KYUNGPOOK Math J. 43(2003), 149-156

On One-point Connectifications of Spaces

Janusz J. Charatonik
Mathematical Institute, University of Wrocław, pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland

Instituto de Matemáticas, Universidad Nacional Autónoma de México, Circuito Exterior, Ciudad Universitaria, 04510 México, D. F., México
e-mail : jjc@matem.unam.mx

Abstract. A connected Hausdorff space Y is called a one-point connectification of a space X if Y contains a copy of X as a dense subspace and $Y \setminus X$ has exactly one point. A generalized linear graph means a connected subset of a linear graph. In a previous paper the subspaces of generalized graphs which have a one-point connectification are characterized by some conditions. In this note relations between these conditions are analyzed if X is embedded in a space belonging to a wider class than one of generalized graphs.

Embedding of a space X as a dense subspace of a space Y which satisfies some special conditions is an old and well known topological construction. The best known and the most important one is the concept of a compactification. The reader is referred to [6, Sections 3.5 and 3.6, p. 166-182] for information about this. Dense embeddings in complete spaces, called completions, are also known, see e.g. [4]. During the last decade a number of papers appeared devoted to connectifications, see [1], [7], [8], [9], [15], [19]. In particular [7] contains characterizations of subsets of the real line which admit a connectification with a one-point set as the remainder. In [2] these results have been extended replacing the real line by an arbitrary generalized linear graph (not necessarily metric). In the present note we provide further investigations in the area.

We will use the standard notation of $\operatorname{card} A$, $\operatorname{cl}_X A$, $\operatorname{bd}_X A$ and $\operatorname{int}_X A$ for the cardinality, the closure, the boundary and the interior of a subset A of a space X. The symbol $\mathbb{N}$ stands for the set of positive integers. The term *continuum* means a compact connected Hausdorff space. A 1-dimensional continuum is called a *curve*. A space is called a *generalized continuum* if it is locally compact, connected and Hausdorff. We say that a property holds for *almost all* points of a set if it holds for

Received August 14, 2001.
2000 Mathematics Subject Classification: 54C25, 54D05, 54F50.
Key words and phrases: component, connected, connectification, embedding, generalized linear graph, non-compact.

all save a finite number of points of the set.

A concept of an *order* of a point p in a space X is used in the sense of Menger-Urysohn, written $\operatorname{ord}(p, X)$. Since the original definition of this concept is formulated for metric spaces, compare e.g. [12, §51, I, p. 274], we recall its definition for the reader convenience in a general setting (as it is used in this paper). Let $\mathfrak{n}$ stand for a cardinal number. We write:

- $\operatorname{ord}(p, X) \le \mathfrak{n}$ provided that for each open neighborhood W of p there is an open neighborhood U of p such that $U \subset W$ and $\operatorname{card} \operatorname{bd}_X U \le \mathfrak{n}$;
- $\operatorname{ord}(p, X) = \mathfrak{n}$ provided that $\operatorname{ord}(p, X) \le \mathfrak{n}$ and for each cardinal number $\mathfrak{m} < \mathfrak{n}$ the condition $\operatorname{ord}(p, X) \le \mathfrak{m}$ does not hold;
- $\operatorname{ord}(p, X) = \omega$ provided that for each open neighborhood W of p there is an open neighborhood U of p such that $U \subset W$, the boundary $\operatorname{bd}_X U$ is finite and, for various U, the cardinalities $\operatorname{card} \operatorname{bd}_X U$ are not bounded by any $n \in \mathbb{N}$.

Points of order 1 are called *end points* of the space X, and points of order $\mathfrak{n} \ge 3$ are called *ramification points* of X.

A continuum Y is said to be a) *rational* or b) *regular* (in the sense of theory of order) if a) $\operatorname{ord}(y, Y) \le \aleph_0$ or b) $\operatorname{ord}(y, Y)$ is finite, respectively, for all $y \in Y$, see [12, §51, I, p. 275].

An *arc* is defined as a continuum A having exactly two points which do not separate A, called the *end points* of the arc; in other words, A is an arc with end points a and b if each point $x \in A \setminus \{a, b\}$ separates a and b in A. For a number $k \in \mathbb{N}$ the union of k arcs emanating from a single point v and otherwise disjoint from one another is called a *k-od*. Then v is called the *vertex* of the k-od. By a *free arc* in a space X we mean an arc ab in X such that the set $ab \setminus \{a, b\}$ is open in X. Recall that a set is called a *linear graph* (see [20, Chapter 10, p. 182]) provided that it is the union of a finite set V of points, called *vertices*, and a finite number of free arcs, called *edges*, such that the two end points of each edge are distinct and belong to V. Thus each linear graph, if connected, is a continuum. The following characterization of linear graphs among continua is known (see [20, Chapter 10, 1, (2), p. 182]).

