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A Note on Unavoidable Sets for a Spherical Curve of Reductivity Four

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Abstract. The reductivity of a spherical curve is the minimal number of times a particular local transformation called an inverse-half-twisted splice is required to obtain a reducible spherical curve from the initial spherical curve. It is unknown if there exists a spherical curve whose reductivity is four. In this paper, an unavoidable set of configurations for a spherical curve with reductivity four is given by focusing on 5-gons. It has also been unknown if there exists a reduced spherical curve which has no 2-gons and 3-gons of type A, B and C. This paper gives the answer to this question by constructing such a spherical curve.

1. Introduction

A *spherical curve* is a closed curve on S^2, where self-intersections, called *crossings*, are double points intersecting transversely. In this paper, spherical curves are considered up to ambient isotopy of S^2, and two spherical curves which are transformed into each other by a reflection are assumed to be the same spherical curve. A spherical curve is *trivial* if it has no crossings. A spherical curve P is *reducible* if one can draw a circle on S^2 which intersects P transversely at just one crossing of P. Otherwise, it is said to be *reduced.* An *inverse-half-twisted splice*, denoted by HS^{-1}, at a crossing of a spherical curve P is a splice on P which yields another

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spherical curve (not a link projection) as shown in Figure 1. An inverse-half-twisted

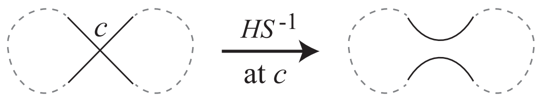

Figure 1: An inverse-half-twisted splice operation at a crossing c. Broken curves represent the outer connections.

splice does not preserve an orientation of a spherical curve. Hence HS^{-1} is a different local transformation from the splice called a "smoothing" in knot theory. Results from [3] show that for every pair of two nontrivial reduced spherical curves P and P', there exists a finite sequence of HS^{-1}s and its inverses which transform P into P' such that a spherical curve at each step of the sequence is also reduced. This implies that all nontrivial reduced spherical curves are connected by HS^{-1}s and its inverses. The *reductivity* of a nontrivial spherical curve P is defined to be the minimal number of inverse-half-twisted splices, HS^{-1}s, which are required to obtain a reducible spherical curve from P. The reductivity tells us how reduced a spherical curve is, like the connectivity in graph theory. In [6], it is shown that every nontrivial spherical curve has the reductivity four or less. Also, in [5] and [6], it is mentioned that there are infinitely many spherical curves with reductivity 0, 1, 2 and 3. We still don't know the answer to the following question:

Problem 1.1.([6]) *For any nontrivial spherical curve, is the reductivity three or less?*

In other words, it is unknown if there exists a spherical curve whose reductivity is four. An *unavoidable set* of configurations for a spherical curve in a class is a set of configurations with the property that any spherical curve in the class has at least one member of the set (see, for example, [2]). It is important to find unavoidable sets for a spherical curve of reductivity four from various viewpoints. In [6], 3-gons were classified into four types considering outer connections as shown in Figure 2 and the unavoidable set U_1, shown in Figure 3, of configurations with outer connections for a spherical curve with reductivity four was given. The unavoidable set U_1 was

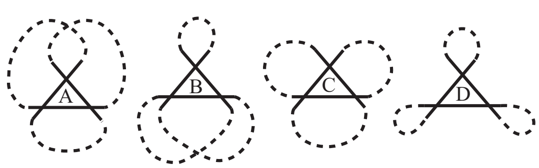

Figure 2: The 3-gons of type A, B, C and D. Broken curves represent outer connections.

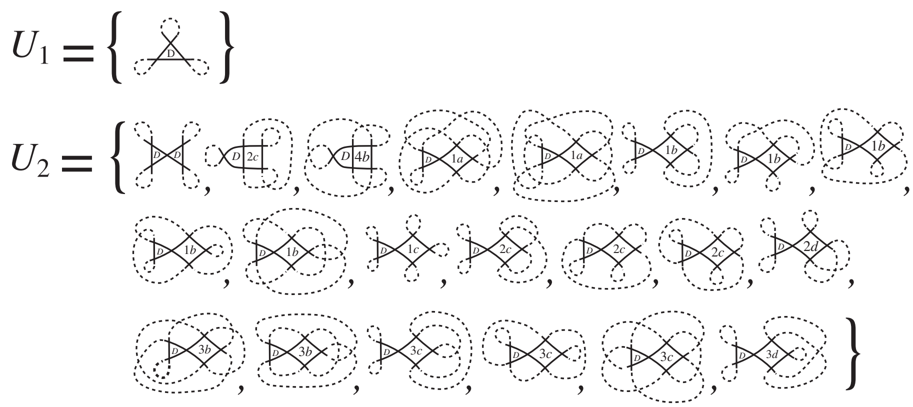

Figure 3: Unavoidable sets with outer connections U_1 and U_2 for a spherical curve of reductivity four. Broken curves represent outer connections.

