

## Real Hypersurfaces with Invariant Normal Jacobi Operator in the Complex Hyperbolic Quadric

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**ABSTRACT.** We introduce the notion of Lie invariant normal Jacobi operators for real hypersurfaces in the complex hyperbolic quadric  $Q^{m*} = SO_{m,2}^o/SO_m SO_2$ . The invariant normal Jacobi operator implies that the unit normal vector field  $N$  becomes  $\mathfrak{A}$ -principal or  $\mathfrak{A}$ -isotropic. Then in each case, we give a complete classification of real hypersurfaces in  $Q^{m*} = SO_{m,2}^o/SO_m SO_2$  with Lie invariant normal Jacobi operators.

### 1. Introduction

When we consider Hermitian symmetric spaces of rank 2, we can usually give examples of Riemannian symmetric spaces  $SU_{m+2}/S(U_2 U_m)$  and  $SU_{2,m}/S(U_2 U_m)$ , which are said to be complex two-plane Grassmannians and complex hyperbolic two-plane Grassmannians respectively (see [21, 22, 23]). These are viewed as Hermitian symmetric spaces and quaternionic Kähler symmetric spaces equipped with the Kähler structure  $J$  and the quaternionic Kähler structure  $\mathfrak{J}$ .

In the complex projective space  $\mathbb{C}P^{m+1}$  and the quaternionic projective space  $\mathbb{Q}P^{m+1}$  some classifications related to commuting Ricci tensor were investigated by Kimura [5, 6], Pérez [13] and Pérez and Suh [14, 15] respectively. The classification

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problems of the complex 2-plane Grassmannian  $G_2(\mathbb{C}^{m+2}) = SU_{m+2}/S(U_2U_m)$  with various geometric conditions were discussed in Jeong, Kim and Suh [4], Pérez [13], and Suh [21, 22, 29], where the classification of *contact hypersurfaces*, *parallel Ricci tensor*, *harmonic curvature* and *Jacobi operator* of a real hypersurface in  $G_2(\mathbb{C}^{m+2})$  were extensively studied.

Another example of Hermitian symmetric space with rank 2 having non-compact type different from the above ones, is the complex hyperbolic quadric  $SO_{2,m}^0/SO_2SO_m$ . It is a simply connected Riemannian manifold whose curvature tensor is the negative of the curvature tensor of the complex quadric  $Q^m$  (see Besse [2], Helgason [3], and Knapp [10]). The complex hyperbolic quadric also can be regarded as a kind of real Grassmann manifolds of non-compact type with rank 2. Accordingly, the complex hyperbolic quadric  $Q^{m*}$  admits two important geometric structures, a complex conjugation structure  $A$  and a Kähler structure  $J$ , which anti-commute with each other, that is,  $AJ = -JA$ . For  $m \geq 2$  the triple  $(Q^{m*}, J, g)$  is a Hermitian symmetric space of non-compact type and its maximal sectional curvature is equal to  $-4$  (see Klein [7], Kobayashi and Nomizu [11], and Reckziegel [16]).

Two last examples of different Hermitian symmetric spaces with rank 2 in the class of compact type or non-compact type, are the complex quadric  $Q^m = SO_{m+2}/SO_mSO_2$  or the complex hyperbolic quadric  $Q^{m*} = SO_{2,m}^0/SO_mSO_2$ , which are a complex hypersurface in complex projective space  $CP^{m+1}$  or in complex hyperbolic space respectively (see Romero [17, 18], Suh [24, 25], and Smyth [19]). The complex quadric  $Q^m$  or the complex hyperbolic quadric  $Q^{m*}$  can be regarded as a kind of real Grassmann manifold of compact or non-compact type with rank 2 respectively (see Helgason [3], Kobayashi and Nomizu [11]). Accordingly, the complex quadric  $Q^m$  and the complex hyperbolic quadric  $Q^{m*}$  both admit two important geometric structures, a complex conjugation structure  $A$  and a Kähler structure  $J$ , which anti-commute with each other, that is,  $AJ = -JA$  (see Klein [7] and Reckziegel [16]).

Now let us introduce a complex hyperbolic quadric  $Q^{m*} = SO_{m,2}^0/SO_2SO_m$ , which can be regarded as a Hermitian symmetric space with rank 2 of noncompact type. Montiel and Romero [12] proved that the complex hyperbolic quadric  $Q^{m*}$  can be immersed in the indefinite complex hyperbolic space  $CH_1^{m+1}(-c)$ ,  $c > 0$ , by interchanging the Kähler metric with its opposite. Changing the Kähler metric of  $CP_{n-s}^{n+1}$  with its opposite, we have that  $Q_{n-s}^n$  endowed with its opposite metric  $g' = -g$  is also an Einstein hypersurface of  $CH_{s+1}^{n+1}(-c)$ . When  $s = 0$ , we know that  $(Q_n^n, g' = -g)$  can be regarded as the complex hyperbolic quadric  $Q^{m*} = SO_{m,2}^0/SO_2SO_m$ , which is immersed in the indefinite complex hyperbolic quadric  $CH_1^{m+1}(-c)$ ,  $c > 0$  as a space-like complex Einstein hypersurface.

Apart from the complex structure  $J$  there is another distinguished geometric structure on  $Q^{m*}$ , namely a parallel rank two vector bundle  $\mathfrak{A}$  which contains a  $S^1$ -bundle of real structures. Note that these real structures are complex conjugations  $A$  on the tangent spaces of the complex hyperbolic quadric  $Q^{m*}$ . This geometric

structure determines a maximal  $\mathfrak{A}$ -invariant subbundle  $\mathcal{Q}$  of the tangent bundle  $TM$  of a real hypersurface  $M$  in the complex hyperbolic quadric  $Q^{m*}$ .

Recall that a nonzero tangent vector  $W \in T_{[z]}Q^{m*}$  is called singular if it is tangent to more than one maximal flat in  $Q^{m*}$ . There are two types of singular tangent vectors for the complex hyperbolic quadric  $Q^{m*}$  as follows:

1. If there exists a conjugation  $A \in \mathfrak{A}$  such that  $W \in V(A)$ , then  $W$  is singular. Such a singular tangent vector is called  $\mathfrak{A}$ -principal.
2. If there exist a conjugation  $A \in \mathfrak{A}$  and orthonormal vectors  $X, Y \in V(A)$  such that  $W/\|W\| = (X + JY)/\sqrt{2}$ , then  $W$  is singular. Such a singular tangent vector is called  $\mathfrak{A}$ -isotropic.

Here  $V(A) = \{X \in T_{[z]}Q^{m*} : AX = X\}$  and  $JV(A) = \{X \in T_{[z]}Q^{m*} : AX = -X\}$ ,  $[z] \in Q^{m*}$ , are the  $(+1)$ -eigenspace and  $(-1)$ -eigenspace for the involution  $A$  on  $T_{[z]}Q^{m*}$ ,  $[z] \in Q^{m*}$ .

When we consider a hypersurface  $M$  in the complex hyperbolic quadric  $Q^{m*}$ , under the assumption of some geometric properties the unit normal vector field  $N$  of  $M$  in  $Q^{m*}$  can be divided into two cases depending on whether  $N$  is  $\mathfrak{A}$ -isotropic or  $\mathfrak{A}$ -principal (see [27, 28, 30, 31]). In the first case where  $N$  is  $\mathfrak{A}$ -isotropic, we have shown in [27] that  $M$  is locally congruent to a tube over a totally geodesic complex hyperbolic space  $\mathbb{C}H^k$  in the complex hyperbolic quadric  $Q^{2k*}$ . In the second case, when the unit normal  $N$  is  $\mathfrak{A}$ -principal, we proved that a contact hypersurface  $M$  in the complex hyperbolic quadric  $Q^{m*}$  is locally congruent to a tube over a totally geodesic and totally real submanifold  $\mathbb{R}H^m$  in  $Q^{m*}$  or a horosphere (see Suh [9], and Suh and Hwang [30]).