Proposition 1. *A continuum is a linear graph if and only if each of its points is of some finite order and almost all of its points are of order less than or equal to* 2.

This implies that for every connected subset B of a linear graph G the boundary of B (with respect to G) is finite.

A space Z is called a *generalized linear graph* if it is connected and if it can be embedded in a linear graph, i.e., if there exists a linear graph G and an embedding $h : Z \to G$ of Z into G. In other words, a space is a generalized linear graph if it is homeomorphic to a connected subset of a linear graph. Metric generalized linear graphs are characterized by conditions (i)-(v) of [3, Theorem 1, p. 337]. It can be observed from the proof of this result given in [3] that metrizability is not used to show the equivalence of conditions (i)-(iii). Thus the following is a consequence of [3, Theorem 1, p. 337] and of the above observation.

Proposition 2. *The following conditions are equivalent if Z is a connected space:*

(2.1) *Z is a generalized linear graph;*

(2.2) *Z can be embedded in a linear graph G_1 by an embedding $h_1 : Z \to G_1$ in such a way that the difference $G_1 \setminus h_1(Z)$ is a compact set;*

(2.3) *Z can be embedded in a linear graph G_2 by an embedding $h_2 : Z \to G_2$ in such a way that $h_2(Z)$ is a dense subset of G_2 and the remainder $G_2 \setminus h_2(Z)$ is a finite set of end points of G_2.*

As an immediate consequence of Proposition 2 we get the following characterizations of noncompact generalized linear graphs:

Proposition 3. *The following conditions are equivalent if Z is a connected space:*

(3.1) *Z is a noncompact generalized linear graph;*

(3.2) *Z can be embedded in a linear graph G_1 by an embedding $h_1 : Z \to G_1$ in such a way that the difference $G_1 \setminus h_1(Z)$ is a nonempty compact set;*

(3.3) *Z can be embedded in a linear graph G_2 by an embedding $h_2 : Z \to G_2$ in such a way that $h_2(Z)$ is a dense subset of G_2 and the remainder $G_2 \setminus h_2(Z)$ is a finite nonempty set of end points of G_2.*

It follows that the real line can be seen as a generalized linear graph.

A (topological) space X is said to be *connectifiable* provided that it can be embedded in a connected Hausdorff space Y as a dense subset; then Y is called a *connectification* of X, and the difference $Y \setminus X$ is called the *remainder* of the connectification. In case when $\operatorname{card}(Y \setminus X) = 1$, i.e., when the remainder of the connectification is a one-point set, the term of a *one-point connectification* is used. Similarly, a space X is said to be *pathwise connectifiable* provided that X can be densely embedded in a pathwise connected Hausdorff space Y, and then Y is called a *pathwise connectification* of X. A pathwise connected Hausdorff space Y is called a *one-point pathwise connectification* of X if it is a pathwise connectification of X with $\operatorname{card}(Y \setminus X) = 1$.

Note that a subspace of a generalized linear graph is connected if and only if it is pathwise connected. Thus it is natural to ask if a subspace of a generalized linear graph is connectifiable if and only if it is pathwise connectifiable. In general the equivalence is not true (see [8, Example 2.4, p. 17] and compare [7, p. 678]), but it is shown to be true for one-point connectifications of the real line, see [7, Theorem, p. 678]. This result has been extended to the one-point connectifications of generalized linear graphs in [2] as follows.

Theorem 4. *Let X be a subspace of a generalized linear graph Z. Then the following conditions are equivalent:*

(4.1) X has a one-point connectification;

(4.2) X has a one-point pathwise connectification;

(4.3) each component of X is open and non-compact;

(4.4) X is locally connected and each component of X is non-compact.

Questions are asked in [2, Question 6] about a possibility to weaken the assumption in Theorem 4 concerning the space Z (of being a generalized linear graph) and about particular implications between conditions (4.1)-(4.4). The aim of this note is to discuss these problems. In the results presented below we consider *all* possible implications between the conditions under various assumptions concerning Z.

We start with recalling some two examples mentioned in Remark 5 of [2], to indicate their new properties needed in the analysis of interrelations between the considered conditions.

