Recursion Formulas for Exton's triple Hypergeometric Functions

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ABSTRACT. This paper continues the study of recursion formulas of multivariable hypergeometric functions. Earlier, in [4], the authors have given the recursion formulas for three variable Lauricella functions, Srivastava's triple hypergeometric functions and k -variable Lauricella functions. Further, in [5], we have obtained recursion formulas for the general triple hypergeometric function. We present here the recursion formulas for Exton's triple hypergeometric functions.

1. Introduction

Recursion formulas for multivariable hypergeometric functions have received considerable attention recently. In this direction, Opps, Saad and Srivastava [2] and Wang [8] have studied the recursion relations of Appell functions. The authors, in [4], have started a systematic study of recursion formulas of multivariable hypergeometric functions including fourteen three variable Lauricella functions, three Srivastava's triple hypergeometric functions and four k -variable Lauricella functions. In [5], we have obtained recursion formulas for the general triple hypergeometric function [6]. These results unify and generalize the results in [4] for the three variable hypergeometric function. In this paper, we obtain recursion formulas for Exton's triple hypergeometric functions [1].

Exton [1] gave the following hypergeometric functions of three variables, denoted by $X_1, X_2, \dots, X_{20}$. We recall their definition below:

(1.1)

$$X_1(a_1, a_2; c_1, c_2; x_1, x_2, x_3) = \sum_{m_1, m_2, m_3=0}^{\infty} \frac{(a_1)_{2m_1+2m_2+m_3} (a_2)_{m_3}}{(c_1)_{m_2+m_3} (c_2)_{m_1}} \prod_{i=1}^3 \frac{x_i^{m_i}}{m_i!},$$

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Received October 18, 2015; revised March 11, 2016; accepted March 16, 2016.

2010 Mathematics Subject Classification: 33C70.

Key words and phrases: Recursion formula, Hypergeometric functions.

(1.2)

$$X_2(a_1, a_2; c_1, c_2, c_3; x_1, x_2, x_3) = \sum_{m_1, m_2, m_3=0}^{\infty} \frac{(a_1)_{2m_1+2m_2+m_3} (a_2)_{m_3}}{(c_1)_{m_1} (c_2)_{m_2} (c_3)_{m_3}} \prod_{i=1}^3 \frac{x_i^{m_i}}{m_i!},$$

(1.3)

$$X_3(a_1, a_2; c_1, c_2; x_1, x_2, x_3) = \sum_{m_1, m_2, m_3=0}^{\infty} \frac{(a_1)_{2m_1+m_2+m_3} (a_2)_{m_2+m_3}}{(c_1)_{m_1+m_2} (c_2)_{m_3}} \prod_{i=1}^3 \frac{x_i^{m_i}}{m_i!},$$

(1.4)

$$X_4(a_1, a_2; c_1, c_2, c_3; x_1, x_2, x_3) = \sum_{m_1, m_2, m_3=0}^{\infty} \frac{(a_1)_{2m_1+m_2+m_3} (a_2)_{m_2+m_3}}{(c_1)_{m_1} (c_2)_{m_2} (c_3)_{m_3}} \prod_{i=1}^3 \frac{x_i^{m_i}}{m_i!},$$

(1.5)

$$X_5(a_1, a_2, a_3; c; x_1, x_2, x_3) = \sum_{m_1, m_2, m_3=0}^{\infty} \frac{(a_1)_{2m_1+m_2+m_3} (a_2)_{m_2} (a_3)_{m_3}}{(c)_{m_1+m_2+m_3}} \prod_{i=1}^3 \frac{x_i^{m_i}}{m_i!},$$

(1.6)

$$X_6(a_1, a_2, a_3; c_1, c_2; x_1, x_2, x_3) = \sum_{m_1, m_2, m_3=0}^{\infty} \frac{(a_1)_{2m_1+m_2+m_3} (a_2)_{m_2} (a_3)_{m_3}}{(c_1)_{m_1+m_2} (c_2)_{m_3}} \prod_{i=1}^3 \frac{x_i^{m_i}}{m_i!},$$

