(LCS)_n$ -manifold.

Finally, in the last section we study weakly ϕ -Ricci symmetric $(LCS)_n$ manifold and proved that it is an η -Einstein manifold. Moreover we have seen that ϕ -Ricci symmetric $(LCS)_n$ -manifold is an Einstein manifold.

2. Preliminaries

In 2003, A.A. Shaikh [9] introduced the notion of Lorentzian concircular structure manifolds (briefly $(LCS)_n$ -manifolds). An n -dimensional Lorentzian manifold M^n is a smooth connected paracompact Hausdorff manifold with a Lorentzian metric g , that is, M^n admits a smooth symmetric tensor field g of type $(0, 2)$ such that for each point $p \in M$, the tensor $g_p : T_p M^n \times T_p M^n \rightarrow R$ is a non-degenerate inner product of signature $(-, +, \dots, +)$, where $T_p M^n$ denotes the tangent vector space of M^n at p and R is the real number space. A non-zero vector $v \in T_p M^n$ is said to be timelike if it satisfies $g_p(v, v) < 0$.

Definition 2.1. In a Lorentzian manifold (M^n, g) a vector field P defined by

$$g(X, P) = A(X),$$

for any $X \in \chi(M^n)$ is said to be a concircular vector field

$$(\nabla_X A)(Y) = \alpha(g(X, Y) + \omega(X)A(Y)),$$

where α is a non zero scalar and ω is a closed 1-form.

Let M^n be an n -dimensional Lorentzian manifold admitting a unit timelike concircular vector field ξ , called the characteristic vector field of the manifold. Then we have

$$(2.1) \quad g(\xi, \xi) = -1.$$

Since ξ is a unit concircular vector field, there exists a non-zero 1-form η such that

$$(2.2) \quad g(X, \xi) = \eta(X).$$

The equation of the following form holds

$$(2.3) \quad (\nabla_X \eta)(Y) = \alpha[g(X, Y) + \eta(X)\eta(Y)], \quad (\alpha \neq 0)$$

for all vector fields X, Y , where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g and α is a non-zero scalar function satisfying

$$(2.4) \quad \nabla_X \alpha = (X\alpha) = d\alpha(X) = \rho\eta(X),$$

ρ being a certain scalar function given by $\rho = -(\xi\alpha)$. If we put

$$(2.5) \quad \phi X = \frac{1}{\alpha} \nabla_X \xi,$$

then from (2.3) and (2.4), we have

$$(2.6) \quad \phi X = X + \eta(X)\xi,$$

from which it follows that ϕ is a symmetric $(1, 1)$ tensor. Thus the Lorentzian manifold M together with the unit timelike concircular vector field ξ , its associated 1-form η and $(1, 1)$ tensor field ϕ is said to be a Lorentzian concircular structure manifold (briefly $(LCS)_n$ -manifold)[9]. Especially, if we take $\alpha = 1$, then we can obtain the Lorentzian para-Sasakian structure of Matsumoto [6]. In a $(LCS)_n$ -manifold, the following relations hold:

$$(2.7) \quad \eta(\xi) = -1, \quad \phi\xi = 0, \quad \eta(\phi X) = 0,$$

$$(2.8) \quad g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y),$$

$$(2.9) \quad R(X, Y)Z = (\alpha^2 - \rho)[g(Y, Z)X - g(X, Z)Y],$$

$$(2.10) \quad \eta(R(X, Y)Z) = (\alpha^2 - \rho)[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)],$$

$$(2.11) \quad (\nabla_X \phi)(Y) = \alpha[g(X, Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X],$$

$$(2.12) \quad S(X, \xi) = (n-1)(\alpha^2 - \rho)\eta(X),$$

$$(2.13) \quad S(\phi X, \phi Y) = S(X, Y) + (n-1)(\alpha^2 - \rho)\eta(X)\eta(Y),$$

$$(2.14) \quad Q\xi = (n-1)(\alpha^2 - \rho)\xi,$$

for any vector fields X, Y, Z , where R, S denotes respectively the curvature tensor and the Ricci tensor of the manifold.

Definition 2.2. An $(LCS)_n$ -manifold M^n is said to be η -Einstein if its Ricci tensor S is of the form

$$(2.15) \quad S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y),$$

for any vector fields X and Y , where a and b are some functions.

