KYUNGPOOK Math. J. 58(2018), 615-625
https://doi.org/10.5666/KMJ.2018.58.4.615
pISSN 1225-6951 eISSN 0454-8124

Certain Models of the Lie Algebra $\mathcal{K}_5$ and Their Connection with Special Functions

Sarasvati Yadav and Geeta Rani*
Department of Mathematics, NIT, Kurukshetra-136119, India
e-mail : sarasvati1201@nitkkr.ac.in *and* geetarani.5121992@gmail.com

Abstract. In this paper, we discuss the connection between the 5-dimensional complex Lie algebra $\mathcal{K}_5$ and Special functions. We construct certain two variable models of the irreducible representations of $\mathcal{K}_5$. We also use an Euler type integral transformation to obtain the new transformed models, in which the basis function appears as ${}_2F_1$. Further, we utilize these models to get some generating functions and recurrence relations.

1. Introduction

The study of Lie theory and its connection with special functions have been a rich source of interesting results in mathematical analysis. For example, authors such as Miller [4], Manocha [2], Manocha and Sahai [3], Sahai [6], Sahai and Yadav [7, 8], etc. have studied the Lie theory of several special functions and deformation of Lie algebras and special functions. The theory of special functions and group representations has been well discussed by the authors such as Manocha and Srivastava [10], Vilenkin [11], Wawrzynczyk [12] etc. Also, tools of the underlying theory have been studied in [1, 5, 9].

In the present paper, we study the connection between the 5-dimensional complex Lie algebra $\mathcal{K}_5$ and certain special functions. We construct new two variable models of the irreducible representations $\mathrm{R}(\omega, m_0, \mu)$ and $\uparrow_{\omega,\mu}$ of $\mathcal{K}_5$ corresponding to $\mu \neq 0$, where basis function appears in the form of ${}_1F_0$. We also use an Euler type integral transformation to obtain the transformed models of the irreducible representations $\mathrm{R}(\omega, m_0, \mu)$ and $\uparrow_{\omega,\mu}$ of $\mathcal{K}_5$. In these models, the basis functions appear in terms of ${}_2F_1$. As an application, these models result in several generating functions and recurrence relations. Section-wise treatment is as follows:

In Section 2, we present some prelimnaries. We define [4] the Lie algebra $\mathcal{K}_5$ and give its one variable model in the representations $\mathrm{R}(\omega, m_0, \mu)$ and $\uparrow_{\omega,\mu}$. We also take

* Corresponding Author.
Received July 15, 2018; revised October 28, 2018; accepted November 6, 2018.
2010 Mathematics Subject Classification: 22E60, 33C20.
Key words and phrases: Lie algebra, hypergeometric function.

an Euler type integral transformation to be used later on. In Section 3, we frame new two variable models of the irreducible representations of $\mathcal{K}_5$ in which the basis functions appear as ${}_1F_0(-n;-;x)y^n$ and ${}_1F_0(n;-;x)y^n$. To make our discussion more fruitful, we exponentiate these models into the local multiplier representations of the corresponding Lie group K_5. In Section 4, we obtain the transformed models by using the Euler type integral transformation, defined in Section 2. The basis functions for these models are given in terms of ${}_2F_1$. In Section 5, we derive several generating functions for ${}_1F_1$ and recurrence relations for ${}_2F_1$ that are believed to be new.

2. Prelimnaries

2.1. Lie Algebra $\mathcal{K}_5$

The complex Lie algebra $\mathcal{K}_5 = \mathrm{L}\{K_5\}$ as defined in [4] is the 5-dimensional complex Lie algebra with basis $\mathcal{J}^{\pm}$, $\mathcal{J}^3$, $\mathcal{Q}$, $\mathcal{E}$ and commutation relations:

$$\begin{aligned} &[\mathcal{J}^3, \mathcal{J}^{\pm}] = \pm\mathcal{J}^{\pm}, \quad [\mathcal{J}^3, \mathcal{Q}] = 2\mathcal{Q}, \\ &[\mathcal{J}^-, \mathcal{J}^+] = \mathcal{E}, \quad [\mathcal{J}^-, \mathcal{Q}] = 2\mathcal{J}^+, \quad [\mathcal{J}^+, \mathcal{Q}] = 0, \\ &[\mathcal{J}^+, \mathcal{E}] = [\mathcal{J}^-, \mathcal{E}] = [\mathcal{J}^3, \mathcal{E}] = [\mathcal{Q}, \mathcal{E}] = 0. \end{aligned} \tag{2.1}$$

Let $g \in K_5$ i.e. $g = \begin{bmatrix} 1 & ce^{\tau} & be^{-\tau} & 2a-bc & \tau \\ 0 & e^{\tau} & 2qe^{-\tau} & b-2qc & 0 \\ 0 & 0 & e^{-\tau} & -c & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$, where a, b, c, q, $\tau \in \mathbb{C}$.

