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# On the Fekete−Szegö Problem for a Certain Class of Meromorphic Functions Using $q-$Derivative Operator

Mohamed Kamal Aouf
*Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt*
*e-mail* : mkaouf127@yahoo.com

Halit Orhan*
*Department of Mathematics, Faculty of Science, Ataturk University, Erzurum 25240, Turkey*
*e-mail* : orhanhalit607@gmail.com

Abstract. In this paper, we obtain Fekete-Szegö inequalities for certain class of meromorphic functions $f(z)$ for which

$$-\frac{(1-\frac{\alpha}{q})qzD_qf(z)+\alpha qzD_q\left[zD_qf(z)\right]}{(1-\frac{\alpha}{q})f(z)+\alpha zD_qf(z)} \prec \varphi(z)(\alpha\in\mathbb{C}\backslash(0,1],\ 0<q<1).$$

Sharp bounds for the Fekete-Szegö functional $\left|a_1-\mu a_0^2\right|$ are obtained.

## 1. Introduction

The theory of $q-$analysis has important role in many areas of mathematics and physics, for example, in the areas of ordinary fractional calculus, optimal control problems, $q-$difference, $q-$integral equations and in $q-$transform analysis (see for instance [1, 6, 8, 9]). The study of $q-$calculus has gained momentum years mainly due to the pioneer work of M. E. H. Ismail et al. [7] in recent years; it was followed by such works as those by S. Kanas and D. Raducanu [10] and S. Sivasubramanian and M. Govindaraj [19]. Let $\Sigma$ denote the class of meromorphic functions of the form:

$$f(z)=\frac{1}{z}+\sum_{k=0}^{\infty}a_kz^k, \tag{1.1}$$

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which are analytic in the open punctured unit disc

$$\mathbb{U}^* = \{z : z \in \mathbb{C} \text{ and } 0 < |z| < 1\} = \mathbb{U}\backslash\{0\}.$$

A function $f \in \Sigma$ is meromorphic starlike of order $\beta$, denoted by $\Sigma^*(\beta)$, if

$$-\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \beta \ (0 \leq \beta < 1;\ z \in \mathbb{U}).$$

The class $\Sigma^*(\beta)$ was introduced and studied by Pommerenke [16] (see also Miller [14]). Let $\varphi(z)$ be an analytic function with positive real part on $\mathbb{U}$ satisfies $\varphi(0) = 1$ and $\varphi'(0) > 0$ which maps $\mathbb{U}$ onto a region starlike with respect to 1 and symmetric with respect to the real axis. Let $\Sigma^*(\varphi)$ be the class of functions $f(z) \in \Sigma$ for which

$$-\frac{zf'(z)}{f(z)} \prec \varphi(z) \ (z \in \mathbb{U}).$$

The class $\Sigma^*(\varphi)$ was introduced and studied by Silverman et al. [18]. The class $\Sigma^*(\beta)$ is the special case of $\Sigma^*(\varphi)$ when $\varphi(z) = \dfrac{1 + (1 - 2\beta)z}{1 - z}$ $(0 \leq \beta < 1)$. Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k,$$

which are analytic in the open unit disc $\mathbb{U}$ and let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of functions which are analytic and univalent in $\mathbb{U}$. Ma and Minda [13] introduced and studied the class $\mathcal{S}^*(\varphi)$ which consists of functions $f(z) \in \mathcal{S}$ for which

$$\frac{zf'(z)}{f(z)} \prec \varphi(z) \ (z \in \mathbb{U}),$$

and the class $\mathcal{C}(\varphi)$ consists of functions $f(z) \in \mathcal{S}$ for which

$$1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z) \ (z \in \mathbb{U}).$$

Following Ma and Minda [13], Shanmugam and Sivasubramanian [17] defined a more general class $\mathcal{M}_\alpha(\varphi)$ consists of functions $f(z) \in \mathcal{S}$ for which

$$\frac{zf'(z) + \alpha z^2 f''(z)}{(1 - \alpha)f(z) + \alpha z f'(z)} \prec \varphi(z) \ (\alpha \geq 0).$$