(1.7)

$$X_7(a_1, a_2, a_3; c_1, c_2; x_1, x_2, x_3) = \sum_{m_1, m_2, m_3=0}^{\infty} \frac{(a_1)_{2m_1+m_2+m_3} (a_2)_{m_2} (a_3)_{m_3}}{(c_1)_{m_2+m_3} (c_2)_{m_1}} \prod_{i=1}^3 \frac{x_i^{m_i}}{m_i!},$$

(1.8)

$$X_8(a_1, a_2, a_3; c_1, c_2, c_3; x_1, x_2, x_3) = \sum_{m_1, m_2, m_3=0}^{\infty} \frac{(a_1)_{2m_1+m_2+m_3} (a_2)_{m_2} (a_3)_{m_3}}{(c_1)_{m_1} (c_2)_{m_2} (c_3)_{m_3}} \prod_{i=1}^3 \frac{x_i^{m_i}}{m_i!},$$

(1.9)

$$X_9(a_1, a_2; c; x_1, x_2, x_3) = \sum_{m_1, m_2, m_3=0}^{\infty} \frac{(a_1)_{2m_1+m_2} (a_2)_{m_2+2m_3}}{(c)_{m_1+m_2+m_3}} \prod_{i=1}^3 \frac{x_i^{m_i}}{m_i!},$$

(1.10)

$$X_{10}(a_1, a_2; c_1, c_2; x_1, x_2, x_3) = \sum_{m_1, m_2, m_3=0}^{\infty} \frac{(a_1)_{2m_1+m_2} (a_2)_{m_2+2m_3}}{(c_1)_{m_1+m_2} (c_2)_{m_3}} \prod_{i=1}^3 \frac{x_i^{m_i}}{m_i!},$$

(1.11)

$$X_{11}(a_1, a_2; c_1, c_2; x_1, x_2, x_3) = \sum_{m_1, m_2, m_3=0}^{\infty} \frac{(a_1)_{2m_1+m_2} (a_2)_{m_2+2m_3}}{(c_1)_{m_1+m_3} (c_2)_{m_2}} \prod_{i=1}^3 \frac{x_i^{m_i}}{m_i!},$$

(1.12)

$$X_{12}(a_1, a_2; c_1, c_2, c_3; x_1, x_2, x_3) = \sum_{m_1, m_2, m_3=0}^{\infty} \frac{(a_1)_{2m_1+m_2} (a_2)_{m_2+2m_3}}{(c_1)_{m_1} (c_2)_{m_2} (c_3)_{m_3}} \prod_{i=1}^3 \frac{x_i^{m_i}}{m_i!},$$

(1.13)

$$X_{13}(a_1, a_2, a_3; c; x_1, x_2, x_3) = \sum_{m_1, m_2, m_3=0}^{\infty} \frac{(a_1)_{2m_1+m_2} (a_2)_{m_2+m_3} (a_3)_{m_3}}{(c)_{m_1+m_2+m_3}} \prod_{i=1}^3 \frac{x_i^{m_i}}{m_i!},$$

(1.14)

$$X_{14}(a_1, a_2, a_3; c_1, c_2; x_1, x_2, x_3) = \sum_{m_1, m_2, m_3=0}^{\infty} \frac{(a_1)_{2m_1+m_2} (a_2)_{m_2+m_3} (a_3)_{m_3}}{(c_1)_{m_1+m_2} (c_2)_{m_3}} \prod_{i=1}^3 \frac{x_i^{m_i}}{m_i!},$$

(1.15)

$$X_{15}(a_1, a_2, a_3; c_1, c_2; x_1, x_2, x_3) = \sum_{m_1, m_2, m_3=0}^{\infty} \frac{(a_1)_{2m_1+m_2} (a_2)_{m_2+m_3} (a_3)_{m_3}}{(c_1)_{m_2+m_3} (c_2)_{m_1}} \prod_{i=1}^3 \frac{x_i^{m_i}}{m_i!},$$