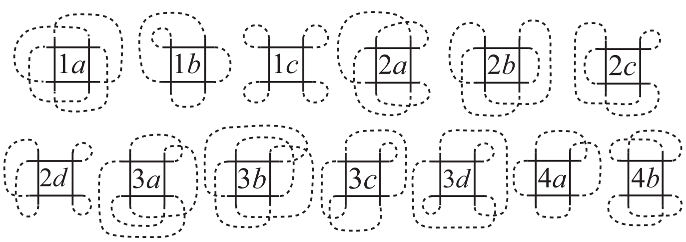

Figure 4: The 13 types of 4-gons.

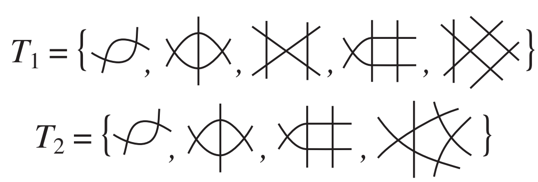

Figure 5: Unavoidable sets of configurations (with any outer connections) T_1 and T_2 for a reduced spherical curve.

obtained by the following two facts; the first one is that every nontrivial reduced spherical curve has a 2-gon or 3-gon [1]. The second one is that if a spherical curve has a 2-gon or a 3-gon of type A, B or C, then the reductivity is three or less [6]. The following problem was also posed in [6]:

Problem 1.2.([6]) *Is the set consisting of a 2-gon, 3-gons of type A, B and C an unavoidable set for a reduced spherical curve?*

If the answer to Problem 1.2 is "yes", then the answer to Problem 1.1 is also "yes". However, the following theorem gives the negative answer to Problem 1.2:

Theorem 1.3. *There exists a reduced spherical curve which has no 2-gons and 3-gons of type A, B and C.*

(See Figures 7 and 8 in Section 2.) In [5], 4-gons were classified into 13 types as shown in Figure 4 and the unavoidable set U_2, in Figure 3, for a spherical curve with reductivity four was given by combining 3-gons and 4-gons based on an unavoidable set T_1 in Figure 5 for a nontrivial reduced spherical curve which was obtained in [6] in the same way to the four-color-theorem. Note that a necessary condition for a spherical curve with reductivity four was also given using the notion of the warping degree in [4]. In this paper, 5-gons are classified in a systematic way which can be used for general n-gons (in Section 3) and another unavoidable set for a spherical curve with reductivity four is given:

Theorem 1.4. *The set U_3 shown in Figure 6 is an unavoidable set for a spherical curve with reductivity four.*

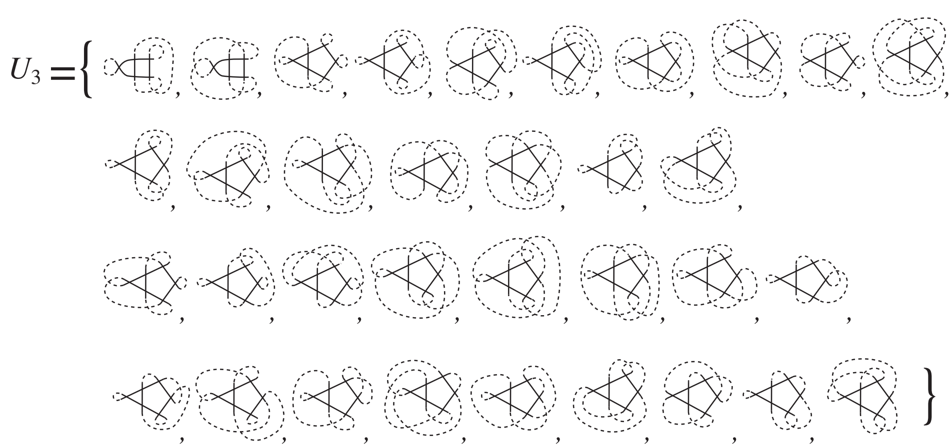

Figure 6: An unavoidable set of configurations with outer connections for a spherical curve of reductivity four.

Theorem 1.4 would be useful for constructing a spherical curve with reductivity four (or showing that there are no such spherical curves), or detecting the reductivity for spherical curves which have no 2-gons and 3-gons of type A, B and C. The rest of the paper is organized as follows: In Section 2, Theorem 1.3 is shown. In Section 3, 5-gons are classified into 56 types. In Section 4, Theorem 1.4 is proved. In Appendix, the 5-gons on chord diagrams are listed.