Analogous to the class $\mathcal{M}_\alpha(\varphi)$, Aouf et al. [4] defined the class $\mathcal{F}_\alpha^*(\varphi)$ as follows: For $\alpha \in \mathbb{C}\backslash(0, 1]$, let $\mathcal{F}_\alpha^*(\varphi)$ be the subclass of $\Sigma$ consisting of functions $f(z)$ of the form (1.1) and satisfying the analytic criterion:

$$-\frac{zf'(z) + \alpha z^2 f''(z)}{(1 - \alpha)f(z) + \alpha z f'(z)} \prec \varphi(z).$$

For a function $f(z) \in \Sigma$ given by (1.1) and $0 < q < 1$, the $q-$derivative of a function $f(z)$ is defined by (see Gasper and Rahman [6])

$$D_q f(z) = \frac{f(qz) - f(z)}{(q-1)z} \text{ if } z \in \mathbb{U}^*. \tag{1.2}$$

From (1.2), we deduce that $D_q f(z)$ for a function $f(z)$ of the form (1.1) is given by

$$D_q f(z) = -\frac{1}{qz^2} + \sum_{k=0}^{\infty} [k]_q \; a_k z^{k-1} \; (z \neq 0),$$

where

$$[i]_q = \frac{1 - q^i}{1 - q}.$$

As $q \to 1^-, \; [k]_q \to k$, we have

$$\lim_{q \to 1^-} \; D_q f(z) = f'(z).$$

Making use of the $q-$derivative $D_q$, we introduce the subclass $\mathcal{F}^*_{q,\alpha}(\varphi)$ as follows: For $\alpha \in \mathbb{C}\backslash(0,1]$, $0 < q < 1$, a function $f(z) \in \Sigma$ is said to be in the class $\mathcal{F}^*_{q,\alpha}(\varphi)$, if and only if

$$-\frac{(1 - \frac{\alpha}{q})qzD_q f(z) + \alpha qzD_q \left[zD_q f(z)\right]}{(1 - \frac{\alpha}{q})f(z) + \alpha zD_q f(z)} \prec \varphi(z) \; (z \in \mathbb{U}). \tag{1.3}$$

We note that:

(i) $\lim\limits_{q \to 1^-} \mathcal{F}^*_{q,\alpha}(\varphi) = \mathcal{F}^*_{\alpha}(\varphi)$ (see Aouf et al. [4]);

(ii) $\lim\limits_{q \to 1^-} \mathcal{F}^*_{q,0}(\varphi) = \Sigma^*(\varphi)$ (see Silverman et al. [18] and Ali and Ravichandran [2]);

(iii) $\lim\limits_{q \to 1^-} \mathcal{F}^*_{q,0}\left(\dfrac{1+z}{1-z}\right) = \mathcal{F}^*(1) = \mathcal{F}^*$ (see Aouf [3, with $b = 1$]);

(iv) $\lim\limits_{q \to 1^-} \mathcal{F}^*_{q,0}\left(\dfrac{1+(1-2\beta)z}{1-z}\right) = \Sigma^*(\beta)$ $(0 \leq \beta < 1)$ (see Pommerenke [16]);

(v) $\lim\limits_{q \to 1^-} \mathcal{F}^*_{q,0}\left(\dfrac{1+\beta(1-2\gamma\eta)z}{1+\beta(1-2\gamma)z}\right) = \Sigma(\eta, \beta, \gamma)$ $(0 \leq \eta < 1, \; 0 < \beta \leq 1, \; \dfrac{1}{2} \leq \gamma \leq 1)$ (see Kulkarni and Joshi [12]);

(vi) $\lim\limits_{q \to 1^-} \mathcal{F}^*_{q,0}\left(\dfrac{1+Az}{1+Bz}\right) = K_1(A, B)$ $(0 \leq B < 1, \; -B < A < B)$ (see Karunakaran [11]).

## 2. Fekete-Szegö Problem

To prove our results, we need the following lemmas.

**Lemma 1.**([13]) *If $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$ is a function with positive real part in $\mathbb{U}$ and $\mu$ is a complex number, then*

$$\left|c_2 - \mu c_1^2\right| \leq 2 \max\{1; |2\mu - 1|\}.$$

*The result is sharp for the functions given by*

$$p(z) = \frac{1+z^2}{1-z^2} \text{ and } p(z) = \frac{1+z}{1-z}.$$

**Lemma 2.**([13]) *If $p_1(z) = 1 + c_1 z + c_2 z^2 + ...$ is a function with positive real part in $\mathbb{U}$, then*

$$\left|c_2 - \nu c_1^2\right| \leq \begin{cases} -4\nu + 2 & \text{if } \nu \leq 0, \\ 2 & \text{if } 0 \leq \nu \leq 1, \\ 4\nu - 2 & \text{if } \nu \geq 1. \end{cases}$$