(1.16)

$$X_{16}(a_1, a_2, a_3; c_1, c_2; x_1, x_2, x_3) = \sum_{m_1, m_2, m_3=0}^{\infty} \frac{(a_1)_{2m_1+m_2} (a_2)_{m_2+m_3} (a_3)_{m_3}}{(c_1)_{m_1+m_3} (c_2)_{m_2}} \prod_{i=1}^3 \frac{x_i^{m_i}}{m_i!},$$

(1.17)

$$X_{17}(a_1, a_2, a_3; c_1, c_2, c_3; x_1, x_2, x_3) = \sum_{m_1, m_2, m_3=0}^{\infty} \frac{(a_1)_{2m_1+m_2} (a_2)_{m_2+m_3} (a_3)_{m_3}}{(c_1)_{m_1} (c_2)_{m_2} (c_3)_{m_3}} \prod_{i=1}^3 \frac{x_i^{m_i}}{m_i!},$$

(1.18)

$$X_{18}(a_1, a_2, a_3, a_4; c; x_1, x_2, x_3) = \sum_{m_1, m_2, m_3=0}^{\infty} \frac{(a_1)_{2m_1+m_2} (a_2)_{m_2} (a_3)_{m_3} (a_4)_{m_3}}{(c)_{m_1+m_2+m_3}} \prod_{i=1}^3 \frac{x_i^{m_i}}{m_i!},$$

(1.19)

$$X_{19}(a_1, a_2, a_3, a_4; c_1, c_2; x_1, x_2, x_3) = \sum_{m_1, m_2, m_3=0}^{\infty} \frac{(a_1)_{2m_1+m_2} (a_2)_{m_2} (a_3)_{m_3} (a_4)_{m_3}}{(c_1)_{m_2+m_3} (c_2)_{m_1}} \prod_{i=1}^3 \frac{x_i^{m_i}}{m_i!},$$

(1.20)

$$X_{20}(a_1, a_2, a_3, a_4; c_1, c_2; x_1, x_2, x_3) = \sum_{m_1, m_2, m_3=0}^{\infty} \frac{(a_1)_{2m_1+m_2} (a_2)_{m_2} (a_3)_{m_3} (a_4)_{m_3}}{(c_1)_{m_1+m_3} (c_2)_{m_2}} \prod_{i=1}^3 \frac{x_i^{m_i}}{m_i!}.$$

Here, $(a)_m$ is the Pochhammer symbol [3, 7]. For convergence of the above listed twenty Exton's triple hypergeometric functions, see [6].

The paper is organised as follows. There are 20 further sections, one for each of the twenty Exton's triple hypergeometric functions. Following abbreviated notations are used in the paper. We, for example, write X_1 for the series $X_1(a_1, a_2; c_1, c_2; x_1, x_2, x_3)$, $X_1(a_1+n)$ for the series $X_1(a_1+n, a_2; c_1, c_2; x_1, x_2, x_3)$ and $X_1(a_1+n, c_1+n_1)$ stands for the series $X_1(a_1+n, a_2; c_1+n_1, c_2; x_1, x_2, x_3)$, etc. Throughout n denotes a non-negative integer.

2. Recursion Formulas for X_1

Theorem 2.1. *The following recursion formulas hold true for the numerator parameters a_1, a_2 of the Exton's triple hypergeometric function X_1 :*

$$X_1(a_1+n) = X_1 + \frac{2x_1}{c_2} \sum_{n_1=1}^n (a_1+n_1) X_1(a_1+1+n_1, c_2+1)$$

$$\begin{aligned}
& + \frac{2x_2}{c_1} \sum_{n_1=1}^n (a_1 + n_1) X_1(a_1 + 1 + n_1, c_1 + 1) \\
(2.1) \quad & + \frac{a_2 x_3}{c_1} \sum_{n_1=1}^n X_1(a_1 + n_1, a_2 + 1, c_1 + 1), \\
X_1(a_1 - n) = & X_1 - \frac{2x_1}{c_2} \sum_{n_1=1}^n (a_1 + 1 - n_1) X_1(a_1 + 2 - n_1, c_2 + 1) \\
& - \frac{2x_2}{c_1} \sum_{n_1=1}^n (a_1 + 1 - n_1) X_1(a_1 + 2 - n_1, c_1 + 1) \\
(2.2) \quad & - \frac{a_2 x_3}{c_1} \sum_{n_1=1}^n X_1(a_1 + 1 - n_1, a_2 + 1, c_1 + 1).
\end{aligned}$$