Usually, Jacobi fields along geodesics of a given Riemannian manifold  $\bar{M}$  satisfy a well known differential equation. Naturally the classical differential equation inspires the so-called *Jacobi operator*. That is, if  $\bar{R}$  is the curvature operator of  $\bar{M}$ , the Jacobi operator with respect to  $X$  at  $z \in M$ , is defined by

$$(\bar{R}_X Y)(z) = (\bar{R}(Y, X)X)(z)$$

for any  $Y \in T_z \bar{M}$ . Then  $\bar{R}_X \in \text{End}(T_z \bar{M})$  becomes a symmetric endomorphism of the tangent bundle  $T\bar{M}$  of  $\bar{M}$ . Clearly, each tangent vector field  $X$  to  $\bar{M}$  provides a Jacobi operator with respect to  $X$ .

From such a view point, in the complex hyperbolic quadric  $Q^{m*}$  the *normal Jacobi operator*  $\bar{R}_N$  is defined by

$$\bar{R}_N = \bar{R}(\cdot, N)N \in \text{End}(T_z M), \quad z \in M$$

for a real hypersurface  $M$  in the complex hyperbolic quadric  $Q^{m*}$  with unit normal vector field  $N$ , where  $\bar{R}$  denotes the curvature tensor of the complex hyperbolic quadric  $Q^{m*}$ . Of course, the normal Jacobi operator  $\bar{R}_N$  is a symmetric endomorphism of  $M$  in the complex hyperbolic quadric  $Q^{m*}$ .

The normal Jacobi operator  $\bar{R}_N$  of  $M$  in the complex hyperbolic quadric  $Q^{m*}$  is said to be *Lie invariant* if the operator  $\bar{R}_N$  satisfies

$$0 = (\mathfrak{L}_X \bar{R}_N)Y$$

for any  $X, Y \in T_z M$ ,  $z \in M$ , where the Lie derivative  $(\mathfrak{L}_X \bar{R}_N)Y$  is defined by

$$\begin{aligned} (1.1) \quad (\mathfrak{L}_X \bar{R}_N)Y &= [X, \bar{R}_N(Y)] - \bar{R}_N([X, Y]) \\ &= \nabla_X(\bar{R}_N(Y)) - \nabla_{\bar{R}_N(Y)}X - \bar{R}_N(\nabla_X Y - \nabla_Y X) \\ &= (\nabla_X \bar{R}_N)Y - \nabla_{\bar{R}_N(Y)}X + \bar{R}_N(\nabla_Y X). \end{aligned}$$

For real hypersurfaces in the complex quadric  $Q^m$  we investigated the notions of parallel Ricci tensor, harmonic curvature and commuting Ricci tensor, which are respectively given by  $\nabla \text{Ric} = 0$ ,  $\delta \text{Ric} = 0$  and  $\text{Ric} \cdot \phi = \phi \cdot \text{Ric}$  (see Suh [25], [26], and Suh and Hwang [29]). But from the assumption of Ricci parallel or harmonic curvature, it was difficult for us to derive the fact that either the unit normal vector field  $N$  is  $\mathfrak{A}$ -isotropic or  $\mathfrak{A}$ -principal. So in [25] and [26] we gave a classification with the further assumption of  $\mathfrak{A}$ -isotropic. Also in the study of complex hyperbolic quadric  $Q^{m*}$  we also have some obstructions to get the fact that the unit normal  $N$  is singular.

In the paper due to Suh [27] we investigate this problem of isometric Reeb flow for the complex hyperbolic quadric  $Q^{m*} = SO_{2,m}^o/SO_m SO_2$ . In view of the previous results, naturally, we expected that the classification might include at least the totally geodesic  $Q^{m-1*} \subset Q^{m*}$ . But, the results are quite different from our expectations. The totally geodesic submanifolds of the above type are not included. Now we introduce the classification as follows:

**Theorem 1.1.** *Let  $M$  be a real hypersurface of the complex hyperbolic quadric  $Q^{m*} = SO_{2,m}^o/SO_m SO_2$ ,  $m \geq 3$ . The Reeb flow on  $M$  is isometric if and only if  $m$  is even, say  $m = 2k$ , and  $M$  is an open part of a tube around a totally geodesic  $\mathbb{C}H^k \subset Q^{2k*}$  or a horosphere whose center at infinity is  $\mathfrak{A}$ -isotropic singular.*

But fortunately, when we consider Lie invariant normal Jacobi operator, that is,  $\mathfrak{L}_X \bar{R}_N = 0$  for any tangent vector field  $X$  on  $M$  in  $Q^{m*}$ , we can assert that the unit normal vector field  $N$  becomes either  $\mathfrak{A}$ -isotropic or  $\mathfrak{A}$ -principal as follows:

**Theorem 1.2.** *Let  $M$  be a Hopf real hypersurface in the complex hyperbolic quadric  $Q^{m*}$ ,  $m \geq 3$ , with Lie invariant normal Jacobi operator. Then the unit normal vector field  $N$  is singular, that is,  $N$  is  $\mathfrak{A}$ -isotropic or  $\mathfrak{A}$ -principal.*

Then motivated by Theorem 1.1 and Theorem 1.2, we can give a complete classification for real hypersurfaces in  $Q^{m*}$  with invariant normal Jacobi operator as follows:

**Theorem 1.3.** *Let  $M$  be a Hopf real hypersurface in the complex hyperbolic quadric  $Q^{m*}$ ,  $m \geq 3$  with Lie invariant normal normal Jacobi operator. Then  $M$  is locally congruent to a tube of radius  $r$  over a totally geodesic  $\mathbb{C}H^k$  in  $Q^{2k*}$  or a horosphere*

whose center at infinity is  $\mathfrak{A}$ -isotropic singular.

## 2. The Complex Hyperbolic Quadric

In this section, let us introduce a new known result of the complex hyperbolic quadric  $Q^{m*}$  different from the complex quadric  $Q^m$ . This section is due to Klein and Suh [9], and Suh [28].

The  $m$ -dimensional complex hyperbolic quadric  $Q^{m*}$  is the non-compact dual of the  $m$ -dimensional complex quadric  $Q^m$ , which is a kind of Hermitian symmetric space of non-compact type with rank 2 (see Besse [2], and Helgason [3]).

The complex hyperbolic quadric  $Q^{m*}$  cannot be realized as a homogeneous complex hypersurface of the complex hyperbolic space  $\mathbb{C}H^{m+1}$ . In fact, Smyth [20, Theorem 3(ii)] has shown that every homogeneous complex hypersurface in  $\mathbb{C}H^{m+1}$  is totally geodesic. This is in marked contrast to the situation for the complex quadric  $Q^m$ , which can be realized as a homogeneous complex hypersurface of the complex projective space  $\mathbb{C}P^{m+1}$  in such a way that the shape operator for any unit normal vector to  $Q^m$  is a real structure on the corresponding tangent space of  $Q^m$ , see [7] and [16]. Another related result by Smyth, [20, Theorem 1], which states that any complex hypersurface  $\mathbb{C}H^{m+1}$  for which the square of the shape operator has constant eigenvalues (counted with multiplicity) is totally geodesic, also precludes the possibility of a model of  $Q^{m*}$  as a complex hypersurface of  $\mathbb{C}H^{m+1}$  with the analogous property for the shape operator.