2. Proof of Theorem 1.3

In this section, Theorem 1.3 is shown.

Proof of Theorem 1.3. The spherical curves depicted in Figure 7 are reduced, and have no 2-gons and 3-gons of type A, B and C. The point is that there are no 2-gons, and all the 3-gons are of type D. □

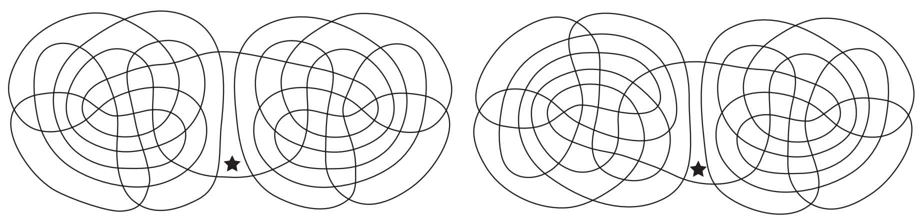

Figure 7: Reduced spherical curves without 2-gons, 3-gons of type A, B and C.

Note that the spherical curves shown in Figure 7 have the reductivity one, not four, because an inverse-half-twisted splice at a crossing at the middle 4-gons with a star derives a reducible spherical curve. Another example is shown in Figure 8. The reductivity of the spherical curve in Figure 8 is not four because it has a 4-gon, with a star in the figure, of type 4a; it is shown in [5] that if a spherical curve has a 4-gon of type 4a, then the reductivity is three or less.

In [6], a reduced spherical curve which has no 2-gons and 3-gons of type A and B was given. Further spherical curves are shown in Figure 9.

3. 5-gons

In this section, 5-gons are classified with respect to the outer connections by a systematic way which can be used for 6-gons or more:

Lemma 3.1. *5-gons of a spherical curve are divided into the 56 types in Figure 21 with respect to outer connections.*

Proof. There are four types of 5-gons when relative orientations of the five sides are considered. The 5-gons of type 1 to 4 are illustrated in Figure 10, where one of

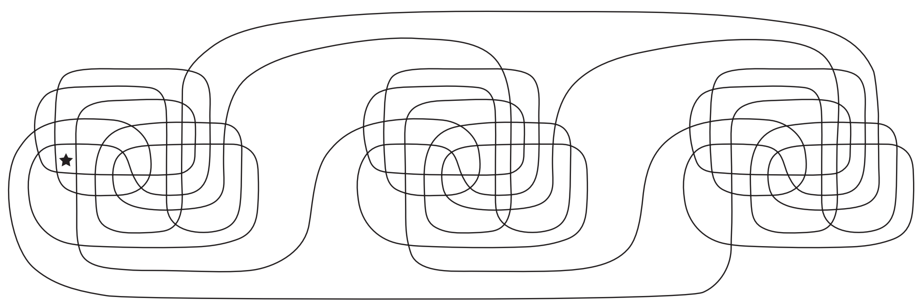

Figure 8: Reduced spherical curve without 2-gons, 3-gons of type A, B and C.

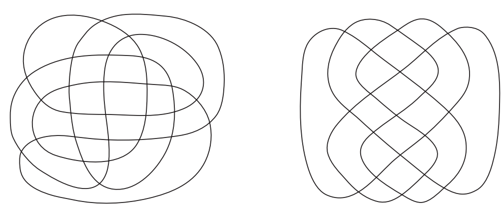

Figure 9: Reduced spherical curves without 2-gons, 3-gons of type A and B.

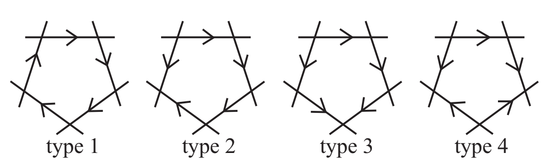

Figure 10: 5-gons of type 1 to 4 with relative orientations of the sides.

the relative orientations are shown by arrows. Let a, b, c, d and e be the sides of a 5-gon as illustrated in Figure 11. The 5-gon of type 1 has two types of symmetries:

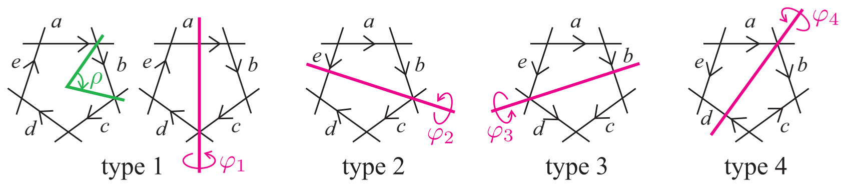

Figure 11: The rotation and reflections on the 5-gons. (One of the relative orientations is shown by arrows at each type.)