and

$$(2.3) \quad X_1(a_2 + n) = X_1 + \frac{a_1 x_3}{c_1} \sum_{n_1=1}^n X_1(a_1 + 1, a_2 + n_1, c_1 + 1),$$

$$(2.4) \quad X_1(a_2 - n) = X_1 - \frac{a_1 x_3}{c_1} \sum_{n_1=1}^n X_1(a_1 + 1, a_2 + 1 - n_1, c_1 + 1),$$

Results (2.3) and (2.4) can also be represented as follows

$$(2.5) \quad X_1(a_2 + n) = \sum_{n_1=0}^n \binom{n}{n_1} \frac{(a_1)_{n_1} x_3^{n_1}}{(c_1)_{n_1}} X_1(a_1 + n_1, a_2 + n_1, c_1 + n_1),$$

$$(2.6) \quad X_1(a_2 - n) = \sum_{n_1=0}^n \binom{n}{n_1} \frac{(a_1)_{n_1} (-x_3)^{n_1}}{(c_1)_{n_1}} X_1(a_1 + n_1, c_1 + n_1).$$

Proof. From the definition of Exton's triple hypergeometric function X_1 and the relation

$$(2.7) \quad (a_1 + 1)_{2m_1+2m_2+m_3} = (a_1)_{2m_1+2m_2+m_3} \left(1 + \frac{2m_1}{a_1} + \frac{2m_2}{a_1} + \frac{m_3}{a_1} \right)$$

we obtain the following contiguous relation:

$$\begin{aligned}
(2.8) \quad X_1(a_1 + 1) = & X_1 + \frac{2x_1}{c_2} (a_1 + 1) X_1(a_1 + 2, c_2 + 1) + \frac{2x_2}{c_1} (a_1 + 1) X_1(a_1 + 2, c_1 + 1) \\
& + \frac{a_2 x_3}{c_1} X_1(a_1 + 1, a_2 + 1, c_1 + 1).
\end{aligned}$$

To obtain contiguous relation for $X_1(a_1 + 2)$, we replace $a_1 \rightarrow a_1 + 1$ in (2.8) and simplify. This leads to

$$\begin{aligned}
 X_1(a_1 + 2) &= X_1 + \frac{2x_1}{c_2} \sum_{n_1=1}^2 (a_1 + n_1) X_1(a_1 + n_1 + 1, c_2 + 1) \\
 &\quad + \frac{2x_2}{c_1} \sum_{n_1=1}^2 (a_1 + n_1) X_1(a_1 + n_1 + 1, c_1 + 1) \\
 (2.9) \quad &+ \frac{a_2 x_3}{c_1} \sum_{n_1=1}^2 X_1(a_1 + n_1, a_2 + 1, c_1 + 1).
 \end{aligned}$$

Iterating this process n times, we obtain (2.1). For the proof of (2.2), replace the parameter $a_1 \rightarrow a_1 - 1$ in (2.8). This gives

$$\begin{aligned}
 X_1(a_1 - 1) &= X_1 - \frac{2x_1}{c_2} a_1 X_1(a_1 + 1, c_2 + 1) - \frac{2x_2}{c_1} a_1 X_1(a_1 + 1, c_1 + 1) \\
 (2.10) \quad &- \frac{a_2 x_3}{c_1} X_1(a_2 + 1, c_1 + 1).
 \end{aligned}$$

Iteratively, we get (2.2).

The recursion formulas (2.3) and (2.4) can be proved in similar manner. Now, we give the proof of (2.5).