3. On Special Weakly Ricci-Symmetric $(LCS)_n$ -manifolds

An n -dimensional Riemannian manifold (M^n, g) is called a special Weakly Ricci-Symmetric manifold $(SWRS)_n$ if

$$(3.1) \quad (\nabla_X S)(Y, Z) = 2\gamma(X)S(Y, Z) + \gamma(Y)S(X, Z) + \gamma(Z)S(Y, X),$$

for any vector fields X, Y on M^n , where γ is a 1-form defined by

$$(3.2) \quad \gamma(X) = g(X, W),$$

where W is the associated vector field.

Theorem 3.1. *In a special weakly Ricci-symmetric $(LCS)_n$ -manifold, the Ricci tensor is parallel.*

Proof. Putting $Z = \xi$ in (3.1), we get

$$(3.3) \quad (\nabla_X S)(Y, \xi) = 2\gamma(X)S(Y, \xi) + \gamma(Y)S(X, \xi) + \gamma(\xi)S(Y, X).$$

The left hand side of above equation can be written as

$$(3.4) \quad (\nabla_X S)(Y, \xi) = \nabla_X S(Y, \xi) - S(\nabla_X Y, \xi) - S(Y, \nabla_X \xi).$$

By using (2.13), (3.2) and (3.4), the equation (3.3) can be written as

$$(3.5) \quad \begin{aligned} \nabla_X S(Y, \xi) - S(\nabla_X Y, \xi) - S(Y, \nabla_X \xi) \\ = 2(n-1)(\alpha^2 - \rho)\gamma(X)\eta(Y) + (n-1)(\alpha^2 - \rho)\gamma(Y)\eta(X) + \eta(W)S(Y, X). \end{aligned}$$

Taking $Y = \xi$ in (3.5) and using (2.10), (2.12), (2.13) and (3.2), we get

$$(3.6) \quad \begin{aligned} -2(n-1)(\alpha^2 - \rho)\gamma(X) &+ (n-1)(\alpha^2 - \rho)\eta(W)\eta(X) \\ &+ (n-1)(\alpha^2 - \rho)\eta(W)\eta(X) = 0. \end{aligned}$$

Putting $X = \xi$ in (3.6), we obtain

$$(3.7) \quad \eta(W) = 0.$$

Using (3.7) in (3.6) gives

$$(3.8) \quad \gamma(X) = 0,$$

for any vector fields X on M^n .

Hence in view of (3.8), we obtain from (3.1) that

$$\nabla_X S = 0. \quad \square$$

Theorem 3.2. *Let (M^n, g) be a special Weakly Ricci-Symmetric $(LCS)_n$ -manifold with a cyclic parallel Ricci tensor. Then the 1-form γ must vanish.*

Proof. Taking the cyclic sum of (3.1), we get

$$(3.9) \quad (\nabla_X S)(Y, Z) + (\nabla_Y S)(Z, X) + (\nabla_Z S)(X, Y) \\ = 4(\gamma(X)S(Y, Z) + \gamma(Y)S(X, Z) + \gamma(Z)S(X, Y)).$$

Let M^n admit a cyclic parallel Ricci tensor. Then (3.9) reduces to

$$(3.10) \quad \gamma(X)S(Y, Z) + \gamma(Y)S(X, Z) + \gamma(Z)S(X, Y) = 0.$$

Taking $Z = \xi$ in (3.10) and using (2.7) and (3.2), we get

$$(3.11) \quad (n-1)(\alpha^2 - \rho)\gamma(X)\eta(Y) + (n-1)(\alpha^2 - \rho)\gamma(Y)\eta(X) \\ + \eta(W)S(X, Y) = 0.$$

Now putting $Y = \xi$ in (3.11) and using (2.7), (2.12) and (3.2), we get

$$(3.12) \quad -(n-1)(\alpha^2 - \rho)\gamma(X) + (n-1)(\alpha^2 - \rho)\eta(W)\eta(X) \\ + (n-1)(\alpha^2 - \rho)\eta(W)\eta(X) = 0.$$

Further, taking $X = \xi$ in (3.12) and using (2.7) and (3.2), we obtain

$$(3.13) \quad \eta(W) = 0.$$

So by the use of (3.13) in (3.12), we have $\gamma(X) = 0$, for any vector field X on M^n . This completes the theorem. $\square$

Theorem 3.3. *If a special Weakly Ricci-symmetric $(LCS)_n$ -manifold is not an Einstein manifold, then 1-form $\gamma \neq 0$.*