The elements of $\mathcal{K}_5$ are of the type $i = \begin{bmatrix} 0 & \alpha_4 & \alpha_3 & \alpha_2 & \alpha_5 \\ 0 & \alpha_5 & 2\alpha_1 & \alpha_3 & 0 \\ 0 & 0 & -\alpha_5 & -\alpha_4 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$,

where $\alpha_1 = \frac{dq}{dt}$, $\alpha_2 = \frac{da}{dt}$, $\alpha_3 = \frac{db}{dt}$, $\alpha_4 = \frac{dc}{dt}$ and $\alpha_5 = \frac{d\tau}{dt}$ in the neighbourhood of identity. The basis elements of the algebra are:

$$\mathcal{Q} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathcal{E} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathcal{J}^+ = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathcal{J}^- = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathcal{J}^3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Let us consider an irreducible representation ρ of $\mathcal{K}_5$ on the vector space V and let

$$J^{\pm} = \rho(\mathcal{J}^{\pm}),\ J^3 = \rho(\mathcal{J}^3),\ Q = \rho(\mathcal{Q}),\ E = \rho(\mathcal{E}),$$

then the operators $J^{\pm}$, J^3, Q, E obey the commutation relations same as (2.1). Define the set of all eigenvalues of J^3 to be the spectrum S of J^3. Further, let the irreducible representation ρ satisfy the conditions:

(i) Each eigenvalue of J^3 has multiplicity equal to one.

(ii) There exists a denumerable basis for V consisting of all the eigenvalues of J^3.

This guarantees that S is denumerable and that there exists a basis for V consisting of vectors f_m such that $J^3 f_m = \mathrm{m} f_m$. From Miller [4], a one variable model of the irreducible representations is given by:

Representation $\mathbf{R}(\omega, m_0, \mu)$

(2.2)
$$\begin{aligned} &J^+ = \mu z, \\ &J^- = (m_0 + \omega) z^{-1} + \frac{d}{dz}, \\ &J^3 = m_0 + z\frac{d}{dz}, \\ &Q = \mu z^2, \\ &E = \mu, \\ &f_m(z) = z^n, \end{aligned}$$

where m∈S = $\{m_0$ + n : n is an integer$\}$.

Representation $\uparrow_{\omega,\mu}$

(2.3)
$$\begin{aligned} &J^+ = \mu z, \\ &J^- = \frac{d}{dz}, \\ &J^3 = z\frac{d}{dz} - \omega, \\ &Q = \mu z^2, \\ &E = \mu, \\ &f_m(z) = z^n, \end{aligned}$$

where m∈S =$\{-\omega$+ n : n is nonnegative integer$\}$.

For these representations there is a basis of V consisting of vectors f_m, defined for each m∈S, such that

$$
\begin{aligned}
J^3 f_m &= m f_m,\\
J^+ f_m &= \mu f_{m+1},\\
J^- f_m &= (m+\omega) f_{m-1},\\
E f_m &= \mu f_m,\\
Q f_m &= \mu f_{m+2}.
\end{aligned} \tag{2.4}
$$

2.2. Euler Integral Transformation

Let us consider the complex vector space V of all functions $f(x)$, representable as a power series about $x = 0$. We use

$$
\begin{aligned}
h(\beta,\gamma) &= I[f(x)]\\
&= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 x^{\beta-1}(1-x)^{\gamma-\beta-1} f(x)dx,
\end{aligned} \tag{2.5}
$$

$\text{Re}\ \gamma > \text{Re}\ \beta > 0.$

Then the isomorphic image of V under the transformation I:f(x) $\to$ h(β, γ) is W=IV. Now according to our requirement, we obtain transform of some expressions under the transformation I in terms of difference operators defined as

$$
\begin{aligned}
E_\beta h(\beta,\gamma) &= h(\beta+1,\gamma),\\
L_\beta h(\beta,\gamma) &= h(\beta-1,\gamma),\\
\Delta_\beta h(\beta,\gamma) &= (E_\beta - 1)h(\beta,\gamma).
\end{aligned} \tag{2.6}
$$