the $(2\pi/5)$-rotation ρ and the reflection φ_1 defined by the following permutations

$$\rho = \begin{pmatrix} a & b & c & d & e \\ b & c & d & e & a \end{pmatrix}, \ \varphi_1 = \begin{pmatrix} a & b & c & d & e \\ a & e & d & c & b \end{pmatrix}.$$

The 5-gons of type 2, 3 and 4 have the reflection symmetries φ_2, φ_3 and φ_4 defined by the following permutations, respectively:

$$\varphi_2 = \begin{pmatrix} a & b & c & d & e \\ d & c & b & a & e \end{pmatrix}, \ \varphi_3 = \begin{pmatrix} a & b & c & d & e \\ c & b & a & e & d \end{pmatrix}, \ \varphi_4 = \begin{pmatrix} a & b & c & d & e \\ b & a & e & d & c \end{pmatrix}.$$

Now let a 5-gon be a part of a spherical curve on S^2. Let a, b, c, d and e be sides of the 5-gon located as same as Figure 11. Fix the orientation of a as e to b. By reading the sides up as one passes the spherical curve, a cyclic sequence consisting of a, b, c, d and e is obtained. In particular, a sequence starting with a is called a *standard sequence*. With the type of relative orientations of the sides, a 5-gon with outer connections is represented by a sequence uniquely. There are $4! = 24$ standard sequences on each type, and we remark that there are some multiplicity by symmetries as a 5-gon of a spherical curve.

Type 1: A 5-gon of type 1 has two symmetries ρ and φ_1. Two cyclic sequences which can be transformed into each other by some ρs represent the same 5-gon with outer connections. For example, $abced$ and $aebcd$ represent the same 5-gon because $\rho(abced) = bcdae = aebcd$. Since the orientation is fixed, two sequences represent the same 5-gon when they are transformed into each other by a single φ_1 and orientation reversing (denoted by γ). For example, $abced$ and $acbde$ represent the same 5-gon because $\gamma(\varphi_1(abced)) = \gamma(aedbc) = cbdea = acbde$. There are 8 equivalent classes of standard sequences up to some ρs and a pair of φ_1 and γ:

$abcde$, $abced = abdce = acbde = acdeb = aebcd$,
$abdec = abecd = acdba = adbce = adebc$,
$abedc = adcbe = adecb = aecdb = aedbc$,

acebd, $acedb = acbed = adceb = aebdc = aecbd$, *adbec*, *aedcb*.

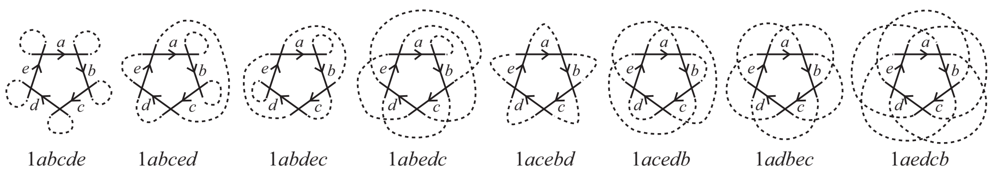

Figure 12: The 5-gons of type 1.

Type 2: A 5-gon of type 2 has the reflection symmetry φ_2. Two cyclic sequences represent the same 5-gon when they are transformed into each other by a single φ_2 and orientation reversing γ. There are 16 equivalent classes of standard sequences up to a pair of φ_2 and γ:

$abcde, abced = aebcd, abdce = acdeb, abdec = acdbe, abecd,$
$abedc = aecdb, acbde, acbed = aecbd, acebd, acedb = aebdc,$
$adbce = adebc, adbec, adcbe = adecb, adceb, aedbc, aedcb.$

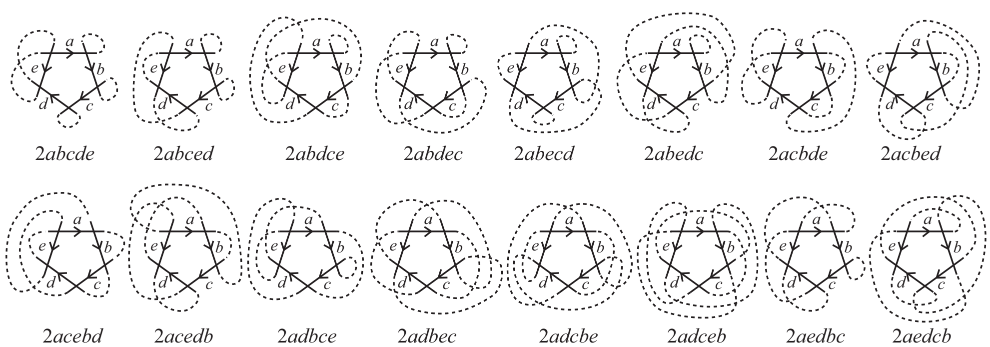

Figure 13: The 5-gons of type 2.