*When $\nu < 0$ or $\nu > 1$, the equality holds if and only if $p_1(z) = \dfrac{1+z}{1-z}$ or one of its rotations. If $0 < \nu < 1$, then the equality holds if and only if $p_1(z) = \dfrac{1+z^2}{1-z^2}$ or one of its rotations. If $\nu = 0$, the equality holds if and only if*

$$p_1(z) = \left(\frac{1}{2} + \frac{\lambda}{2}\right)\frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{\lambda}{2}\right)\frac{1-z}{1+z} \ (0 \leq \lambda \leq 1),$$

*or one of its rotations. If $\nu = 1$, the equality holds if and only if*

$$\frac{1}{p_1(z)} = \left(\frac{1}{2} + \frac{\lambda}{2}\right)\frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{\lambda}{2}\right)\frac{1-z}{1+z} \ (0 \leq \lambda \leq 1),$$

*or one of its rotations. Also the above upper bound is sharp and it can be improved as follows when $0 < \nu < 1$ :*

$$\left|c_2 - \nu c_1^2\right| + \nu \left|c_1\right|^2 \leq 2 \ \left(0 < \nu \leq \frac{1}{2}\right),$$

*and*

$$\left|c_2 - \nu c_1^2\right| + (1-\nu) \left|c_1\right|^2 \leq 2 \ \left(\frac{1}{2} < \nu < 1\right).$$

Unless otherwise mentioned, we assume throughout this paper that $\alpha \in \mathbb{C}\backslash(0, 1]$ and $0 < q < 1$.

**Theorem 1.** *Let $\varphi(z) = 1 + B_1 z + B_2 z^2 + \ldots$. If $f(z)$ given by (1.1) belongs to the class $\mathcal{F}^*_{q,\alpha}(\varphi)$ and $\mu$ is a complex number, then*

$$\text{(2.1)}\quad \text{(i)}\quad \left|a_1 - \mu a_0^2\right| \le \frac{1}{1+q}\left|\frac{(q-2\alpha)\,B_1}{(q-\alpha+\alpha q)}\right| \times \max\left\{1, \left|\frac{B_2}{B_1} - \left[1 - \mu\frac{(q-2\alpha)\,(q-\alpha+\alpha q)\,(q+1)}{(q-\alpha)^2}\right] B_1\right|\right\}\ (B_1 \neq 0),$$

*and*

$$\text{(2.2)}\quad \text{(ii)}\qquad |a_1| \le \frac{1}{1+q}\left|\frac{(q-2\alpha)\,B_2}{(q-\alpha+\alpha q)}\right|\ (B_1 = 0).$$

*The result is sharp.*

*Proof.* If $f(z) \in \mathcal{F}^*_{\alpha}(\varphi)$, then there is a Schwarz function $w(z)$ in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ in $\mathbb{U}$ and such that

$$\text{(2.3)}\qquad -\frac{(1-\frac{\alpha}{q})qzD_q f(z) + \alpha qzD_q\left[zD_q f(z)\right]}{(1-\frac{\alpha}{q})f(z) + \alpha zD_q f(z)} = \varphi(w(z)).$$

Define the function $p_1(z)$ by

$$\text{(2.4)}\qquad p_1(z) = \frac{1+w(z)}{1-w(z)} = 1 + c_1 z + c_2 z^2 + \ldots.$$

Since $w(z)$ is a Schwarz function, we see that $\Re\{p_1(z)\} > 0$ and $p_1(0) = 1$. Define

$$\text{(2.5)}\qquad p(z) = -\frac{(1-\frac{\alpha}{q})qzD_q f(z) + \alpha qzD_q\left[zD_q f(z)\right]}{(1-\frac{\alpha}{q})f(z) + \alpha zD_q f(z)} = 1 + b_1 z + b_2 z^2 + \ldots.$$

In view of (2.3), (2.4) and (2.5), we have

$$\text{(2.6)}\qquad p(z) = \varphi\left(\frac{p_1(z)-1}{p_1(z)+1}\right).$$

Since

$$\frac{p_1(z)-1}{p_1(z)+1} = \frac{1}{2}\left[c_1 z + \left(c_2 - \frac{c_1^2}{2}\right)z^2 + \left(c_3 + \frac{c_1^3}{4} - c_1 c_2\right)z^3 + \ldots\right].$$