The following are the transforms under I:

$$
\begin{aligned}
I[xf] &= \frac{\beta}{\gamma} E_{\gamma\beta}.h\\
I[x\partial_x f] &= \beta \Delta_\beta .h\\
I[(1-x)f] &= \frac{(\gamma-\beta)}{\gamma} E_\gamma .h\\
I[(1-x)^{-1} f] &= \frac{(\gamma-1)}{(\gamma-\beta-1)} L_\gamma .h.
\end{aligned} \tag{2.7}
$$

3. Two Variable Models

We give below the two-variable models of the Lie algebra $\mathcal{K}_5$. Also, given along with are the multiplier representations of the local Lie group K_5 induced by the operators on $\mathcal{F}$, the space of all analytic functions in a neighbourhood of (x_0, y_0).

Model IA
Representation R(ω, m_0, μ):

$$J^- = (m_0+\omega)y^{-1}(1-x)^{-1} + \frac{\partial}{\partial y} - y^{-1}x\frac{\partial}{\partial x}, \tag{3.1}$$
$$J^3 = m_0 + y\frac{\partial}{\partial y},$$
$$J^+ = \mu y(1-x),$$
$$Q = \mu y^2(1-x)^2,$$
$$E = \mu,$$
$$f_m(x,y) = {}_1F_0(-n;-;x)y^n,$$

where $m \in S = \{m_0 + n : n \text{ is an integer}\}$.

The multiplier representation $T_1(g)f$ of the Lie group K_5 induced by the operators on $\mathcal{F}$ is

$$[T_1(g)f](x,y) = \exp\left(\mu(y^2(1-x)^2q + a + y(1-x)b) + m_0\tau\right) \times \left(1+\frac{c}{y(1-x)}\right)^{m_0+\omega} f\left(\frac{xy}{y+c}, (y+c)e^{\tau}\right), \tag{3.2}$$

where $|c/y(1-x)| < 1$ and g$\in \mathcal{K}_5$ lies in a small enough neighbourhood of **e**.

Model IIA
Representation R$(\omega,\ m_0,\ \mu)$:

$$J^- = (m_0+\omega)y^{-1}(1-x) + \frac{\partial}{\partial y} - y^{-1}x(1-x)\frac{\partial}{\partial x}, \tag{3.3}$$
$$J^3 = m_0 + y\frac{\partial}{\partial y},$$
$$J^+ = \mu y(1-x)^{-1},$$
$$Q = \mu y^2(1-x)^{-2},$$
$$E = \mu,$$
$$f_m(x,y) = {}_1F_0(n;-;x)y^n,$$

where $m \in S = \{m_0 + n : n \text{ is an integer}\}$.

The multiplier representation $T_2(g)f$ of the Lie group K_5 induced by the operators on $\mathcal{F}$ is

$$[T_2(g)f](x,y) = \exp(\mu(y^2(1-x)^{-2}q + a + y(1-x)^{-1}b) + m_0\tau) \times \left(\frac{y+c(1-x)}{y}\right)^{m_0+\omega} f\left(\frac{xy}{y+c(1-x)}, (y+c)e^{\tau}\right), \tag{3.4}$$

where $|c(1-x)/y| < 1$ and $g \in \mathcal{K}_5$ lies in a small enough neighbourhood of **e**.

Model IIIA
Representation $\uparrow_{\omega,\mu}$:

$$\begin{aligned}
J^- &= \frac{\partial}{\partial y} - y^{-1}x\frac{\partial}{\partial x}, \\
J^3 &= y\frac{\partial}{\partial y} - \omega, \\
J^+ &= \mu y(1-x), \\
Q &= \mu y^2(1-x)^2, \\
E &= \mu, \\
f_m(x,y) &= {}_1F_0(-n;-;x)y^n,
\end{aligned} \tag{3.5}$$

where $m \in S = \{-\omega + n : n \text{ is nonnegative integer}\}$.

The multiplier representation $T_3(g)f$ of the Lie group K_5 induced by the operators on $\mathcal{F}$ is

$$[T_3(g)f](x,y) = \exp(\mu(y^2(1-x)^2q + a + y(1-x)b) - \omega\tau) f\left(\frac{xy}{y+c}, (y+c)e^{\tau}\right), \tag{3.6}$$

where g$\in\mathcal{K}_5$ lies in a small enough neighbourhood of **e**.