Proof. For an Einstein manifold $(\nabla_X S)(Y, Z) = 0$ and $S(Y, Z) = kg(Y, Z)$, By (3.1), we have

$$(3.14) \quad 2\gamma(X)S(Y, Z) + \gamma(Y)S(X, Z) + \gamma(Z)S(Y, X) = 0.$$

Taking $Z = \xi$ in (3.14) and using (2.13) and (3.2), we get

$$(3.15) \quad 2(n-1)(\alpha^2 - \rho)\gamma(X)\eta(Y) + (n-1)(\alpha^2 - \rho)\eta(X)\gamma(Y) \\ + \eta(W)S(Y, X) = 0.$$

Taking $X = \xi$ in (3.15) and using (2.12) (2.13) and (3.2), we get

$$(3.16) \quad \begin{aligned} 2(n-1)(\alpha^2 - \rho)\eta(W)\eta(Y) &= (n-1)(\alpha^2 - \rho)\gamma(Y) \\ &+ (n-1)(\alpha^2 - \rho)\eta(W)\eta(Y) = 0. \end{aligned}$$

Again taking $Y = \xi$ in (3.16) and by virtue of (2.12) and (3.2), we get

$$(3.17) \quad \eta(W) = 0.$$

Using (3.17) in (3.16), we get

$$\gamma(Y) = 0,$$

for any vector field Y on M^n . □

4. Weakly ϕ -Ricci Symmetric $(LCS)_n$ -manifolds

Definition 4.3. A $(LCS)_n$ -manifold is said to be weakly ϕ -Ricci Symmetric if the Ricci operator satisfies

$$(4.1) \quad \phi^2((\nabla_X Q)(Y)) = A(X)\phi^2(QY) + B(Y)\phi^2(QX) + S(Y, X)\phi^2(\sigma).$$

Especially, if the 1-forms $A = B = D = 0$, then (4.1) turns into the notion of ϕ -Ricci symmetric introduced by Shukla and Shukla [12].

Let us take $(LCS)_n$ -manifold, which is weakly ϕ -Ricci symmetric. Then from (2.8), equation (4.1) becomes

$$\begin{aligned} (\nabla_X Q)(Y) + \eta((\nabla_X Q)(Y))\xi &= A(X)[QY + \eta(QY)\xi] \\ &+ B(Y)[QX + \eta(QX)\xi] + S(Y, X)[\sigma + \eta(\sigma)\xi], \end{aligned}$$

from which it follows that

$$(4.2) \quad \begin{aligned} g(\nabla_X Q(Y), Z) - S(\nabla_X Y, Z) + \eta((\nabla_X Q)(Y))\eta(Z) &= A(X)[S(Y, Z) \\ &+ \eta(QY)\eta(Z)] + B(Y)[S(X, Z) + \eta(QX)\eta(Z)] + S(Y, X)[\sigma + \eta(\sigma)\xi]. \end{aligned}$$

Putting $Y = \xi$ in (4.2) and using (2.9), (2.12) and (2.14), we get

$$(4.3) \quad \begin{aligned} (\alpha + B(\xi))S(X, Z) &= -(n-1)(\alpha^2 - \rho)[- \alpha g(X, Z) \\ &+ \eta(X)D(Z) + (B(\xi) + \eta(\sigma))\eta(X)\eta(Z)]. \end{aligned}$$

Replacing X by ϕX and Z by ϕZ in (4.3), we have

$$(4.4) \quad [\alpha + B(\xi)]S(\phi X, \phi Z) = \alpha(n-1)(\alpha^2 - \rho)g(\phi X, \phi Z).$$

By virtue of (2.8) and (2.13), equation (4.4) becomes

$$(4.5) \quad S(X, Z) = kg(X, Z) + l\eta(X)\eta(Z),$$

where

$$k = \frac{\alpha(n-1)(\alpha^2 - \rho)}{\alpha + B(\xi)} \quad \text{and} \quad l = \frac{(n-1)(\alpha^2 - \rho)B(\xi)}{\alpha + B(\xi)},$$

provided $\alpha + B(\xi) \neq 0$.

Hence we can state,

Theorem 4.4. *A weakly ϕ -Ricci symmetric $(LCS)_n$ -manifolds is an η -Einstein manifold.*

From Theorem 4.4., it can be easily seen that

Corollary 4.1. *A ϕ -Ricci symmetric $(LCS)_n$ -manifold is an Einstein manifold.*

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