Type 3: Two cyclic sequences represent the same 5-gon when they are transformed into each other by a single φ_3 and orientation reversing γ. There are 16 equivalent classes of standard sequences up to a pair of φ_3 and γ:

$abcde, abced, abdce = aebcd, abdec = adebc, abecd = adbce,$
$abedc = aedbc, acbde = acdeb, acbed = acedb, acdbe, acebd, adbec,$

$adcbe = aecdb, adceb = aecbd, adecb, aebdc, aedcb.$

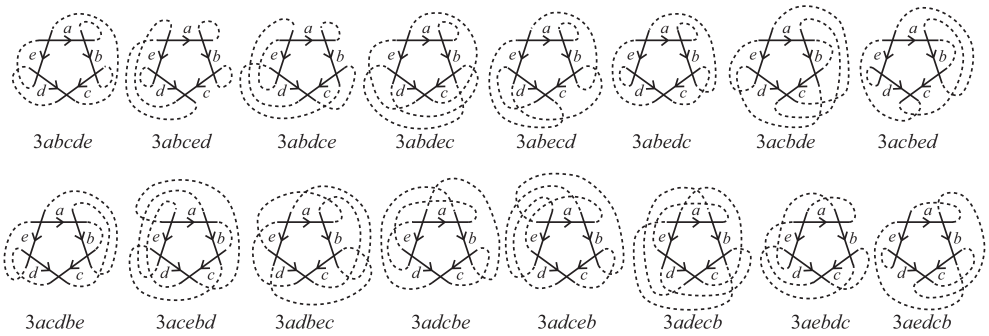

Figure 14: The 5-gons of type 3.

Type 4: Two cyclic sequences represent the same 5-gon when they are transformed into each other by a single φ_4 and orientation reversing γ. There are 16 equivalent classes of standard sequences up to a pair of φ_4 and γ:

$abcde, abced = abdce, abdec = abecd, abedc, acbde = aebcd,$
$acbed = aebdc, acdbe = adebc, acdeb, acebd, acedb = adceb,$
$adbce, adbec, adcbe = aedbc, adecb = aecdb, aecbd, aedcb.$

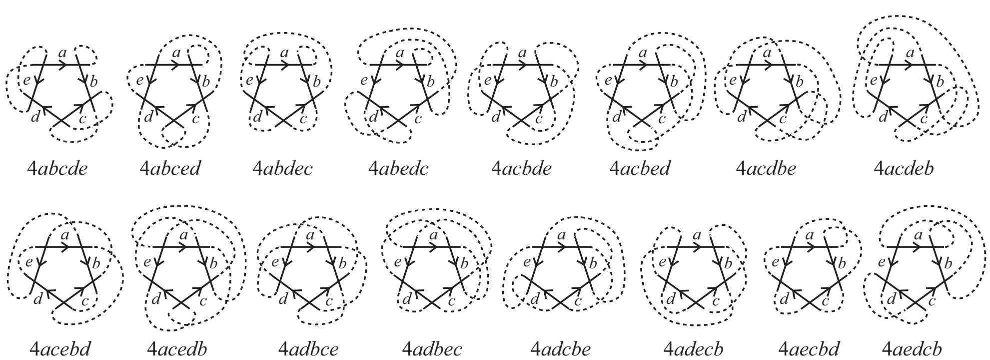

Figure 15: The 5-gons of type 4.

Thus, 5-gons are classified into the 56 types shown in Figure 21. □

4. Proof of Theorem 1.4

In this section, Theorem 1.4 is proved.

Proof of Theorem 1.4. Let P be a spherical curve with reductivity four. Since P is reduced, the set T_2 in Figure 5 is also an unavoidable set for P. Here, P can not have the first and second configuration because they make reductivity three or less as discussed in [5] and [6]. The third one of T_2 has already been discussed in Theorem 1 in [5]. Hence just the fourth one needs to be discussed here. Since the 3-gon should be of type D because 3-gons of type A, B and C make reductivity three or less, the 5-gon should be of type 2 or 4 with respect to the relative orientations of the sides (see Figure 16). Let a, b, c, d and e be the sides of a 5-gon of type 2

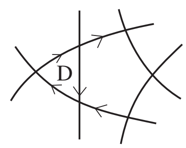

Figure 16: Type 2 and 4.