Using the definition of Exton's triple hypergeometric function X_1 and the relation

$$(2.11) \quad (a_2 + 1)_{m_3} = (a_2)_{m_3} \left(1 + \frac{m_3}{a_2}\right)$$

we get

$$(2.12) \quad X_1(a_2 + 1) = X_1 + \frac{a_1 x_3}{c_1} X_1(a_1 + 1, a_2 + 1, c_1 + 1).$$

Replace $a_2 \rightarrow a_2 + 1$ in (2.12) to get

$$\begin{aligned}
 X_1(a_2 + 2) &= X_1 + \frac{a_1 x_3}{c_1} X_1(a_1 + 1, a_2 + 1, c_1 + 1) \\
 (2.13) \quad &+ \frac{a_1 x_3}{c_1} \left[X_1(a_1 + 1, a_2 + 1, c_1 + 1) + \frac{(a_1 + 1)x_3}{(c_1 + 1)} X_1(a_1 + 2, a_2 + 2, c_1 + 2) \right].
 \end{aligned}$$

Simplifying, we get

$$X_1(a_2 + 2)$$

$$(2.14) \quad = X_1 + \frac{2a_1x_3}{c_1} X_1(a_1 + 1, a_2 + 1, c_1 + 1) + \frac{(a_1)_2 x_3^2}{(c_1)_2} X_1(a_1 + 2, a_2 + 2, c_1 + 2)$$

Iterating this process n times, we get (2.5). Proof of (2.6) is similar. $\square$

By using Pascal's identity the recursion formulas (2.5) and (2.6) can also be proved by induction method.

Theorem 2.2. *The following recursion formulas hold true for the denominator parameters c_1, c_2 of the Exton's triple hypergeometric function X_1 :*

$$(2.15) \quad X_1(c_1 - n) = X_1 + (a_1)_2 x_2 \sum_{n_1=1}^n \frac{1}{(c_1 - n_1)(c_1 + 1 - n_1)} X_1(a_1 + 2, c_1 + 2 - n_1)$$

$$+ a_1 a_2 x_3 \sum_{n_1=1}^n \frac{1}{(c_1 - n_1)(c_1 + 1 - n_1)} X_1(a_1 + 1, a_2 + 1, c_1 + 2 - n_1),$$

$$(2.16) \quad X_1(c_2 - n) = X_1 + (a_1)_2 x_1 \sum_{n_1=1}^n \frac{1}{(c_2 - n_1)(c_2 + 1 - n_1)} X_1(a_1 + 2, c_2 + 2 - n_1),$$

The above result can also be expressed as

$$(2.17) \quad X_1(c_2 - n) = \sum_{n_1=0}^n \binom{n}{n_1} \frac{x_1^{n_1} (a_1)_{2n_1}}{(c_2)_{n_1} (c_2 - n)_{n_1}} X_1(a_1 + 2n_1, c_2 + n_1).$$

Proof. Using the definition of Exton's triple hypergeometric function X_1 and the relation

$$(2.18) \quad \frac{1}{(c_1 - 1)_{m_2+m_3}} = \frac{1}{(c_1)_{m_2+m_3}} \left(1 + \frac{m_2}{c_1 - 1} + \frac{m_3}{c_1 - 1} \right)$$

we have

$$(2.19) \quad \begin{aligned} X_1(c_1 - 1) &= X_1 + \frac{(a_1)_2 x_2}{c_1(c_1 - 1)} X_1(a_1 + 2, c_1 + 1) \\ &+ \frac{a_1 a_2 x_3}{c_1(c_1 - 1)} X_1(a_1 + 1, a_2 + 1, c_1 + 1). \end{aligned}$$

Using this contiguous relation to the X_1 with the parameter $c_1 - n$ for n times, we get result (2.15). Second recursion formula (2.16) can be proved in a similar manner.

Now, we give the proof of (2.17). Apply the definition of X_1 and the relation

$$(2.20) \quad \frac{1}{(c_2 - 1)_{m_1}} = \frac{1}{(c_2)_{m_1}} \left(1 + \frac{m_1}{c_2 - 1} \right)$$

to get

$$(2.21) \quad X_1(c_2 - 1) = X_1 + \frac{(a_1)_2 x_1}{c_2(c_2 - 1)} X_1(a_1 + 2, c_2 + 1)$$

Replacing $c_2 \rightarrow c_2 - 1$ in (2.21) and simplifying leads to

$$(2.22) \quad \begin{aligned} X_1(c_2 - 2) &= X_1 + \frac{2(a_1)_2 x_1}{c_2(c_2 - 2)} X_1(a_1 + 2, c_2 + 1) \\ &\quad + \frac{(a_1)_4 x_1^2}{(c_2)_2(c_2 - 2)_2} X_1(a_1 + 4, c_2 + 2) \end{aligned}$$

Iteratively, we get (2.17). $\square$

Now, we present the recursion formulas for other Exton's triple hypergeometric functions. We omit the proof of the given below theorems.