Therefore, we have

$$\varphi\left(\frac{p_1(z)-1}{p_1(z)+1}\right) = 1 + \frac{1}{2}B_1 c_1 z + \left[\frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2 c_1^2\right]z^2 + \ldots,$$

and from this equation and (2.6), we obtain

$$b_1 = \frac{1}{2}B_1 c_1,$$

and

$$b_2 = \frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2c_1^2.$$

Then, from (2.5) and (1.1), we see that

$$b_1 = -\left(\frac{q-\alpha}{q-2\alpha}\right)a_0,$$

and

$$b_2 = \left(\frac{q-\alpha}{q-2\alpha}\right)^2 a_0^2 - \frac{(q+1)\,(q-\alpha+\alpha q)}{q-2\alpha}a_1,$$

or, equivalently, we have

$$a_0 = -\left(\frac{q-2\alpha}{q-\alpha}\right), \tag{2.7}$$

and

$$a_1 = -\frac{(q-2\alpha)\,B_1}{2(1+q)\,(q-\alpha+\alpha q)}\left[c_2 - \frac{c_1^2}{2}\left(1 - \frac{B_2}{B_1} + B_1\right)\right]. \tag{2.8}$$

Therefore

$$a_1 - \mu a_0^2 = -\frac{(q-2\alpha)\,B_1}{2(1+q)\,(q-\alpha+\alpha q)}\left\{c_2 - \nu c_1^2\right\},$$

where

$$\nu = \frac{1}{2}\left[1 - \frac{B_2}{B_1} + B_1 - \mu\frac{(q-2\alpha)\,(q-\alpha+\alpha q)\,(q+1)B_1}{(q-\alpha)^2}\right]. \tag{2.9}$$

Now, the result (2.1) follows by an application of Lemma 1. Also, if $B_1 = 0$, then

$$a_0 = 0 \text{ and } a_1 = -\frac{(q-2\alpha)\,B_2c_1^2}{4(1+q)\,(q-\alpha+\alpha q)}.$$

Since $p(z)$ has positive real part, $|c_1| \leq 2$ (see Nehari [15]), so that

$$|a_1| \leq \frac{1}{1+q}\left|\frac{(q-2\alpha)\,B_2}{(q-\alpha+\alpha q)}\right|,$$

this proving (2.2). The result is sharp for the functions

$$-\frac{(1-\frac{\alpha}{q})qzD_qf(z) + \alpha qzD_q\left[D_qf(z)\right]}{(1-\frac{\alpha}{q})f(z) + \alpha zD_qf(z)} = \varphi(z^2),$$

and

$$-\frac{(1-\frac{\alpha}{q})qzD_qf(z) + \alpha qzD_q\left[D_qf(z)\right]}{(1-\frac{\alpha}{q})f(z) + \alpha zD_qf(z)} = \varphi(z).$$

This completes the proof of Theorem 1. □

**Remark 1**.

(i) For $q \to 1^-$ in Theorem 1, we obtain the result obtained by Aouf et al. [4, Theorem 2.1];

(ii) For $q \to 1^-$ and $\alpha = 0$ in Theorem 1, we obtain the result obtained by Silverman et al. [18, Theorem 2.1].

By using Lemma 2, we can obtain the following theorem.

**Theorem 2.** *Let* $\varphi(z) = 1 + B_1 z + B_2 z^2 + \ldots \left(B_i > 0,\ i \in \{1,2\}, 0 < \alpha < \frac{q}{1+q}\right)$. *If* $f(z)$ *given by (1.1) belongs to the class* $\mathcal{F}^*_{q,\alpha}(\varphi)$, *then*

(2.10)
$$\left|a_1 - \mu a_0^2\right| \le \begin{cases} \frac{(q-2\alpha)B_1^2}{(1+q)(q-\alpha+\alpha q)}\left\{-B_2 + \left[1 - \mu\frac{[q-\alpha(1+q)](1+q)(q-2\alpha)}{q(1-\alpha)^2}\right]B_1^2\right\} & \text{if } \mu \le \sigma_1, \\ \frac{(q-2\alpha)B_1}{(1+q)(q-\alpha+\alpha q)} & \text{if } \sigma_1 \le \mu \le \sigma_2, \\ \frac{(q-2\alpha)B_1^2}{(1+q)(q-\alpha+\alpha q)}\left\{B_2 - \left[1 - \mu\frac{[q-\alpha(1+q)](1+q)(q-2\alpha)}{q(1-\alpha)^2}\right]B_1^2\right\} & \text{if } \mu \ge \sigma_2, \end{cases}$$