Model IVA
Representation $\uparrow_{\omega,\mu}$:

$$\begin{aligned}
J^- &= \frac{\partial}{\partial y} - y^{-1}x(1-x)\frac{\partial}{\partial x}, \\
J^3 &= y\frac{\partial}{\partial y} - \omega, \\
J^+ &= \mu y(1-x)^{-1}, \\
Q &= \mu y^2(1-x)^{-2}, \\
E &= \mu, \\
f_m(x,y) &= {}_1F_0(n;-;x)y^n,
\end{aligned} \tag{3.7}$$

where $m \in S = \{-\omega + n : n \text{ is nonnegative integer}\}$.

The multiplier representation $T_4(g)f$ of the Lie group K_5 induced by the operators on $\mathcal{F}$ is

$$\begin{aligned}
[T_4(g)f](x,y) &= \exp(\mu(y^2(1-x)^{-2}q + a + y(1-x)^{-1}b) - \omega\tau) \\
&\quad \times f\left(\frac{xy}{y+c(1-x)}, (y+c)e^{\tau}\right),
\end{aligned} \tag{3.8}$$

where g$\in\mathcal{K}_5$ lies in a sufficiently small neighbourhood of **e**.

4. Transformed Models Of Lie Algebra $\mathcal{K}_5$

To get the models, with the basis function appearing as hypergeometric functions, we reproduce a theorem as in [6]:

Theorem 4.1. *Let the basis operators* $\{J^{\pm},\ J^3,\ Q,\ E\}$ *on a representation space* V *with basis functions* $\{f_m : m\in S\}$ *gives the irreducible representation* ρ *of the Lie algebra* $\mathcal{K}_5$*. Then the transformation* I *induces another irreducible representation* σ *of* $\mathcal{K}_5$ *on the representation space* $W = IV$ *having basis functions* $\{h_m : m\in S\}$ *in terms of operators* $\{K^{\pm},\ K^3,\ Q_1,\ E_1\}$*, where*

$$K^+ = IJ^+I^-, \quad K^- = IJ^-I^-, \quad K^3 = IJ^3I^- \tag{4.1}$$
$$Q_1 = IQI^-, \qquad E_1 = IEI^-$$
$$h_m = If_m, \quad m \in S,$$

i.e. ρ *and* σ *are isomorphic.*

We give below the transforms of models IA, IIA, IIIA and IVA introduced in section 3.

Model IB

$$K^- = (m_0+\omega)y^{-1}\left(\frac{\gamma-1}{\gamma-\beta-1}\right)L_\gamma + \frac{d}{dy} - y^{-1}\beta\Delta_\beta, \tag{4.2}$$
$$K^+ = \mu y\left(\frac{\gamma-\beta}{\gamma}\right)E_\gamma,$$
$$K^3 = m_0 + y\frac{d}{dy},$$
$$Q_1 = \mu y^2\frac{(\gamma-\beta)(\gamma-\beta+1)}{\gamma(\gamma+1)}E_\gamma^2,$$
$$E_1 = \mu,$$
$$h_m(\beta,\gamma,y) = {}_2F_1(-n,\beta;\gamma;1)y^n,$$

where $m \in S = \{m_0 + n : n \text{ is an integer}\}$.

Model IIB

$$K^- = (m_0+\omega)y^{-1}\frac{(\gamma-\beta)}{\gamma}E_\gamma + \frac{d}{dy} - y^{-1}\frac{\beta(\gamma-\beta)}{\gamma}\Delta_\beta E_\gamma, \tag{4.3}$$
$$K^+ = \mu y\frac{(\gamma-1)}{(\gamma-\beta-1)}L_\gamma,$$
$$K^3 = m_0 + y\frac{d}{dy},$$
$$Q_1 = \mu y^2\frac{(\gamma-1)(\gamma-2)}{(\gamma-\beta-1)(\gamma-\beta-2)}L_\gamma^2,$$
$$E_1 = \mu,$$
$$h_m(\beta,\gamma,y) = {}_2F_1(n,\beta;\gamma;1)y^n,$$

where $m \in S = \{m_0 + n : n \text{ is an integer}\}$.

Model IIIB

$$K^- = \frac{d}{dy} - y^{-1}\beta\Delta_\beta, \tag{4.4}$$
$$K^+ = \mu y\left(\frac{\gamma-\beta}{\gamma}\right)E_\gamma,$$
$$K^3 = y\frac{d}{dy} - \omega,$$
$$Q_1 = \mu y^2\frac{(\gamma-\beta)(\gamma-\beta+1)}{\gamma(\gamma+1)}E_\gamma^2,$$
$$E_1 = \mu,$$
$$h_m(\beta,\gamma,y) = {}_2F_1(-n,\beta;\gamma;1)y^n,$$

where $m \in S = \{-\omega + n : n \text{ is nonnegative integer}\}$.