Therefore we realize the complex hyperbolic quadric  $Q^{m*}$  as the quotient manifold  $SO_{2,m}^0/SO_2SO_m$ . As  $Q^{1*}$  is isomorphic to the real hyperbolic space  $\mathbb{R}H^2 = SO_{1,2}^0/SO_2$ , and  $Q^{2*}$  is isomorphic to the Hermitian product of complex hyperbolic spaces  $\mathbb{C}H^1 \times \mathbb{C}H^1$ , we suppose  $m \geq 3$  in the sequel and throughout this paper. Let  $G := SO_{2,m}^0$  be the transvection group of  $Q^{m*}$  and  $K := SO_2SO_m$  be the isotropy group of  $Q^{m*}$  at the “origin”  $p_0 := eK \in Q^{m*}$ . Then

$$\sigma : G \rightarrow G, g \mapsto sgs^{-1} \quad \text{with} \quad s := \begin{pmatrix} -1 & & & & \\ & -1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & \ddots \\ & & & & & 1 \end{pmatrix}$$

is an involutive Lie group automorphism of  $G$  with  $\text{Fix}(\sigma)_0 = K$ , and therefore  $Q^{m*} = G/K$  is a Riemannian symmetric space. The center of the isotropy group  $K$  is isomorphic to  $SO_2$ , and therefore  $Q^{m*}$  is in fact a Hermitian symmetric space.

The Lie algebra  $\mathfrak{g} := \mathfrak{so}_{2,m}$  of  $G$  is given by

$$\mathfrak{g} = \{X \in \mathfrak{gl}(m+2, \mathbb{R}) : X^t \cdot s = -s \cdot X\}$$

(see [10, p. 59]). In the sequel we will write members of  $\mathfrak{g}$  as block matrices with respect to the decomposition  $\mathbb{R}^{m+2} = \mathbb{R}^2 \oplus \mathbb{R}^m$ , i.e. in the form

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix},$$

where  $X_{11}$ ,  $X_{12}$ ,  $X_{21}$ ,  $X_{22}$  are real matrices of the dimension  $2 \times 2$ ,  $2 \times m$ ,  $m \times 2$  and  $m \times m$ , respectively. Then

$$\mathfrak{g} = \left\{ \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} : X_{11}^t = -X_{11}, X_{12}^t = X_{21}, X_{22}^t = -X_{22} \right\}.$$

The linearisation  $\sigma_L = \text{Ad}(s) : \mathfrak{g} \rightarrow \mathfrak{g}$  of the involutive Lie group automorphism  $\sigma$  induces the Cartan decomposition  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ , where the Lie subalgebra

$$\begin{aligned} \mathfrak{k} &= \text{Eig}(\sigma_*, 1) = \{X \in \mathfrak{g} : sXs^{-1} = X\} \\ &= \left\{ \begin{pmatrix} X_{11} & 0 \\ 0 & X_{22} \end{pmatrix} : X_{11}^t = -X_{11}, X_{22}^t = -X_{22} \right\} \\ &\cong \mathfrak{so}_2 \oplus \mathfrak{so}_m \end{aligned}$$

is the Lie algebra of the isotropy group  $K$ , and the  $2m$ -dimensional linear subspace

$$\mathfrak{m} = \text{Eig}(\sigma_*, -1) = \{X \in \mathfrak{g} : sXs^{-1} = -X\} = \left\{ \begin{pmatrix} 0 & X_{12} \\ X_{21} & 0 \end{pmatrix} : X_{12}^t = X_{21} \right\}$$

is canonically isomorphic to the tangent space  $T_{p_0}Q^{m*}$ . Under the identification  $T_{p_0}Q^{m*} \cong \mathfrak{m}$ , the Riemannian metric  $g$  of  $Q^{m*}$  (where the constant factor of the metric is chosen so that the formulae become as simple as possible) is given by

$$g(X, Y) = \frac{1}{2} \text{tr}(Y^t \cdot X) = \text{tr}(Y_{12} \cdot X_{21}) \quad \text{for } X, Y \in \mathfrak{m}.$$

$g$  is clearly  $\text{Ad}(K)$ -invariant, and therefore corresponds to an  $\text{Ad}(G)$ -invariant Riemannian metric on  $Q^{m*}$ . The complex structure  $J$  of the Hermitian symmetric space is given by

$$JX = \text{Ad}(j)X \quad \text{for } X \in \mathfrak{m}, \quad \text{where } j := \begin{pmatrix} 0 & 1 \\ -1 & 0 & & \\ & 1 & 1 & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \in K.$$

Because  $j$  is in the center of  $K$ , the orthogonal linear map  $J$  is  $\text{Ad}(K)$ -invariant, and thus defines an  $\text{Ad}(G)$ -invariant Hermitian structure on  $Q^{m*}$ . By identifying the multiplication with the unit complex number  $i$  with the application of the linear map  $J$ , the tangent spaces of  $Q^{m*}$  thus become  $m$ -dimensional complex linear spaces, and we will adopt this point of view in the sequel.

Like for the complex quadric (again compare [7, 8, 16]), there is another important structure on the tangent bundle of the complex quadric besides the Riemannian metric and the complex structure, namely an  $S^1$ -bundle  $\mathfrak{A}$  of real structures. The situation here differs from that of the complex quadric in that for  $Q^{m*}$ , the real structures in  $\mathfrak{A}$  cannot be interpreted as the shape operator of a complex hypersurface in a complex space form, but as the following considerations will show,  $\mathfrak{A}$  still plays an important role in the description of the geometry of  $Q^{m*}$ .

Let

$$a_0 := \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & 1 \\ & & & \ddots \\ & & & & 1 \end{pmatrix}.$$

Note that we have  $a_0 \notin K$ , but only  $a_0 \in O_2 SO_m$ . However,  $\text{Ad}(a_0)$  still leaves  $\mathfrak{m}$  invariant, and therefore defines an  $\mathbb{R}$ -linear map  $A_0$  on the tangent space  $\mathfrak{m} \cong T_{p_0} Q^{m*}$ .  $A_0$  turns out to be an involutive orthogonal map with  $A_0 \circ J = -J \circ A_0$  (i.e.  $A_0$  is anti-linear with respect to the complex structure of  $T_{p_0} Q^{m*}$ ), and hence a real structure on  $T_{p_0} Q^{m*}$ . But  $A_0$  commutes with  $\text{Ad}(g)$  not for all  $g \in K$ , but only for  $g \in SO_m \subset K$ . More specifically, for  $g = (g_1, g_2) \in K$  with  $g_1 \in SO_2$  and  $g_2 \in SO_m$ , say  $g_1 = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}$  with  $t \in \mathbb{R}$  (so that  $\text{Ad}(g_1)$  corresponds to multiplication with the complex number  $\mu := e^{it}$ ), we have

$$A_0 \circ \text{Ad}(g) = \mu^{-2} \cdot \text{Ad}(g) \circ A_0 .$$

This equation shows that the object which is  $\text{Ad}(K)$ -invariant and therefore geometrically relevant is not the real structure  $A_0$  by itself, but rather the “circle of real structures”

$$\mathfrak{A}_{p_0} := \{\lambda A_0 | \lambda \in S^1\} .$$

$\mathfrak{A}_{p_0}$  is  $\text{Ad}(K)$ -invariant, and therefore generates an  $\text{Ad}(G)$ -invariant  $S^1$ -subbundle  $\mathfrak{A}$  of the endomorphism bundle  $\text{End}(TQ^{m*})$ , consisting of real structures on the tangent spaces of  $Q^{m*}$ . For any  $A \in \mathfrak{A}$ , the tangent line to the fibre of  $\mathfrak{A}$  through  $A$  is spanned by  $JA$ .