and 4 as same as Figures 13 and 15. When the 5-gon is of type 2, only the side e can be shared with the 3-gon. In this case, by considering the outer connections of the 3-gon of type D, the 5-gon should be the one whose sequence includes a, e, d with this cyclic order, which are the type of 2*abced*, 2*abecd*, 2*abedc*, 2*acbed*, 2*acebd*, 2*acedb*, 2*aedbc* and 2*aedcb*. Hence the eight configurations with outer connections illustrated in Figure 17 are obtained. When the 5-gon is of type 4, the sides e,

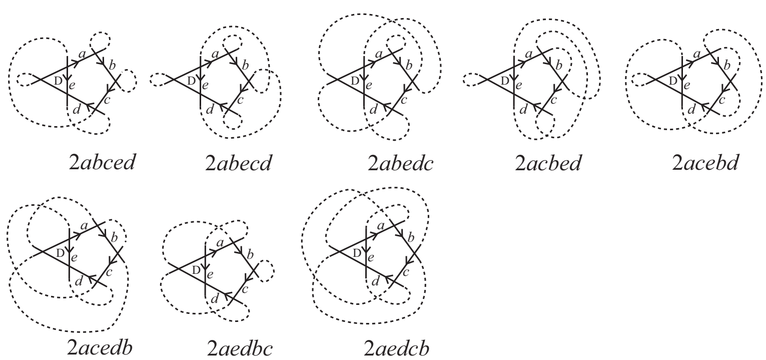

Figure 17: The case that a 3-gon of type D and a 5-gon of type 2 share the side e.

d and c can be shared with the 3-gon. When e is shared, the 5-gon should be the one whose sequence includes a, e, d with this cyclic order, which are the type of 4*abced*, 4*abedc*, 4*acbed*, 4*acebd*, 4*acedb*, 4*aecbd* and 4*aedcb*. Hence the seven configurations with outer connections in Figure 18 are obtained. When d is shared,

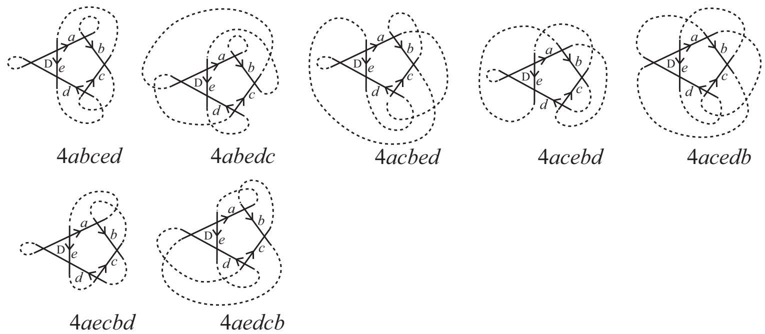

Figure 18: The case that a 3-gon of type D and a 5-gon of type 4 share the side e.

the 5-gon should be the one whose sequence includes c, d, e with this cyclic order, which are the type of 4*abcde*, 4*abdec*, 4*acbde*, 4*acdbe*, 4*acdeb*, 4*adbec*, 4*adecb* and 4*aecbd*. Hence the eight configurations with outer connections in Figure 19 are obtained. When c is shared, the 5-gon should be the one whose sequence includes

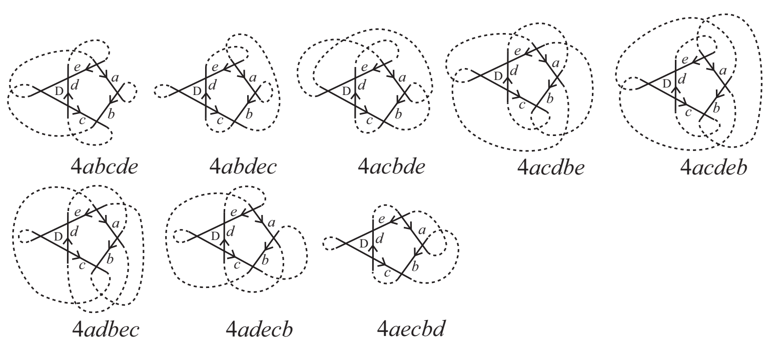

Figure 19: The case that a 3-gon of type D and a 5-gon of type 4 share the side d.

b, d, c with this cyclic order, which are the type of 4*abdec*, 4*abedc*, 4*acbde*, 4*acbed*, 4*acebd*, 4*adcbe*, 4*adecb*, 4*aecbd* and 4*aedcb*. Hence the nine configurations with outer connections in Figure 20 are obtained.
Hence, the set U_3 is an unavoidable set for a spherical curve of reductivity four. □