Example 5. *The Knaster-Kuratowski biconnected space.* Let Y denote the standard Cantor fan in the plane $\mathbb{R}^2$ (i.e., the cone with the vertex $v = (\frac{1}{2}, \frac{1}{2})$ over the Cantor ternary set located on the segment $[0,1] \times \{0\}$), and let F be the Knaster-Kuratowski biconnected space (see e.g. [12, §46, II, Remark, p. 135] or [17, Example 129, p. 145]; the space is called the Knaster-Kuratowski fan in [6, 6.3.23, p. 380]; more about biconnected spaces in [5] and [18]; compare also [16]). Define $X = F \setminus \{v\}$. Then:

(5.1) $X \subset F \subset Y$;

(5.2) F is a one-point connectification of X;

(5.3) Y is a pathwise connectification of X;

(5.4) Y is a 1-dimensional continuum;

(5.5) X has no one-point pathwise connectification (see [7, Example 1, p. 679]);

(5.6) X is locally conneted at no of its points;

(5.7) each component of X is a one-point set (see [12, §46, II, Remark, p. 135]; note that there are biconnected spaces that do not have this property, i.e., biconnected spaces without dispersion points: see [13, Theorem 8, p. 128] or [17, Example 131, p. 148]).

Example 6. *The topologists sine-curve.* Let $A = \{(x, \sin \frac{\pi}{x}) : x \in (0,1]\}$, $X = A \cup \{(0,0)\}$, $p = (0,y)$ with $0 \neq y \in [-1,1]$, and $Y = \mathrm{cl}_{\mathbb{R}^2} A$. Then:

(6.1) Y is the well known $\sin(1/x)$-curve;

(6.2) $X \cup \{p\}$ is a one-point connectification of X;

(6.3) X has no pathwise connectification at all (see [7, Remark 2, p. 680]);

(6.4) X is not locally connected at the point $(0,0)$.

Example 7. *The harmonic fan.* In the plane $\mathbb{R}^2$ let $H = \{(0,0)\} \cup \{(\frac{1}{n}, 0) : n \in \mathbb{N}\}$ and let Y be the cone over H with the vertex $v = (0,1)$. Put $X = Y \setminus \{v\}$. Denote by K the limit component of X, i.e., $K = \{0\} \times [0,1)$. Then:

(7.1) Y is the homeomorphic to the well known harmonic fan, so it is a rational plane curve;

(7.2) Y is a one-point (pathwise) connectification of X;

(7.3) the limit component K is not open in X;

(7.3) X is locally connected at no point of the limit component K.

Statement 8. *If a generalized linear graph Z in Theorem* 4 *is replaced by a locally connected plane curve, then condition* (4.1) *need not imply any of the conditions* (4.2)*,* (4.3) *and* (4.4)*.*

Proof. Let Z denote the Sierpiński universal plane curve. Then Z is locally connected. If X and Y are as in Example 5, then Y, being a plane curve by (5.4), can be embedded in Z. So we can assume that $X \subset Y \subset Z$. Thus X satisfies (4.1) according to (5.2) and it does not satisfy (4.2) by (5.5). Further, no component of X is open and each one of them is compact. Thus X satisfies neither (4.3) nor (4.4) even in such strong form as indicated. □

Statement 9. *If a generalized linear graph Z in Theorem* 4 *is replaced by a rational plane curve, then condition* (4.1) *need not imply any of the conditions* (4.2)*,* (4.3) *and* (4.4)*.*

Proof. a) To see that (4.1) need not imply (4.2) and (4.4), let X and Y be as in Example 6, and define $Z = Y$. Then Z is a rational plane curve by (6.1), X satisfies (4.1) according to (6.2) and it does not satisfy (4.2) by (6.3). Further, X is not locally connected by (6.4), so (4.4) is not satisfied.

b) To see that (4.1) need not imply (4.3), let $Z = Y$ be the harmonic fan with the vertex v and let $X = Y \setminus \{v\}$ as in Example 7. Then the conclusion holds by (7.2) and (7.3). □

Remark 10. Observe that Example 6 shows that if a rational curve is substituted in place of a generalized linear graph Z in Theorem 4, then condition (4.1) need not imply the existence of *any* (not only one-point) pathwise connectification, according to (6.3).

Note that if X is a subspace of an arbitrary space Z, then (4.2) trivially implies (4.1). The other two implications from (4.2) are not true. The next statement shows this.