For any  $p \in Q^{m*}$  and  $A \in \mathfrak{A}_p$ , the real structure  $A$  induces a splitting

$$T_p Q^{m*} = V(A) \oplus JV(A)$$

into two orthogonal, maximal totally real subspaces of the tangent space  $T_p Q^{m*}$ . Here  $V(A)$  resp.  $JV(A)$  are the  $(+1)$ -eigenspace resp. the  $(-1)$ -eigenspace of  $A$ . For every unit vector  $Z \in T_p Q^{m*}$  there exist  $t \in [0, \frac{\pi}{4}]$ ,  $A \in \mathfrak{A}_p$  and orthonormal vectors  $X, Y \in V(A)$  so that

$$Z = \cos(t) \cdot X + \sin(t) \cdot JY$$

holds; see [16, Proposition 3]. Here  $t$  is uniquely determined by  $Z$ . The vector  $Z$  is singular, i.e. contained in more than one Cartan subalgebra of  $\mathfrak{m}$ , if and only if either  $t = 0$  or  $t = \frac{\pi}{4}$  holds. The vectors with  $t = 0$  are called  $\mathfrak{A}$ -principal, whereas the vectors with  $t = \frac{\pi}{4}$  are called  $\mathfrak{A}$ -isotropic. If  $Z$  is regular, i.e.  $0 < t < \frac{\pi}{4}$  holds, then also  $A$  and  $X, Y$  are uniquely determined by  $Z$ .

Like for the complex quadric, the Riemannian curvature tensor  $\bar{R}$  of  $Q^{m*}$  can be fully described in terms of the “fundamental geometric structures”  $g$ ,  $J$  and  $\mathfrak{A}$ . In fact, under the correspondence  $T_{p_0} Q^{m*} \cong \mathfrak{m}$ , the curvature  $\bar{R}(X, Y)Z$  corresponds to  $-[[X, Y], Z]$  for  $X, Y, Z \in \mathfrak{m}$ , see [11, Chapter XI, Theorem 3.2(1)]. By evaluating the latter expression explicitly, one can show that one has

$$\begin{aligned} (2.1) \quad \bar{R}(X, Y)Z &= -g(Y, Z)X + g(X, Z)Y \\ &\quad - g(JY, Z)JX + g(JX, Z)JY + 2g(JX, Y)JZ \\ &\quad - g(AY, Z)AX + g(AX, Z)AY \\ &\quad - g(JAY, Z)JAX + g(JAX, Z)JAY \end{aligned}$$

for arbitrary  $A \in \mathfrak{A}_{p_0}$ . Therefore the curvature of  $Q^{m*}$  is the negative of that of the complex quadric  $Q^m$ , compare [16, Theorem 1]. This confirms that the symmetric space  $Q^{m*}$  which we have constructed here is indeed the non-compact dual of the complex quadric.

Let  $M$  be a real hypersurface in complex hyperbolic quadric  $Q^{m*}$  and denote by  $(\phi, \xi, \eta, g)$  the induced almost contact metric structure on  $M$  and by  $\nabla$  the induced Riemannian connection on  $M$ . Note that  $\xi = -JN$ , where  $N$  is a (local) unit normal vector field of  $M$ . The vector field  $\xi$  is known as the Reeb vector field of  $M$ . If the integral curves of  $\xi$  are geodesics in  $M$ , the hypersurface  $M$  is called a Hopf hypersurface. The integral curves of  $\xi$  are geodesics in  $M$  if and only if  $\xi$  is a principal curvature vector of  $M$  everywhere. The tangent bundle  $TM$  of  $M$  splits orthogonally into  $TM = \mathcal{C} \oplus \mathcal{F}$ , where  $\mathcal{C} = \ker(\eta)$  is the maximal complex subbundle of  $TM$  and  $\mathcal{F} = \mathbb{R}\xi$ . The structure tensor field  $\phi$  restricted to  $\mathcal{C}$  coincides with the complex structure  $J$  restricted to  $\mathcal{C}$ , and we have  $\phi\xi = 0$ . We denote by  $\nu M$  the normal bundle of  $M$ .

We first introduce some notations. For a fixed real structure  $A \in \mathfrak{A}_{[z]}$  and  $X \in T_{[z]}M$  we decompose  $AX$  into its tangential and normal component, that is,

$$AX = BX + \rho(X)N$$

where  $BX$  is the tangential component of  $AX$  and

$$\rho(X) = g(AX, N) = g(X, AN) = g(X, AJ\xi) = g(JX, A\xi).$$

Since  $JX = \phi X + \eta(X)N$  and  $A\xi = B\xi + \rho(\xi)N$  we also have

$$\rho(X) = g(\phi X, B\xi) + \eta(X)\rho(\xi) = \eta(B\phi X) + \eta(X)\rho(\xi).$$

We also define

$$\delta = g(N, AN) = g(JN, JAN) = -g(JN, AJN) = -g(\xi, A\xi).$$

At each point  $[z] \in M$  we define

$$\mathcal{Q}_{[z]} = \{X \in T_{[z]}M : AX \in T_{[z]}M \text{ for all } A \in \mathfrak{A}_{[z]}\},$$

which is the maximal  $\mathfrak{A}_{[z]}$ -invariant subspace of  $T_{[z]}M$ . Then by using the same method for real hypersurfaces in complex hyperbolic quadric  $Q^{m*}$  as in Berndt and Suh [1] we get the following

**Lemma 2.1.** *Let  $M$  be a real hypersurface in complex hyperbolic quadric  $Q^{m*}$ . Then the following statements are equivalent:*

- (i) *The normal vector  $N_{[z]}$  of  $M$  is  $\mathfrak{A}$ -principal,*
- (ii)  $\mathcal{Q}_{[z]} = \mathcal{C}_{[z]}$ ,
- (iii) *There exists a real structure  $A \in \mathfrak{A}_{[z]}$  such that  $AN_{[z]} \in \mathcal{C}\nu_{[z]}M$ .*



Assume now that the normal vector  $N_{[z]}$  of  $M$  is not  $\mathfrak{A}$ -principal. Then there exists a real structure  $A \in \mathfrak{A}_{[z]}$  such that

$$N_{[z]} = \cos(t)Z_1 + \sin(t)JZ_2$$

for some orthonormal vectors  $Z_1, Z_2 \in V(A)$  and  $0 < t \leq \frac{\pi}{4}$ . This implies

$$(2.2) \quad \begin{aligned} AN_{[z]} &= \cos(t)Z_1 - \sin(t)JZ_2, \\ \xi_{[z]} &= \sin(t)Z_2 - \cos(t)JZ_1, \\ A\xi_{[z]} &= \sin(t)Z_2 + \cos(t)JZ_1, \end{aligned}$$

and therefore  $\mathcal{Q}_{[z]} = T_{[z]}Q^m \ominus ([Z_1] \oplus [Z_2])$  is strictly contained in  $\mathcal{C}_{[z]}$ . Moreover, we have

$$A\xi_{[z]} = B\xi_{[z]} \text{ and } \rho(\xi_{[z]}) = 0.$$

We have

$$\begin{aligned} g(B\xi_{[z]} + \delta\xi_{[z]}, N_{[z]}) &= 0, \\ g(B\xi_{[z]} + \delta\xi_{[z]}, \xi_{[z]}) &= 0, \\ g(B\xi_{[z]} + \delta\xi_{[z]}, B\xi_{[z]} + \delta\xi_{[z]}) &= \sin^2(2t), \end{aligned}$$

where the function  $\delta$  denotes  $\delta = -g(\xi, A\xi) = -(\sin^2 t - \cos^2 t) = \cos 2t$ . Therefore

$$U_{[z]} = \frac{1}{\sin(2t)}(B\xi_{[z]} + \delta\xi_{[z]})$$

is a unit vector in  $\mathcal{C}_{[z]}$  and

$$\mathcal{C}_{[z]} = \mathcal{Q}_{[z]} \oplus [U_{[z]}] \text{ (orthogonal direct sum).}$$

If  $N_{[z]}$  is not  $\mathfrak{A}$ -principal at  $[z]$ , then  $N$  is not  $\mathfrak{A}$ -principal in an open neighborhood of  $[z]$ , and therefore  $U$  is a well-defined unit vector field on that open neighborhood. We summarize this in the following

**Lemma 2.2.** *Let  $M$  be a real hypersurface in complex hyperbolic quadric  $Q^{m*}$  whose unit normal  $N_{[z]}$  is not  $\mathfrak{A}$ -principal at  $[z]$ . Then there exists an open neighborhood of  $[z]$  in  $M$  and a section  $A$  in  $\mathfrak{A}$  on that neighborhood consisting of real structures such that*

- (i)  $A\xi = B\xi$  and  $\rho(\xi) = 0$ ,
- (ii)  $U = (B\xi + \delta\xi)/\|B\xi + \delta\xi\|$  is a unit vector field tangent to  $\mathcal{C}$ ,
- (iii)  $\mathcal{C} = \mathcal{Q} \oplus [U]$ .