Model IVB

$$K^- = \frac{d}{dy} - y^{-1}\frac{\beta(\gamma-\beta)}{\gamma}\Delta_\beta E_\gamma, \tag{4.5}$$
$$K^+ = \mu y\frac{(\gamma-1)}{(\gamma-\beta-1)}L_\gamma,$$
$$K^3 = y\frac{d}{dy} - \omega,$$
$$Q_1 = \mu y^2\frac{(\gamma-1)(\gamma-2)}{(\gamma-\beta-1)(\gamma-\beta-2)}L_\gamma^2,$$
$$E_1 = \mu,$$
$$h_m(\beta,\gamma,y) = {}_2F_1(n,\beta;\gamma;1)y^n,$$

where $m \in S = \{-\omega + n : n \text{ is nonnegative integer}\}$.

The models given above satisfy the following:

$$[K^3, K^\pm] = K^\pm, \quad [K^3, Q_1] = 2Q_1, \tag{4.6}$$
$$[K^-, K^+] = E_1, \quad [K^-, Q_1] = 2K^+, \quad [K^+, Q_1] = 0,$$
$$[K^+, E_1] = [K^-, E_1] = [K^3, E_1] = [Q_1, E_1] = 0,$$

and thus represent a representation of $\mathcal{K}_5$.

Also

$$K^3 h_m = m h_m, \tag{4.7}$$
$$K^+ h_m = \mu h_{m+1},$$
$$K^- h_m = (m+\omega)h_{m-1},$$
$$E_1 h_m = \mu h_m,$$
$$Q_1 h_m = \mu h_{m+2}.$$

5. Recurrence Relations and Generating Functions

We will use models IA, IIA, IIIA and IVA for obtaining generating functions and the transformed models IB, IIB, IIIB and IVB for deriving recurrence relations. To obtain generating functions, we follow the method given in Manocha and Sahai [3]. We omit details and present the results only as follows:

5.1. Generating Functions

When we put q=0 in equation (3.2), we get

$$\begin{aligned}(5.1)\quad &\exp\left(\mu(a+y(1-x)b)+m_0\tau\right)\left(1+\frac{c}{y(1-x)}\right)^{m_0+\omega} f\left(\frac{xy}{y+c},(y+c)e^{\tau}\right)\\ &=\sum_{l=-\infty}^{\infty}\exp(\mu a+(m_0+k)\tau)c^{k-l}\frac{\Gamma(k+\rho+1)}{\Gamma(k-l+1)\Gamma(\rho+l+1)}\\ &\quad\times\ {}_1F_1(-\rho-l;k-l+1;-\mu bc)[y(1-x)]^l,\end{aligned}$$

and when we put q = 0 in equation (3.4), we get

(5.2)

$$\begin{aligned}&\exp\left(\mu(a+y(1-x)^{-1}b)+m_0\tau\right)\left(1+\frac{c}{y(1-x)^{-1}}\right)^{m_0+\omega} f\left(\frac{xy}{y+c(1-x)},(y+c)e^{\tau}\right)\\ &=\sum_{l=-\infty}^{\infty}\exp(\mu a+(m_0+k)\tau)c^{k-l}\frac{\Gamma(k+\rho+1)}{\Gamma(k-l+1)\Gamma(\rho+l+1)}\\ &\quad\times\ {}_1F_1(-\rho-l;k-l+1;-\mu bc)[y(1-x)^{-1}]^l.\end{aligned}$$

Also, when we put q=0 in equation (3.6), we get

$$\begin{aligned}(5.3)\quad &\exp\left(\mu(a+y(1-x)b)-\omega\tau\right) f\left(\frac{xy}{y+c},(y+c)e^{\tau}\right)\\ &=\sum_{l=-\infty}^{\infty}\exp(\mu a+(k-\omega)\tau)c^{k-l}\frac{\Gamma(k+1)}{\Gamma(k-l+1)\Gamma(l+1)}\\ &\quad\times\ {}_1F_1(-l;k-l+1;-\mu bc)[y(1-x)]^l,\end{aligned}$$

and q=0 in equation (3.8) gives

$$\begin{aligned}(5.4)\quad &\exp\left(\mu(a+y(1-x)^{-1}b)-\omega\tau\right) f\left(\frac{xy}{y+c(1-x)},(y+c)e^{\tau}\right)\\ &=\sum_{l=-\infty}^{\infty}\exp(\mu a+(k-\omega)\tau)c^{k-l}\frac{\Gamma(k+1)}{\Gamma(k-l+1)\Gamma(l+1)}\\ &\quad\times\ {}_1F_1(-l;k-l+1;-\mu bc)[y(1-x)^{-1}]^l.\end{aligned}$$