Statement 11. *If a generalized linear graph Z in Theorem* 4 *is replaced either by a locally connected plane curve or by a rational plane curve, then condition* (4.2) *need not imply any of the conditions* (4.3) *and* (4.4)*.*

Proof. a) Let Z denote the Sierpiński universal plane curve, and let X and Y be as in Example 7. Without loss of generality we may assume that $X \subset Y \subset Z$. Then (4.2) holds by (7.2), while (4.3) and (4.4) are not satisfied by (7.3) and (7.4), respectively.

b) Defining $Z = Y$ in Example 7 we argue as in a) above. $\square$

The following result is due to J. R. Prajs.

Theorem 12 (Prajs). *For each topological space X having a metrizable compactification condition* (4.3) *implies condition* (4.1).

Proof. Let a topological space X have a metrizable compactification and satisfy condition (4.3). Then X has at most countably many components. Denote by Y a compactification of X such that:

(12.1) Y is a subset of the Hilbert cube;

(12.2) for each component C of $X \subset Y$ its closure $\mathrm{cl}_Y(C)$ (in Y) is an open subset of Y.

Then the difference

$$D = Y \setminus \bigcup\{\mathrm{cl}_Y(C) : C \text{ is a component of } X\}$$

is a closed subset of Y. For each component C of X choose a point $p(C) \in \mathrm{cl}_Y(C) \setminus X$ and let E be the set of the chosen points. Denote $F = D \cup E$. Then the quotient space Y/F is a one-point connectification of X. $\square$

Question 13. Is having a metrizable compactification of the space X an essential assumption in Theorem 12?

Statement 14. *If a generalized linear graph Z in Theorem* 4 *is replaced either by a locally connected plane curve or by a rational plane curve, then condition* (4.3) *need not imply any of the conditions* (4.2) *and* (4.4).

Proof. Define X as in Example 6. Since X is connected, condition (4.3) is satisfied by the definition of X. Take as Z either the Sierpiński universal plane curve, or the topologists sine-curve Y. As previously we can assume that $X \subset Y \subset Z$. Then (4.2) does not hold according to (6.3). By (6.4) condition (4.4) does not hold, too. $\square$

Remark 15. Observe that in Statement 14 one can consider as Z any space containing X of Example 6. Furthermore, one can take as X an arbitrary connected, non-locally connected and non-compact space, and as Z any space containing such X to see that (4.3) does not imply (4.4) in general.

Statement 16. *For each topological space X condition* (4.4) *implies condition* (4.3).

Proof. Indeed, the implication is known, see e.g. [12, §49, II, Theorem 4, p. 230]. $\square$

As a consequence of Theorem 12 and Statement 16 we get the following.

Corollary 17. *For each topological space X having a metrizable compactification condition* (4.4) *implies condition* (4.1)*.*

Example 18. *The Knaster-Kuratowski subset of the triangular Sierpiński curve.* Let Y denote the triangular Sierpiński curve in the plane (see e.g. [12, §51, I, Example 6 and Fig. 9, p. 270] or [11, Sections 2 and 3, p. 106-107]). Let X be the connected and locally connected subset of Y containing no perfect subset, as constructed in [11] (for a connected and locally connected subset of the unit square containing no arc see [14]). Then:

(18.1) Y is a regular plane curve;

(18.2) X contains the (countable) set of vertices of all triangles considered in the construction of Y, so it is a dense subset of Y;

(18.3) X is not compact by its definition, and Y is a compactification of X;

(18.4) X contains no arc (because it contains no perfect subset by construction);

(18.5) X has a one-point connectification (according to (18.3) and Corollary 17).

Statement 19. *If a generalized linear graph Z in Theorem* 4 *is replaced by a regular plane curve, then condition* (4.4) *need not imply condition* (4.2)*.*

Proof. Define X and Y as in Example 18, and put $Z = Y$. Then by (18.1) X is a subset of the regular plane curve Z. Further, X is locally connected by its construction, and since it is connected, it satisfies condition (4.4) according to (18.3). By (18.5) there exists a one-point connectification of X. No such connectification can be pathwise connected because otherwise X contains an arc, contrary to (18.4). Thus X does not satisfy (4.2). □

Remark 20. Note that if metrizability is not assumed, then the space $X = \{(x, y) : x, y \in [0, 1]\} \setminus \{(1, 1)\}$ ordered lexicographically (see e.g. [17, Example 48, p. 73] or [6, 3.12.3 (d), p. 221]) satisfies (4.4) but not (4.2).

Acknowledgements. The author thanks Professor Paweł Krupski for a fruitful discussion on the topic of this paper.

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