### 3. Some General Equations

Let  $M$  be a real hypersurface in the complex hyperbolic quadric  $Q^{m*}$  and denote by  $(\phi, \xi, \eta, g)$  the induced almost contact metric structure. Note that  $\xi = -JN$ , where  $N$  is a (local) unit normal vector field of  $M$  and  $\eta$  the corresponding 1-form defined by  $\eta(X) = g(\xi, X)$  for any tangent vector field  $X$  on  $M$ . The tangent bundle  $TM$  of  $M$  splits orthogonally into  $TM = \mathcal{C} \oplus \mathbb{R}\xi$ , where  $\mathcal{C} = \ker(\eta)$  is the maximal complex subbundle of  $TM$ . The structure tensor field  $\phi$  restricted to  $\mathcal{C}$  coincides with the complex structure  $J$  restricted to  $\mathcal{C}$ , and  $\phi\xi = 0$ .

At each point  $z \in M$  we define a maximal  $\mathfrak{A}$ -invariant subspace of  $T_zM$ ,  $z \in M$  as follows:

$$\mathcal{Q}_z = \{X \in T_zM : AX \in T_zM \text{ for all } A \in \mathfrak{A}_z\}.$$

Then we want to introduce an important lemma which will be used in the proof of our main Theorem in the introduction.

**Lemma 3.1.**([24]) *For each  $z \in M$  we have*

- (i) *If  $N_z$  is  $\mathfrak{A}$ -principal, then  $\mathcal{Q}_z = \mathcal{C}_z$ .*
- (ii) *If  $N_z$  is not  $\mathfrak{A}$ -principal, there exist a conjugation  $A \in \mathfrak{A}$  and orthonormal vectors  $X, Y \in V(A)$  such that  $N_z = \cos(t)X + \sin(t)JY$  for some  $t \in (0, \pi/4]$ . Then we have  $\mathcal{Q}_z = \mathcal{C}_z \ominus \mathbb{C}(JX + Y)$ .*

From the explicit expression of the Riemannian curvature tensor of the complex hyperbolic quadric  $Q^{m*}$  we can easily derive the Codazzi equation for a real hypersurface  $M$  in  $Q^{m*}$ :

$$\begin{aligned} (3.1) \quad & g((\nabla_X S)Y - (\nabla_Y S)X, Z) \\ &= -\eta(X)g(\phi Y, Z) + \eta(Y)g(\phi X, Z) + 2\eta(Z)g(\phi X, Y) \\ &\quad - \rho(X)g(BY, Z) + \rho(Y)g(BX, Z) \\ &\quad + \eta(BX)g(BY, \phi Z) + \eta(BX)\rho(Y)\eta(Z) \\ &\quad - \eta(BY)g(BX, \phi Z) - \eta(BY)\rho(X)\eta(Z). \end{aligned}$$

We now assume that  $M$  is a Hopf hypersurface. Then the shape operator  $S$  of  $M$  in  $Q^{m*}$  satisfies

$$S\xi = \alpha\xi$$

with the smooth function  $\alpha = g(S\xi, \xi)$  on  $M$ . Inserting  $Z = \xi$  into the Codazzi equation leads to

$$g((\nabla_X S)Y - (\nabla_Y S)X, \xi) = 2g(\phi X, Y) - 2\rho(X)\eta(BY) + 2\rho(Y)\eta(BX).$$

On the other hand, we have

$$\begin{aligned} (3.2) \quad & g((\nabla_X S)Y - (\nabla_Y S)X, \xi) \\ &= g((\nabla_X S)\xi, Y) - g((\nabla_Y S)\xi, X) \\ &= d\alpha(X)\eta(Y) - d\alpha(Y)\eta(X) + \alpha g((S\phi + \phi S)X, Y) - 2g(S\phi SX, Y). \end{aligned}$$

Comparing the previous two equations and putting  $X = \xi$  yields

$$d\alpha(Y) = d\alpha(\xi)\eta(Y) + 2\delta\rho(Y),$$

where the function  $\delta = -g(A\xi, \xi)$  and  $\rho(Y) = g(AN, Y)$  for any vector field  $Y$  on  $M$  in  $Q^{m*}$ .

Reinserting this into the previous equation yields

$$\begin{aligned} g((\nabla_X S)Y - (\nabla_Y S)X, \xi) &= -2\delta\eta(X)\rho(Y) + 2\delta\rho(X)\eta(Y) \\ &\quad + \alpha g((\phi S + S\phi)X, Y) - 2g(S\phi SX, Y). \end{aligned}$$

Altogether this implies

$$\begin{aligned} (3.3) \quad 0 &= 2g(S\phi SX, Y) - \alpha g((\phi S + S\phi)X, Y) + 2g(\phi X, Y) \\ &\quad - 2\delta\rho(X)\eta(Y) - 2\rho(X)\eta(BY) + 2\rho(Y)\eta(BX) + 2\delta\eta(X)\rho(Y) \\ &= g((2S\phi S - \alpha(\phi S + S\phi) + 2\phi)X, Y) \\ &\quad - 2\rho(X)\eta(BY + \delta Y) + 2\rho(Y)\eta(BX + \delta X) \\ &= g((2S\phi S - \alpha(\phi S + S\phi) + 2\phi)X, Y) \\ &\quad - 2\rho(X)g(Y, B\xi + \delta\xi) + 2g(X, B\xi + \delta\xi)\rho(Y). \end{aligned}$$

If  $AN = N$  we have  $\rho = 0$ , otherwise we can use Lemma 2.2 to calculate  $\rho(Y) = g(Y, AN) = g(Y, AJ\xi) = -g(Y, JA\xi) = -g(Y, JB\xi) = -g(Y, \phi B\xi)$ . Thus we have proved

**Lemma 3.2.** *Let  $M$  be a Hopf hypersurface in  $Q^{m*}$ ,  $m \geq 3$ . Then we have*

$$(2S\phi S - \alpha(\phi S + S\phi) + 2\phi)X = 2\rho(X)(B\xi + \delta\xi) + 2g(X, B\xi + \delta\xi)\phi B\xi.$$

If the unit normal vector field  $N$  is  $\mathfrak{A}$ -principal, we can choose a real structure  $A \in \mathfrak{A}$  such that  $AN = N$ . Then we have  $\rho = 0$  and  $\phi B\xi = -\phi\xi = 0$ , and therefore

$$2S\phi S - \alpha(\phi S + S\phi) = -2\phi.$$

If  $N$  is not  $\mathfrak{A}$ -principal, we can choose a real structure  $A \in \mathfrak{A}$  as in Lemma 2.2 and get

$$\begin{aligned} &\rho(X)(B\xi + \delta\xi) + g(X, B\xi + \delta\xi)\phi B\xi \\ &= -g(X, \phi(B\xi + \delta\xi))(B\xi + \delta\xi) + g(X, B\xi + \delta\xi)\phi(B\xi + \delta\xi) \\ &= \|B\xi + \delta\xi\|^2(g(X, U)\phi U - g(X, \phi U)U) \\ &= \sin^2(2t)(g(X, U)\phi U - g(X, \phi U)U), \end{aligned}$$

which is equal to 0 on  $\mathcal{Q}$  and equal to  $\sin^2(2t)\phi X$  on  $\mathcal{C} \ominus \mathcal{Q}$ . Altogether we have proved:

**Lemma 3.3.** *Let  $M$  be a Hopf hypersurface in  $Q^{*m}$ ,  $m \geq 3$ . Then the tensor field*

$$2S\phi S - \alpha(\phi S + S\phi)$$

leaves  $\mathcal{Q}$  and  $\mathcal{C} \ominus \mathcal{Q}$  invariant and we have

$$2S\phi S - \alpha(\phi S + S\phi) = -2\phi \text{ on } \mathcal{Q}$$

and

$$2S\phi S - \alpha(\phi S + S\phi) = -2\delta^2\phi \text{ on } \mathcal{C} \ominus \mathcal{Q},$$

where  $\delta = \cos 2t$  as in section 3.