3. Recursion Formulas for X_2

Theorem 3.1. *The following recursion formulas hold true for the numerator parameters a_1, a_2 of the Exton's triple hypergeometric function X_2 :*

$$(3.1) \quad \begin{aligned} X_2(a_1 + n) &= X_2 + \frac{2x_1}{c_1} \sum_{n_1=1}^n (a_1 + n_1) X_2(a_1 + 1 + n_1, c_1 + 1) \\ &\quad + \frac{2x_2}{c_2} \sum_{n_1=1}^n (a_1 + n_1) X_2(a_1 + 1 + n_1, c_2 + 1) \\ &\quad + \frac{a_2 x_3}{c_3} \sum_{n_1=1}^n X_2(a_1 + n_1, a_2 + 1, c_3 + 1), \end{aligned}$$

$$(3.2) \quad \begin{aligned} X_2(a_1 - n) &= X_2 - \frac{2x_1}{c_1} \sum_{n_1=1}^n (a_1 + 1 - n_1) X_2(a_1 + 2 - n_1, c_1 + 1) \\ &\quad - \frac{2x_2}{c_2} \sum_{n_1=1}^n (a_1 + 1 - n_1) X_2(a_1 + 2 - n_1, c_2 + 1) \\ &\quad - \frac{a_2 x_3}{c_3} \sum_{n_1=1}^n X_2(a_1 + 1 - n_1, a_2 + 1, c_3 + 1), \end{aligned}$$

$$(3.3) \quad X_2(a_2 + n) = X_2 + \frac{a_1 x_3}{c_3} \sum_{n_1=1}^n X_2(a_1 + 1, a_2 + n_1, c_3 + 1),$$

$$(3.4) \quad X_2(a_2 - n) = X_2 - \frac{a_1 x_3}{c_3} \sum_{n_1=1}^n X_2(a_1 + 1, a_2 + 1 - n_1, c_3 + 1).$$

Theorem 3.2. *The following recursion formulas hold true for the denominator parameter a_2 of the Exton's triple hypergeometric function X_2 :*

$$(3.5) \quad X_2(a_2 + n) = \sum_{n_1=0}^n \binom{n}{n_1} \frac{(a_1)_{n_1} x_3^{n_1}}{(c_3)_{n_1}} X_2(a_1 + n_1, a_2 + n_1, c_3 + n_1),$$

$$(3.6) \quad X_2(a_2 - n) = \sum_{n_1=0}^n \binom{n}{n_1} \frac{(a_1)_{n_1} (-x_3)^{n_1}}{(c_3)_{n_1}} X_2(a_1 + n_1, c_3 + n_1).$$

Theorem 3.3. *The following recursion formulas hold true for the denominator parameters c_i , $i = 1, 2$, and c_3 of the Exton's triple hypergeometric function X_2 :*

$$(3.7) \quad X_2(c_i - n) = X_2 + (a_1)_2 x_i \sum_{n_1=1}^n \frac{1}{(c_i - n_1)(c_i + 1 - n_1)} X_2(a_1 + 2, c_i + 2 - n_1),$$

$$(3.8) \quad X_2(c_i - n) = \sum_{n_1=0}^n \binom{n}{n_1} \frac{x_i^{n_1} (a_1)_{2n_1}}{(c_i)_{n_1} (c_i - n)_{n_1}} X_2(a_1 + 2n_1, c_i + n_1),$$

$$(3.9) \quad X_2(c_3 - n) = X_2 + a_1 a_2 x_3 \sum_{n_1=1}^n \frac{1}{(c_3 - n_1)(c_3 + 1 - n_1)} X_2(a_1 + 1, a_2 + 1, c_3 + 2 - n_1),$$

$$(3.10) \quad X_2(c_3 - n) = \sum_{n_1=0}^n \binom{n}{n_1} \frac{x_3^{n_1} (a_1)_{n_1} (a_2)_{n_1}}{(c_3)_{n_1} (c_3 - n)_{n_1}} X_2(a_1 + n_1, a_2 + n_1, c_3 + n_1).$$