*where*

$$\sigma_1 = \frac{(q-\alpha)^2\left[-B_1 - B_2 + B_1^2\right]}{(q-2\alpha)(q-\alpha+\alpha q)(1+q)B_1^2} \text{ and } \sigma_2 = \frac{(q-\alpha)^2\left(B_1 - B_2 + B_1^2\right)}{(q-2\alpha)(q-\alpha+\alpha q)(1+q)B_1^2}.$$

*The result is sharp. Further, let*

$$\sigma_3 = \frac{(q-\alpha)^2\left[-B_2 + B_1^2\right]}{(q-2\alpha)(q-\alpha+\alpha q)(1+q)B_1^2}.$$

(i) *If* $\sigma_1 \le \mu \le \sigma_3$, *then*

$$\begin{aligned} &\left|a_1 - \mu a_0^2\right| + \frac{(q-\alpha)^2}{(q-2\alpha)(q-\alpha+\alpha q)(1+q)B_1^2} \\ &\times\left\{(B_1 + B_2) + \left[\mu\frac{(1+q)(q-2\alpha)(q-\alpha+\alpha q)}{q(1-\alpha)^2} - 1\right]B_1^2\right\}|a_0|^2 \\ &\le \frac{(q-2\alpha)B_1}{(1+q)(q-\alpha+\alpha q)}. \end{aligned} \tag{2.11}$$

(ii) *If* $\sigma_3 \le \mu \le \sigma_2$, *then*

$$\begin{aligned} &\left|a_1 - \mu a_0^2\right| + \frac{(q-\alpha)^2}{(q-2\alpha)(q-\alpha+\alpha q)(1+q)B_1^2} \\ &\times\left\{(B_1 - B_2) + \left[1 - \mu\frac{(1+q)(q-2\alpha)(q-\alpha+\alpha q)}{q(1-\alpha)^2}\right]B_1^2\right\}|a_0|^2 \\ &\le \frac{(q-2\alpha)B_1}{(1+q)(q-\alpha+\alpha q)}. \end{aligned} \tag{2.12}$$

*Proof.* First, let $\mu \leq \sigma_1$. Then

$$\begin{aligned}\left|a_1 - \mu a_0^2\right| &\leq \frac{(q-2\alpha)\, B_1}{(1+q)\,(q-\alpha+\alpha q)} \left\{-\frac{B_2}{B_1} + \left[1 - \mu \frac{[q-\alpha(1+q)]\,(1+q)\,(q-2\alpha)}{q(1-\alpha)^2}\right] B_1\right\} \\ &\leq \frac{(q-2\alpha)\, B_1^2}{(1+q)\,(q-\alpha+\alpha q)} \left\{-B_2 + \left[1 - \mu \frac{[q-\alpha(1+q)]\,(1+q)\,(q-2\alpha)}{q(1-\alpha)^2}\right] B_1^2\right\}.\end{aligned}$$

Let, now $\sigma_1 \leq \mu \leq \sigma_2$. Then, using the above calculations, we obtain

$$\left|a_1 - \mu a_0^2\right| \leq \frac{(q-2\alpha)\, B_1}{(1+q)\,(q-\alpha+\alpha q)}.$$

Finally, if $\mu \geq \sigma_2$, then

$$\begin{aligned}\left|a_1 - \mu a_0^2\right| &\leq \frac{(q-2\alpha)\, B_1}{(1+q)\,(q-\alpha+\alpha q)} \left\{\frac{B_2}{B_1} - \left[1 - \mu \frac{[q-\alpha(1+q)]\,(1+q)\,(q-2\alpha)}{q(1-\alpha)^2}\right] B_1\right\} \\ &\leq \frac{(q-2\alpha)\, B_1^2}{(1+q)\,(q-\alpha+\alpha q)} \left\{B_2 - \left[1 - \mu \frac{[q-\alpha(1+q)]\,(1+q)\,(q-2\alpha)}{q(1-\alpha)^2}\right] B_1^2\right\}.\end{aligned}$$