#### 4. Invariant Normal Jacobi Operator and a Key Lemma

By the curvature tensor  $\bar{R}$  of (2.1) for a real hypersurface in the complex hyperbolic quadric  $Q^{m*}$  in section 2, the normal Jacobi operator  $\bar{R}_N$  is defined in such a way that

$$\begin{aligned} \bar{R}_N(X) &= \bar{R}(X, N)N \\ &= -X - g(JN, N)JX + g(JX, N)JN + 2g(JX, N)JN \\ &\quad - g(AN, N)AX + g(AX, N)AN - g(JAN, N)JAX + g(JAX, N)JAN \end{aligned}$$

for any tangent vector field  $X$  in  $T_z M$  and the unit normal  $N$  of  $M$  in  $T_z Q^{m*}$ ,  $z \in Q^{m*}$ . Then the normal Jacobi operator  $\bar{R}_N$  becomes a symmetric operator on the tangent space  $T_z M$ ,  $z \in M$ , of  $Q^{m*}$ . From this, by the complex structure  $J$  and the complex conjugations  $A \in \mathfrak{A}$ , together with the fact that  $g(A\xi, N) = 0$  and  $\xi = -JN$  in section 3, the normal Jacobi operator  $\bar{R}_N$  is given by

$$(4.1) \quad \begin{aligned} \bar{R}_N(Y) &= -Y - 3\eta(Y)\xi - g(AN, N)AY \\ &\quad + g(AY, N)AN + g(AY, \xi)A\xi \end{aligned}$$

for any  $Y \in T_z M$ ,  $z \in M$ . Then the derivative of  $\bar{R}_N$  is given by

$$\begin{aligned} (4.2) \quad (\nabla_X \bar{R}_N)Y &= \nabla_X(\bar{R}_N(Y)) - \bar{R}_N(\nabla_X Y) \\ &= -3(\nabla_X \eta)(Y)\xi - 3\eta(Y)\nabla_X \xi \\ &\quad - \{g(\bar{\nabla}_X(AN), N) + g(AN, \bar{\nabla}_X N)\}AY \\ &\quad - g(AN, N)\{\bar{\nabla}_X(AY) - A\nabla_X Y\} \\ &\quad + \{g(\bar{\nabla}_X(AY) - A\nabla_X Y, N) + g(AY, \bar{\nabla}_X N)\}AN \\ &\quad + g(AY, N)\bar{\nabla}_X(AN) \\ &\quad + \{g(\bar{\nabla}_X(AY) - A\nabla_X Y, \xi)A\xi + g(AY, \bar{\nabla}_X \xi)\}A\xi \\ &\quad + g(AY, \xi)\bar{\nabla}_X(A\xi), \end{aligned}$$

where the connection  $\bar{\nabla}$  on the complex hyperbolic quadric  $Q^{m*}$  is given by

$$\begin{aligned} \bar{\nabla}_X(AY) - A\nabla_X Y &= (\bar{\nabla}_X A)Y + A\bar{\nabla}_X Y - A\nabla_X Y \\ &= q(X)JAY + A\sigma(X, Y) \\ &= q(X)JAY + g(SX, Y)AN. \end{aligned}$$

From this, together with the invariance of  $\mathcal{L}_X \bar{R}_N = 0$  in (1.1), it follows that

$$\begin{aligned}
 (4.3) \quad & \nabla_{\bar{R}_N(Y)} X - \bar{R}_N(\nabla_Y X) \\
 &= (\nabla_X \bar{R}_N) Y \\
 &= -3g(\phi SX, Y)\xi - 3\eta(Y)\phi SX \\
 &\quad - \{q(X)g(JAN, N) - g(ASX, N) - g(AN, SX)\}AY \\
 &\quad - g(AN, N)\{q(X)JAY + g(SX, Y)AN\} \\
 &\quad + \{q(X)g(JAY, N) + g(SX, Y)g(AN, N)\}AN \\
 &\quad - g(AY, SX)AN + g(AY, N)\{(\bar{\nabla}_X A)N + A\bar{\nabla}_X N\} \\
 &\quad + g((\bar{\nabla}_X A)Y, \xi)A\xi + g(AY, \phi SX + \sigma(X, \xi))A\xi \\
 &\quad + g(AY, \xi)\bar{\nabla}_X(A\xi),
 \end{aligned}$$

where we have used the equation of Gauss  $\bar{\nabla}_X \xi = \nabla_X \xi + \sigma(X, \xi)$ ,  $\sigma(X, \xi)$  denotes the normal bundle  $T^\perp M$  valued second fundamental tensor on  $M$  in  $Q^{m*}$ . From this, putting  $Y = \xi$  and using  $(\bar{\nabla}_X A)Y = q(X)JAY$ , and  $\bar{\nabla}_X N = -SX$  we have

$$\begin{aligned}
 (4.4) \quad & \nabla_{\bar{R}_N(\xi)} X - \bar{R}_N(\nabla_\xi X) \\
 &= (\nabla_X \bar{R}_N)\xi = -3\phi SX \\
 &\quad - \{q(X)g(JAN, N) - g(ASX, N) - g(AN, SX)\}A\xi \\
 &\quad - g(AN, N)\{q(X)JA\xi + g(SX, \xi)AN\} \\
 &\quad + \{q(X)g(JA\xi, N) + g(SX, \xi)g(AN, N)\}AN \\
 &\quad - g(A\xi, SX)AN + g(q(X)JA\xi, \xi)A\xi \\
 &\quad + g(A\xi, \phi SX + \sigma(X, \xi))A\xi \\
 &\quad + g(A\xi, \xi)\{q(X)JA\xi + A\phi SX + g(SX, \xi)\}AN.
 \end{aligned}$$

From this, by taking the inner product with the unit normal  $N$ , we have

$$\begin{aligned}
 (4.5) \quad & -g(A\xi, SX)g(AN, N) + g(A\xi, \xi)\{q(X)g(JA\xi, N) \\
 & \quad + g(A\phi SX, N) + g(SX, \xi)g(AN, N)\} = 0.
 \end{aligned}$$

Then by putting  $X = \xi$  and using the assumption of Hopf, we have

$$(4.6) \quad q(\xi)g(A\xi, \xi)^2 = 0.$$

This gives that  $q(\xi) = 0$  or  $g(A\xi, \xi) = 0$ . The latter case implies that the unit normal  $N$  is  $\mathfrak{A}$ -isotropic. Now we only consider the case  $q(\xi) = 0$ .

We put  $Y = \xi$  in (4.1). Then it follows that

$$\bar{R}_N(\xi) = -4\xi - \{g(AN, N) - g(A\xi, \xi)\}A\xi = -4\xi - 2g(AN, N)A\xi,$$

where we have used that  $g(A\xi, \xi) = g(AJN, JN) = -g(JAN, JN) = -g(AN, N)$ .