4. Recursion Formulas for X_3

Theorem 4.1. *The following recursion formulas hold true for the numerator parameters a_1 , a_2 of the Exton's triple hypergeometric function X_3 :*

$$(4.1) \quad \begin{aligned} X_3(a_1 + n) = & X_3 + \frac{2x_1}{c_1} \sum_{n_1=1}^n (a_1 + n_1) X_3(a_1 + 1 + n_1, c_1 + 1) \\ & + \frac{a_2 x_2}{c_1} \sum_{n_1=1}^n X_3(a_1 + n_1, a_2 + 1, c_1 + 1) \\ & + \frac{a_2 x_3}{c_2} \sum_{n_1=1}^n X_3(a_1 + n_1, a_2 + 1, c_2 + 1), \end{aligned}$$

$$\begin{aligned}
 X_3(a_1 - n) = & X_3 - \frac{2x_1}{c_1} \sum_{n_1=1}^n (a_1 + 1 - n_1) X_3(a_1 + 2 - n_1, c_1 + 1) \\
 & - \frac{a_2 x_2}{c_1} \sum_{n_1=1}^n X_3(a_1 + 1 - n_1, a_2 + 1, c_1 + 1) \\
 & - \frac{a_2 x_3}{c_2} \sum_{n_1=1}^n X_3(a_1 + 1 - n_1, a_2 + 1, c_2 + 1),
 \end{aligned}
 \tag{4.2}$$

$$\begin{aligned}
 X_3(a_2 + n) = & X_3 + \frac{a_1 x_2}{c_1} \sum_{n_1=1}^n X_3(a_1 + 1, a_2 + n_1, c_1 + 1) \\
 & + \frac{a_1 x_3}{c_2} \sum_{n_1=1}^n X_3(a_1 + 1, a_2 + n_1, c_2 + 1),
 \end{aligned}
 \tag{4.3}$$

$$\begin{aligned}
 X_3(a_2 - n) = & X_3 - \frac{a_1 x_2}{c_1} \sum_{n_1=1}^n X_3(a_1 + 1, a_2 + 1 - n_1, c_1 + 1) \\
 & - \frac{a_1 x_3}{c_2} \sum_{n_1=1}^n X_3(a_1 + 1, a_2 + 1 - n_1, c_2 + 1).
 \end{aligned}
 \tag{4.4}$$

Theorem 4.2. *The following recursion formulas hold true for the numerator parameter a_2 of the Exton's triple hypergeometric function X_3 :*

$$\begin{aligned}
 X_3(a_2 + n) = & \sum_{n_1=0}^n \sum_{n_2=0}^{n-n_1} \binom{n}{n_1} \binom{n-n_1}{n_2} \frac{(a_1)_{N_2} x_2^{n_1} x_3^{n_2}}{(c_1)_{n_1} (c_2)_{n_2}} \\
 & \times X_3(a_1 + N_2, a_2 + N_2, c_1 + n_1, c_2 + n_2),
 \end{aligned}
 \tag{4.5}$$

$$\begin{aligned}
 X_3(a_2 - n) = & \sum_{n_1=0}^n \sum_{n_2=0}^{n-n_1} \binom{n}{n_1} \binom{n-n_1}{n_2} \frac{(a_1)_{N_2} (-x_2)^{n_1} (-x_3)^{n_2}}{(c_1)_{n_1} (c_2)_{n_2}} \\
 & \times X_3(a_1 + N_2, c_1 + n_1, c_2 + n_2),
 \end{aligned}
 \tag{4.6}$$

where $N_2 = n_1 + n_2$.