5.2. Recurrence Relations

By using the Model IB, we get

$$(n+\beta)\,{}_2F_1(-n,\beta;\gamma;1)+\left(\frac{(m_0+\omega)(\gamma-1)}{\gamma-\beta-1}\right){}_2F_1(-n,\beta;\gamma-1;1) - \beta\,{}_2F_1(-n,\beta+1;\gamma;1)=(m+\omega)\,{}_2F_1(-n+1,\beta;\gamma;1). \tag{5.5}$$

Similarly, the more recurrence relations which we get from Models IB, IIB, IIIB and IVB are as follows:

$$\frac{(\gamma-\beta)(\gamma-\beta+1)}{\gamma(\gamma+1)}\,{}_2F_1(-n,\beta;\gamma+2;1)={}_2F_1(-n-2,\beta;\gamma;1), \tag{5.6}$$

$$\frac{(\gamma-\beta)}{\gamma}\,{}_2F_1(-n,\beta;\gamma+1;1)={}_2F_1(-n-1,\beta;\gamma;1), \tag{5.7}$$

$$\frac{(m_0+\omega)(\gamma-\beta)}{\gamma}\,{}_2F_1(n,\beta;\gamma+1;1)-\frac{\beta(\gamma-\beta)}{\gamma}[\,{}_2F_1(n,\beta+1;\gamma+1;1) - {}_2F_1(n,\beta;\gamma+1;1)]+n\,{}_2F_1(n,\beta;\gamma;1)=(m+\omega)\,{}_2F_1(n-1,\beta;\gamma;1), \tag{5.8}$$

$$\frac{(\gamma-1)}{(\gamma-\beta-1)}\,{}_2F_1(n,\beta;\gamma-1;1)={}_2F_1(n+1,\beta;\gamma;1), \tag{5.9}$$

$$\frac{(\gamma-1)(\gamma-2)}{(\gamma-\beta-1)(\gamma-\beta-2)}\,{}_2F_1(n,\beta;\gamma-2;1)={}_2F_1(n+2,\beta;\gamma;1). \tag{5.10}$$

Acknowledgements. The financial assistance provided to the second author in the form of a CSIR fellowship (File No.:09/1050(0006)/2016-EMR-I) from CSIR, HRDG, New Delhi, India is gratefully acknowledged.

References

[1] G. Gasper and M. Rahman, *Basic hypergeometric series. Second Edition*, Cambridge University Press, 2004.

[2] H. L. Manocha, *Lie algebras of difference-differential operators and Appell functions* F_1, J. Math. Anal. Appl., **138**(1989), 491–510.

[3] H. L. Manocha and V. Sahai, *On models of irreducible unitary representations of compact Lie group* $SU(2)$ *involving special functions*, Indian J. pure Appl. Math., **21(12)**(1990), 1059–1071.

[4] W. Miller, *Lie theory and special functions*, Mathematics in Science and Engineering **43**, Academic Press, 1968.

[5] E. D. Rainville, *Special functions*, Macmillan, New York, 1960.

[6] V. Sahai, *On certain models of irreducible representations of* $sl(2,\mathbb{C})$ *and Euler integral transformation*, Indian J. pure appl. Math., **26(12)**(1995), 1177–1189.

[7] V. Sahai and S. Yadav, *Representations of two parameter quantum algebras and* p, q*-special functions*, J. Math. Anal. Appl., **335**(2007), 268-279.

[8] V. Sahai and S. Yadav, *On models of certain* p, q*-algebra representations: The* p, q*-oscillator algebra*, J. Math. Phys., **49**(2008), 053504, 12 pp.

[9] I. N. Sneddon, *Special functions of mathematical physics and chemistry*, Oliver and Boyd, Ltd., Edinburgh, 1961.

[10] H. M. Srivastava and H. L. Manocha, *A treatise on generating functions*, Ellis Horwood, Chichester, Halsted Press, New York, 1984.

[11] N. Ja. Vilenkin, *Special functions and the theory of group representations*, American Mathematical Society, 1968.

[12] A. Wawrzynczyk, *Group representations and special functions: Examples and Problems prepared by Aleksander Strasburger*, Polish Scientific Publishers - Warszawa, 1984.