Differentiating this one, it follows that

$$\begin{aligned}
 (4.7) \quad \nabla_{\bar{R}_N(\xi)} X - \bar{R}_N(\nabla_\xi X) &= (\nabla_X \bar{R}_N) \xi \\
 &= -4\nabla_X \xi - 2\{g(\bar{\nabla}_X(AN), N)A\xi + g(AN, \bar{\nabla}_X N)A\xi\} \\
 &\quad - 2g(AN, N)\bar{\nabla}_X(A\xi).
 \end{aligned}$$

Then, by putting  $Y = \xi$ , and taking the inner product of (4.7) with the unit normal  $N$ , we have

$$g(AN, N)\{q(\xi)g(A\xi, \xi) - \alpha g(A\xi, \xi)\} = 0.$$

From this, together with  $q(\xi) = 0$ , it follows that

$$(4.8) \quad \alpha g(A\xi, \xi)g(AN, N) = 0.$$

Then from (4.8) we can assert the following lemma.

**Lemma 4.1.** *Let  $M$  be a Hopf real hypersurface in the complex hyperbolic quadric  $Q^{m*}$ ,  $m \geq 3$ , with parallel normal Jacobi operator. Then the unit normal vector field  $N$  is  $\mathfrak{A}$ -principal or  $\mathfrak{A}$ -isotropic.*

*Proof.* When the Reeb function  $\alpha$  is non-vanishing, the unit normal  $N$  is  $\mathfrak{A}$ -isotropic. When the Reeb function  $\alpha$  identically vanishes, let us show that  $N$  is  $\mathfrak{A}$ -isotropic or  $\mathfrak{A}$ -principal. In order to do this, from the condition of Hopf, we can differentiate  $S\xi = \alpha\xi$  and use the equation of Codazzi (3.1) in section 3, then we get the formula

$$Y\alpha = (\xi\alpha)\eta(Y) - 2g(\xi, AN)g(Y, A\xi) + 2g(Y, AN)g(\xi, A\xi).$$

From this, if we put  $\alpha = 0$ , together with the fact  $g(\xi, AN) = 0$  in section 3, we know  $g(Y, AN)g(\xi, A\xi) = 0$  for any  $Y \in T_z M$ ,  $z \in M$ . This gives that the vector  $AN$  is normal, that is,  $AN = g(AN, N)N$  or  $g(A\xi, \xi) = 0$ , which implies respectively the unit normal  $N$  is  $\mathfrak{A}$ -principal or  $\mathfrak{A}$ -isotropic. This completes the proof of our Lemma.  $\square$

By virtue of this Lemma, we distinguish between two classes of real hypersurfaces in the complex hyperbolic quadric  $Q^{m*}$  with invariant normal Jacobi operator : those that have  $\mathfrak{A}$ -principal unit normal, and those that have  $\mathfrak{A}$ -isotropic unit normal vector field  $N$ . We treat the respective cases in sections 5 and 6.

## 5. Invariant Normal Jacobi Operator with $\mathfrak{A}$ -principal Normal

In this section let us consider a real hypersurface  $M$  in the complex hyperbolic quadric  $Q^{m*}$  with  $\mathfrak{A}$ -principal unit normal vector field. Then the unit normal vector field  $N$  satisfies  $AN = N$  for a complex conjugation  $A \in \mathfrak{A}$ . This also implies that  $A\xi = -\xi$  for the Reeb vector field  $\xi = -JN$ .

Then the normal Jacobi operator  $\bar{R}_N$  in section 4 becomes

$$(5.1) \quad \bar{R}_N(X) = -X - 3\eta(X)\xi - AX + \eta(X)\xi = -X - 2\eta(X)\xi - AX,$$

where we have used that  $AN = N$  and

$$\begin{aligned} g(AX, \xi)A\xi &= g(AX, JN)AJN = g(X, AJN)AJN \\ &= g(X, JAN)JAN = g(X, JN)JN \\ &= \eta(X)\xi. \end{aligned}$$

On the other hand, we can put

$$AY = BY + \rho(Y)N,$$

where  $BY$  denotes the tangential component of  $AY$  and  $\rho(Y) = g(AY, N) = g(Y, AN) = g(Y, N) = 0$ . So it becomes always  $AY = BY$  for any vector field  $Y$  on  $M$  in  $Q^{m*}$ . Then by differentiating (5.1) along any direction  $X$ , we have

$$\begin{aligned} (5.2) \quad (\nabla_X \bar{R}_N)Y &= \nabla_X(\bar{R}_N(Y)) - \bar{R}_N(\nabla_X Y) \\ &= -2(\nabla_X \eta)(Y) - 2\eta(Y)\nabla_X \xi - (\nabla_X B)Y. \end{aligned}$$

Now let us consider that the normal Jacobi operator  $\bar{R}_N$  is invariant, that is,  $\mathcal{L}_X \bar{R}_N = 0$ . This is given by

$$\begin{aligned} 0 &= (\mathcal{L}_X \bar{R}_N)Y \\ &= \mathcal{L}_X(\bar{R}_N Y) - \bar{R}_N(\mathcal{L}_X Y) \\ &= [X, \bar{R}_N Y] - \bar{R}_N[X, Y] \\ &= \nabla_X(\bar{R}_N Y) - \nabla_{\bar{R}_N(Y)} X - \bar{R}_N(\nabla_X Y - \nabla_Y X) \\ &= \nabla_X(\bar{R}_N Y) - \nabla_{\bar{R}_N(Y)} X + \bar{R}_N(\nabla_Y X). \end{aligned}$$

Then it follows that

$$\begin{aligned} -2g(\phi SX, Y)\xi - 2\eta(Y)\phi SX - (\nabla_X B)Y &= \{\nabla_Y X + 2\eta(\nabla_Y X)\xi + A\nabla_Y X\} \\ &\quad - \{\nabla_Y X + 2\eta(Y)\nabla_\xi X + \nabla_{AY} X\}. \end{aligned}$$

From this putting  $Y = \xi$  and using  $A\xi = -\xi$ , it follows that

$$\begin{aligned} (5.3) \quad -2\phi SX - (\nabla_X B)\xi &= 2\eta(\nabla_\xi X)\xi + A\nabla_\xi X + \nabla_\xi X \\ &= -2\phi SX - \{q(X)JA\xi - \sigma(X, A\xi) + \eta(SX)N\}. \end{aligned}$$

where we have used the following

$$\begin{aligned} (\nabla_X A)\xi &= \nabla_X(A\xi) - A\nabla_X \xi \\ &= \bar{\nabla}_X(A\xi) - A\nabla_X \xi \\ &= \{(\bar{\nabla}_X A)\xi + A\bar{\nabla}_X \xi\} - A\phi SX \\ &= q(X)JA\xi + A\phi SX - \sigma(X, A\xi) + g(SX, \xi)AN - A\phi SX \\ &= q(X)JA\xi - \sigma(X, A\xi) + \alpha\eta(X)N. \end{aligned}$$

Then by taking the inner product of (5.3) with the unit normal  $N$ , we have

$$q(X) = 2\alpha\eta(X).$$

This implies  $q(\xi) = 2\alpha$ , and the 1-form  $q$  is given by

$$(5.4) \quad q(X) = q(\xi)\eta(X).$$

On the other hand, in section 4 from the Lie invariance of the normal Jacobi operator we have calculated the following

$$(5.5) \quad \begin{aligned} & \nabla_{\bar{R}_N(\xi)}X - \bar{R}_N(\nabla_\xi X) \\ &= (\nabla_X \bar{R}_N)\xi = -3\phi SX \\ & \quad - \{q(X)g(JAN, N) - g(ASX, N) - g(AN, SX)\}A\xi \\ & \quad - g(AN, N)\{q(X)JA\xi + g(SX, \xi)AN\} \\ & \quad + \{q(X)g(JA\xi, N) + g(SX, \xi)g(AN, N)\}AN \\ & \quad - g(A\xi, SX)AN + g(q(X)JA\xi, \xi)A\xi \\ & \quad + g(A\xi, \phi SX + \sigma(X, \xi))A\xi \\ & \quad + g(A\xi, \xi)\{q(X)JA\xi + A\phi SX + g(SX, \xi)AN\}. \end{aligned}$$

From this, by taking the inner product with the unit normal  $N$ , we have

$$(5.6) \quad \begin{aligned} & -g(A\xi, SX)g(AN, N) + g(A\xi, \xi)\{q(X)g(JA\xi, N) \\ & \quad + g(A\xi, \xi)\{g(JA\xi, N) + g(A\phi SX, N) + g(SX, \xi)g(AN, N)\} = 0. \end{aligned}$$

Then by putting  $X = \xi$  and using the assumption of Hopf, we have

$$(5.7) \quad q(\xi)g(A\xi, \xi)^2 = 0.$$

From this, together with (5.4) and  $A\xi = -\xi$ , it follows that the 1-form  $q$  vanishes identically on  $M$ .