To show that the bounds are sharp, we define the functions $K_{\varphi n}$ $(n \geq 2)$ by

$$-\frac{(1-\frac{\alpha}{q})qzD_qK_{\varphi n}(z) + \alpha qzD_q\left[zD_qK_{\varphi n}(z)\right]}{(1-\frac{\alpha}{q})K_{\varphi n}(z) + \alpha zD_qK_{\varphi n}(z)} = \varphi(z^{n-1}),$$

$$z^2K_{\varphi n}(z)\mid_{z=0} = 0 = -z^2K'_{\varphi n}(z)\mid_{z=0} -1,$$

and the functions $F_\gamma$ and $G_\gamma$ $(0 \leq \gamma \leq 1)$ by

$$-\frac{(1-\frac{\alpha}{q})qzD_qF_\gamma(z) + \alpha qzD_q\left[zD_qF_\gamma(z)\right]}{(1-\frac{\alpha}{q})F_\gamma(z) + \alpha zD_qF_\gamma(z)} = \varphi\left(\frac{z(z+\gamma)}{1+\gamma z}\right),$$

$$z^2F_\gamma(z)\mid_{z=0} = 0 = -z^2F'_\gamma(z)\mid_{z=0} -1,$$

and

$$-\frac{(1-\frac{\alpha}{q})qzD_qG_\gamma(z) + \alpha qzD_q\left[zD_qG_\gamma(z)\right]}{(1-\frac{\alpha}{q})G_\gamma(z) + \alpha zD_qG_\gamma(z)} = \varphi\left(-\frac{z(z+\gamma)}{1+\gamma z}\right),$$

$$z^2G_\gamma(z)\mid_{z=0} = 0 = -z^2G'_\gamma(z)\mid_{z=0} -1.$$

Cleary the functions $K_{\varphi n}$, $F_\gamma$ and $G_\gamma \in \mathcal{F}^*_{q,\alpha}(\varphi)$. Also we write $K_\varphi = K_{\varphi 2}$. If $\mu < \sigma_1$ or $\mu > \sigma_2$, then the equality holds if and only if $f(z)$ is $K_\varphi$ or one of its rotations. When $\sigma_1 < \mu < \sigma_2$, then the equality holds if $f(z)$ is $K_{\varphi 3}$ or one of its rotations. If $\mu = \sigma_1$, then the equality holds if and only if $f(z)$ is $F_\gamma$ or one of its rotations. If $\mu = \sigma_2$, then the equality holds if and only if $f(z)$ is $G_\gamma$ or one of its rotations. This completes the proof of Theorem 2. □

**Remark 2.**

(i) For $q \to 1^-$ in Theorem 2, we obtain the result obtained by Aouf et al. [4, Theorem 2];

(ii) Putting $q \to 1^-$ and $\alpha = 0$ in Theorem 2, we obtain the result obtained by Ali and Ravichandran [2, Theorem 5.1].

### 3. Applications to Functions Defined by $q-$Bessel Function

We recall some definitions of $q-$calculus which we will be used in our paper. For any complex number $\alpha$, the $q-$shifted factorials are defined by

$$(\alpha; q)_0 = 1;\ (\alpha; q)_n = \prod_{k=0}^{n-1} \left(1 - \alpha q^k\right)\ \ (n \in \mathbb{N} = \{1, 2, ...\}). \tag{3.1}$$

If $|q| < 1$, the definition (3.1) remains meaningful for $n = \infty$ as a convergent infinite product

$$(\alpha; q)_\infty = \prod_{j=0}^{\infty} \left(1 - \alpha q^j\right).$$

In terms of the analogue of the gamma function

$$(q^\alpha; q)_n = \frac{\Gamma_q(\alpha + n)(1 - q)^n}{\Gamma_q(\alpha)}\ (n > 0),$$

where the $q-$gamma function is defined by

$$\Gamma_q(x) = \frac{(q; q)_\infty (1 - q)^{1-x}}{(q^x; q)_\infty}\ (0 < q < 1).$$

We note that

$$\lim_{q \to 1^-} \frac{(q^\alpha; q)_n}{(1 - q)^n} = (\alpha)_n,$$

where

$$(\alpha)_n = \begin{cases} 1 & \text{if } n = 0, \\ \alpha(\alpha + 1)(\alpha + 2)...(\alpha + n - 1) & \text{if } \ n \in \mathbb{N}. \end{cases}$$