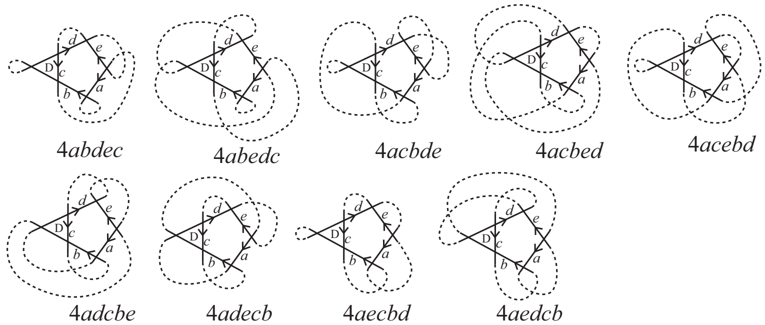

Figure 20: The case a 3-gon of type D and a 5-gon of type 4 share the side c.

Appendix: 5-gons on chord diagrams

A *chord diagram* of a spherical curve P is a preimage of P with each pair of points corresponding to the same double point connected by a segment as P is assumed to be an image of an immersion of a circle to S^2. In Figure 22, all the 5-gons of a spherical curve on chord diagrams are listed.

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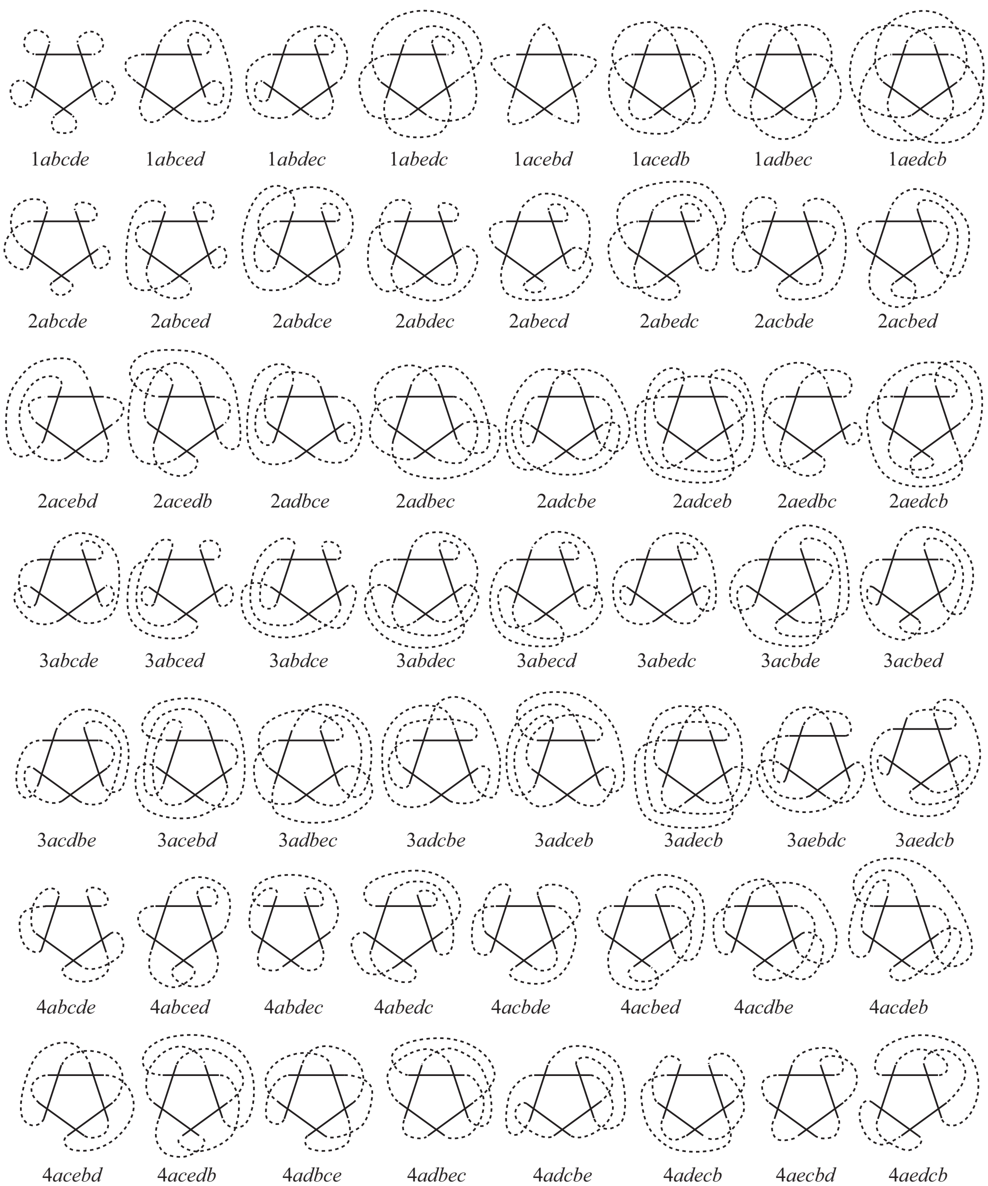

Figure 21: All the 5-gons of a spherical curve with outer connections.