On the other hand, we know that the complex hyperbolic quadric  $Q^{m*}$  can be immersed into the indefinite complex hyperbolic space  $CH_1^{m+1}$  in  $C_2^{m+2}$  (see Montiel and Romero [12], and Kobayashi and Nomizu [11]). Then the same 1-form  $q$  appears in the Weingarten formula

$$\tilde{\nabla}_X \bar{z} = -A_{\bar{z}}X + q(X)J\bar{z}$$

for unit normal vector fields  $\{\bar{z}, J\bar{z}\}$  on the complex hyperbolic quadric  $Q^{m*}$  which can be immersed in indefinite complex hyperbolic space  $CH_1^{m+1}$  as a space-like complex hypersurface, where  $\tilde{\nabla}$  denotes the Riemannian connection on  $CH_1^{m+1}$  induced from the Euclidean connection on  $C_2^{m+2}$  (see Smyth [19] and [20]). But the 1-form  $q$  never vanishes on  $Q^{m*}$ . This gives a contradiction (see Smyth [19]). This means that there do not exist any real hypersurfaces in the complex hyperbolic



quadric  $Q^{m*}$  with invariant normal Jacobi operator, that is,  $\mathcal{L}_X \bar{R}_N = 0$  for the  $\mathfrak{A}$ -principal unit normal vector field  $N$ .

## 6. Invariant Normal Jacobi Operator with $\mathfrak{A}$ -isotropic Normal

In this section let us assume that the unit normal vector field  $N$  is  $\mathfrak{A}$ -isotropic. Then the normal vector field  $N$  can be put

$$N = \frac{1}{\sqrt{2}}(Z_1 + JZ_2)$$

for  $Z_1, Z_2 \in V(A)$ , where  $V(A)$  denotes a  $(+1)$ -eigenspace of the complex conjugation  $A \in \mathfrak{A}$ . Then it follows that

$$AN = \frac{1}{\sqrt{2}}(Z_1 - JZ_2), AJN = -\frac{1}{\sqrt{2}}(JZ_1 + Z_2), \text{ and } JN = \frac{1}{\sqrt{2}}(JZ_1 - Z_2).$$

From this, together with (2.2) and the anti-commuting  $AJ = -JA$ , it follows that

$$g(\xi, A\xi) = g(JN, AJN) = 0, \quad g(\xi, AN) = 0 \text{ and } g(AN, N) = 0.$$

By virtue of these formulas for the  $\mathfrak{A}$ -isotropic unit normal, the normal Jacobi operator  $\bar{R}_N$  in section 4 is given by

$$\bar{R}_N(Y) = -Y - 3\eta(Y)\xi + g(AY, N)AN + g(AY, \xi)A\xi.$$

Then the derivative of the normal Jacobi operator  $\bar{R}_N$  on  $M$  is given as follows:

$$\begin{aligned} (6.1) \quad (\nabla_X \bar{R}_N)Y &= -3(\nabla_X \eta)(Y)\xi - 3\eta(Y)\nabla_X \xi + g(\nabla_X(AN), Y)AN \\ &\quad + g(AN, Y)\nabla_X(AN) + g(Y, \nabla_X(A\xi))A\xi + g(A\xi, Y)\nabla_X(A\xi). \end{aligned}$$

On the other hand, the Lie invariance (4.1) gives that

$$\begin{aligned} (6.2) \quad (\nabla_X \bar{R}_N)Y &= \nabla_X(\bar{R}_N(Y)) - \bar{R}_N(\nabla_X Y) \\ &= \nabla_{\bar{R}_N(Y)} X - \bar{R}_N(\nabla_Y X). \end{aligned}$$

Then by putting  $Y = \xi$  in (6.1) and (6.2), and using  $\bar{R}_N(\xi) = 4\xi$ , we have

$$\begin{aligned} (6.3) \quad -3\phi SX - g(AN, \phi SX)AN - g(\phi SX, A\xi)A\xi \\ = -4\nabla_\xi X + \{\nabla_\xi X + 3\eta(\nabla_\xi X)\xi \\ - g(A\nabla_\xi X, N)AN - g(A\nabla_\xi X, \xi)A\xi\} \end{aligned}$$

From this, taking the inner product (6.3) with the vector field  $AN$ , it follows that

$$4g(\phi SX, AN) = 4g(\nabla_\xi X, AN).$$

Then from this, together with (6.3), we get

$$(6.4) \quad \phi SX = \nabla_\xi X - \eta(\nabla_\xi X)\xi.$$

For any  $X \in \xi^\perp$ , where  $\xi^\perp$  denotes the orthogonal complement of the Reeb vector field  $\xi$  in the tangent space  $T_z M$ ,  $z \in M$ , we know that  $\nabla_\xi X$  is orthogonal to the Reeb vector field  $\xi$ , that is,  $\eta(\nabla_\xi X) = -g(\nabla_\xi \xi, X) = 0$ . Then the formula (6.4) becomes for any tangent vector field  $X \in \xi^\perp$

$$(6.5) \quad \phi SX = \nabla_\xi X.$$

When we consider the  $\mathfrak{A}$ -isotropic unit normal, the vector fields  $A\xi$  and  $AN$  belong to the distribution  $\mathfrak{C} - \mathfrak{Q}$  in section 3.

On the other hand, by virtue of Lemma 3.1, we prove the following for a Hopf hypersurface in  $Q^{m*}$  with  $\mathfrak{A}$ -isotropic unit normal vector field as follows:

**Lemma 6.1.** *Let  $M$  be a Hopf real hypersurface in the complex hyperbolic quadric  $Q^{m*}$ ,  $m \geq 3$ , with  $\mathfrak{A}$ -isotropic unit normal vector field. Then*

$$(6.6) \quad SAN = 0, \quad \text{and} \quad SA\xi = 0.$$

*Proof.* Let us denote by  $\mathfrak{C} - \mathfrak{Q} = \text{Span}[A\xi, AN]$ . Since  $N$  is isotropic,  $g(AN, N) = 0$  and  $g(A\xi, \xi) = 0$ . By differentiating  $g(AN, N) = 0$ , and using  $(\bar{\nabla}_X A)Y = q(X)JAY$  in the introduction and the equation of Weingarten, we know that

$$\begin{aligned} 0 &= g(\bar{\nabla}_X(AN), N) + g(AN, \bar{\nabla}_X N) \\ &= g(q(X)JAN - ASX, N) - g(AN, SX) \\ &= -2g(ASX, N). \end{aligned}$$

Then  $SAN = 0$ . Moreover, by differentiating  $g(A\xi, N) = 0$ , and using  $g(AN, N) = 0$  and  $g(A\xi, \xi) = 0$ , we have the following formula

$$\begin{aligned} 0 &= g(\bar{\nabla}_X(A\xi), N) + g(A\xi, \bar{\nabla}_X N) \\ &= g(q(X)JA\xi + A(\phi SX + g(SX, \xi)N), N) - g(SA\xi, X) \\ &= -2g(SA\xi, X) \end{aligned}$$

for any  $X \in T_z M$ ,  $z \in M$ , where in the third equality we have used  $\phi AN = JAN = -AJN = A\xi$ . Then it follows that

$$SA\xi = 0.$$

It completes the proof of our assertion.  $\square$

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