Now, consider the $q-$analoge of Bessel fnction defined by (Jackson [8])

$$\mathfrak{J}_\upsilon^{(1)}(z; q) = \frac{(q^{\upsilon+1}; q)_\infty}{(q; q)_\infty} \sum_{k=0}^{\infty} \frac{(-1)^k}{(q; q)_k (q^{\upsilon+1}; q)_k} \left(\frac{z}{2}\right)^{2k+\nu}\ \ (0 < q < 1).$$

Also, let us define

$$\begin{aligned} \mathcal{L}_\upsilon(z; q) &= \frac{2^\upsilon (q; q)_\infty}{(q^{\upsilon+1}; q)_\infty (1 - q)^\upsilon z^{\upsilon/2+1}} \mathfrak{J}_\upsilon^{(1)} \left(z^{1/2}(1 - q); q\right) \\ &= \frac{1}{z} + \sum_{k=0}^{\infty} \frac{(-1)^{k+1}(1 - q)^{2(k+1)}}{4^{(k+1)}\ (q; q)_{k+1} (q^{\upsilon+1}; q)_{k+1}} z^k\ \ (z \in \mathbb{U}). \end{aligned}$$

By using the Hadamard product (or convolution), we define the linear operator $\mathcal{L}_{q,\upsilon} : \Sigma \to \Sigma$, as follows:

$$\begin{aligned}(\mathcal{L}_{q,\upsilon}f)(z) &= \mathcal{L}_{\upsilon}(z;q) * f(z) \\ &= \frac{1}{z} + \sum_{k=0}^{\infty} \frac{(-1)^{k+1}(1-q)^{2(k+1)}}{4^{(k+1)}\ (q;q)_{k+1}(q^{\upsilon+1};q)_{k+1}} a_k z^k.\end{aligned}$$

As $q \to 1^-$, the linear operator $\mathcal{L}_{q,\upsilon}$ reduces to the operator $\mathcal{L}_{\upsilon}$ introduced and studied by Aouf et al. [5]. For $0 < q < 1$ and $\alpha \in \mathbb{C}\backslash(0,1]$, let $\mathcal{F}^*_{q,\alpha,\upsilon}(\varphi)$ be the subclass of $\Sigma$ consisting of functions $f(z)$ of the form (1.1) and satisfies the analytic criterion:

$$-\frac{(1-\frac{\alpha}{q})qzD_q(\mathcal{L}_{q,\upsilon}f) + \alpha qzD_q\left[D_q(\mathcal{L}_{q,\upsilon}f)\right]}{(1-\frac{\alpha}{q})(\mathcal{L}_{q,\upsilon}f) + \alpha zD_q(\mathcal{L}_{q,\upsilon}f)} \prec \varphi(z)\ (z \in \mathbb{U}).$$

Using similar arguments to those in the proof of the above theorems, we obtain the following theorems.

**Theorem 3.** *Let $\varphi(z) = 1 + B_1 z + B_2 z^2 + \cdots$. If $f(z)$ given by (1.1) belongs to the class $\mathcal{F}^*_{q,\alpha,\upsilon}(\varphi)$ and $\mu$ is a complex number, then*

(i) $$\left|a_1 - \mu a_0^2\right| \leq \frac{4^2(1-q^{\upsilon+1})(1-q^{\upsilon+2})}{(1-q)^2}\left|\frac{B_1\,(q-2\alpha)}{q+\alpha q-\alpha}\right| \times \max\left\{1, \left|\frac{B_2}{B_1} - \left[1-\mu\frac{(q-2\alpha)\,(1-q^{\upsilon+1})\,(q-\alpha+\alpha q)}{(1-q^{\upsilon+2})(q-\alpha)^2}\right]B_1\right|\right\}\ (B_1 \neq 0),$$

(ii) $$|a_1| \leq \frac{4^2(1-q^{\upsilon+1})(1-q^{\upsilon+2})}{(1-q)^2}\left|\frac{B_2\,(q-2\alpha)}{q+\alpha q-\alpha}\right|\ (B_1 = 0).$$