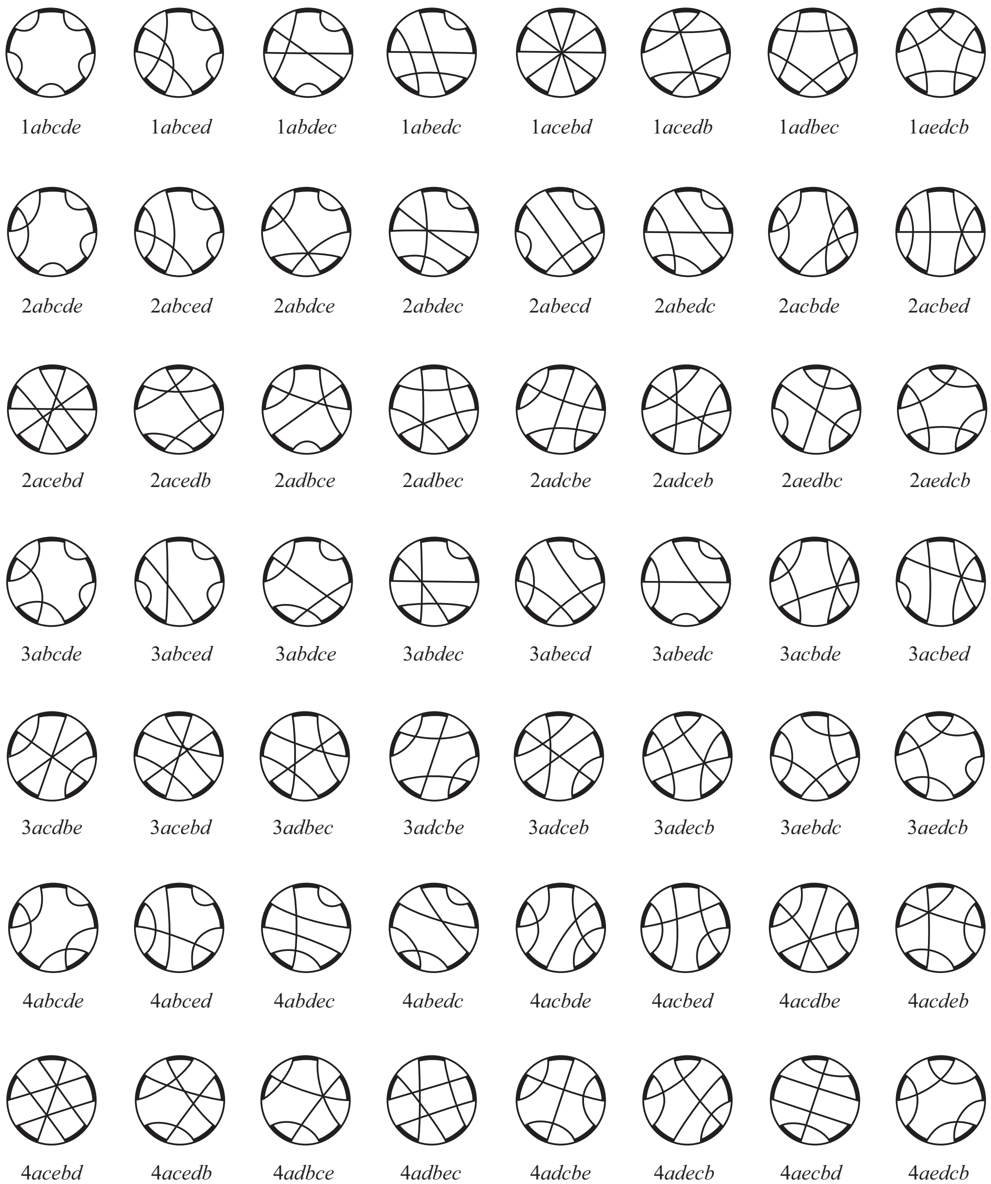

Figure 22: All the 5-gons on chord diagrams. There are no endpoints of segments in the interior of each thick arc.