*The result is sharp.*

**Theorem 4.** *Let $\varphi(z) = 1 + B_1 z + B_2 z^2 + ...,(B_i > 0,\ i \in \{1,2\},\ \alpha > 0)$. If $f(z)$ given by (1.1) belongs to the class $\mathcal{F}^*_{q,\alpha,\upsilon}(\varphi)$, then*

$$\left|a_1 - \mu a_0^2\right| \leq \begin{cases} \dfrac{4^2(1-q^{\upsilon+1})(1-q^{\upsilon+2})\,(q-2\alpha)\,B_1^2}{(1-q)^2\,(q+\alpha q-\alpha)} \\ \quad \times\left\{-B_2 + \left[1-\mu\dfrac{(q-2\alpha)\,(1-q^{\upsilon+1})\,(q-\alpha+\alpha q)}{(1-q^{\upsilon+2})(q-\alpha)^2}\right]B_1^2\right\} & \text{if } \mu \leq \sigma_1^*, \\ \dfrac{4^2(1-q^{\upsilon+1})(1-q^{\upsilon+2})\,[q-\alpha(q+1)]\,B_1}{q(1-q)^2} & \text{if } \sigma_1^* \leq \mu \leq \sigma_2^*, \\ \dfrac{4^2(1-q^{\upsilon+1})(1-q^{\upsilon+2})\,[q-\alpha(q+1)]}{q(1-q)^2} \\ \quad \times\left\{B_2 - \left[1-\mu\dfrac{(q-2\alpha)\,(1-q^{\upsilon+1})\,(q-\alpha+\alpha q)}{(1-q^{\upsilon+2})(q-\alpha)^2}\right]B_1^2\right\} & \text{if } \mu \geq \sigma_2^*, \end{cases}$$

*where*

$$\sigma_1^* = \frac{(q-\alpha)^2(1-q^{\upsilon+2})\left[-B_1-B_2+B_1^2\right]}{(q-2\alpha)\,(q-\alpha+\alpha q)\,(1-q^{\upsilon+1})B_1^2},$$

*and*

$$\sigma_2^* = \frac{(q-\alpha)^2(1-q^{\upsilon+2})\left[B_1-B_2+B_1^2\right]}{(q-2\alpha)\,(q-\alpha+\alpha q)\,(1-q^{\upsilon+1})B_1^2}.$$

*The result is sharp. Further, let*

$$\sigma_3^* = \frac{(q-\alpha)^2(1-q^{\upsilon+2})\left[-B_2+B_1^2\right]}{(q-2\alpha)\,(q-\alpha+\alpha q)\,(1-q^{\upsilon+1})B_1^2}.$$

(i) *If $\sigma_1^* \le \mu \le \sigma_3^*$, then*

$$\begin{aligned}
&\left|a_1-\mu a_0^2\right| + \frac{(1-q^{\upsilon+2})(q-\alpha)^2}{(q-2\alpha)\,(q-\alpha+\alpha q)\,(1-q^{\upsilon+1})B_1^2}\\
&\times\left\{(B_1+B_2)+\left[\mu\frac{(q-2\alpha)\,(1-q^{\upsilon+1})(1-q^{\upsilon+2})\,(q-\alpha+\alpha q)}{(1-q^{\upsilon+2})(q-\alpha)^2}-1\right]B_1^2\right\}|a_0|^2\\
\le\ & \frac{4^2\,(q-2\alpha)\,(1-q^{\upsilon+1})(1-q^{\upsilon+2})B_1}{(1-q)^2\,(q-\alpha+\alpha q)}.
\end{aligned}$$

(ii) *If $\sigma_3^* \le \mu \le \sigma_2^*$, then*

$$\begin{aligned}
&\left|a_1-\mu a_0^2\right| + \frac{(1-q^{\upsilon+2})(q-\alpha)^2}{(q-2\alpha)\,(q-\alpha+\alpha q)\,(1-q^{\upsilon+1})B_1^2}\\
&\times\left\{(B_1-B_2)+\left[1-\mu\frac{(q-2\alpha)\,(1-q^{\upsilon+1})(1-q^{\upsilon+2})\,(q-\alpha+\alpha q)}{(1-q^{\upsilon+2})(q-\alpha)^2}\right]B_1^2\right\}|a_0|^2\\
\le\ & \frac{4^2\,(q-2\alpha)\,(1-q^{\upsilon+1})(1-q^{\upsilon+2})B_1}{(1-q)^2\,(q-\alpha+\alpha q)}.
\end{aligned}$$

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