Theorem 4.3. *The following recursion formulas hold true for the denominator parameters c_1, c_2 of the Exton's triple hypergeometric function X_3 :*

$$\begin{aligned}
 X_3(c_1 - n) = & X_3 + (a_1)_2 x_1 \sum_{n_1=1}^n \frac{1}{(c_1 - n_1)(c_1 + 1 - n_1)} X_3(a_1 + 2, c_1 + 2 - n_1) \\
 & + a_1 a_2 x_2 \sum_{n_1=1}^n \frac{1}{(c_1 - n_1)(c_1 + 1 - n_1)} X_3(a_1 + 1, a_2 + 1, c_1 + 2 - n_1), \\
 & X_3(c_2 - n)
 \end{aligned}
 \tag{4.7}$$

(4.8)

$$= X_3 + a_1 a_2 x_3 \sum_{n_1=1}^n \frac{1}{(c_2 - n_1)(c_2 + 1 - n_1)} X_3(a_1 + 1, a_2 + 1, c_2 + 2 - n_1),$$

(4.9)

$$X_3(c_2 - n) = \sum_{n_1=0}^n \binom{n}{n_1} \frac{x_3^{n_1} (a_1)_{n_1} (a_2)_{n_1}}{(c_2)_{n_1} (c_2 - n)_{n_1}} X_3(a_1 + n_1, a_2 + n_1, c_2 + n_1).$$

5. Recursion Formulas for X_4

Theorem 5.1. *The following recursion formulas hold true for the numerator parameters a_1, a_2 of the Exton's triple hypergeometric function X_4 :*

$$X_4(a_1 + n) = X_4 + \frac{2x_1}{c_1} \sum_{n_1=1}^n (a_1 + n_1) X_4(a_1 + 1 + n_1, c_1 + 1)$$

(5.1)

$$+ \frac{a_2 x_2}{c_2} \sum_{n_1=1}^n X_4(a_1 + n_1, a_2 + 1, c_2 + 1) + \frac{a_2 x_3}{c_3} \sum_{n_1=1}^n X_4(a_1 + n_1, a_2 + 1, c_3 + 1),$$

$$X_4(a_1 - n) = X_4 - \frac{2x_1}{c_1} \sum_{n_1=1}^n (a_1 - n_1 + 1) X_4(a_1 + 2 - n_1, c_1 + 1)$$

$$- \frac{a_2 x_2}{c_2} \sum_{n_1=1}^n X_4(a_1 + 1 - n_1, a_2 + 1, c_2 + 1)$$

(5.2)

$$- \frac{a_2 x_3}{c_3} \sum_{n_1=1}^n X_4(a_1 + 1 - n_1, a_2 + 1, c_3 + 1),$$

$$X_4(a_2 + n) = X_4 + \frac{a_1 x_2}{c_2} \sum_{n_1=1}^n X_4(a_1 + 1, a_2 + n_1, c_2 + 1)$$

(5.3)

$$+ \frac{a_1 x_3}{c_3} \sum_{n_1=1}^n X_4(a_1 + 1, a_2 + n_1, c_3 + 1),$$

$$X_4(a_2 - n) = X_4 - \frac{a_1 x_2}{c_2} \sum_{n_1=1}^n X_4(a_1 + 1, a_2 + 1 - n_1, c_2 + 1)$$

(5.4)

$$- \frac{a_1 x_3}{c_3} \sum_{n_1=1}^n X_4(a_1 + 1, a_2 + 1 - n_1, c_3 + 1).$$

Theorem 5.2. *The following recursion formulas hold true for the numerator parameter a_2 of the Exton's triple hypergeometric function X_4 :*

$$(5.5) \quad X_4(a_2 + n) = \sum_{n_1=0}^n \sum_{n_2=0}^{n-n_1} \binom{n}{n_1} \binom{n-n_1}{n_2} \frac{(a_1)_{N_2} x_2^{n_1} x_3^{n_2}}{(c_2)_{n_1} (c_3)_{n_2}} \\ \times X_4(a_1 + N_2, a_2 + N_2, c_2 + n_1, c_3 + n_2),$$

$$(5.6) \quad X_4(a_2 - n) = \sum_{n_1=0}^n \sum_{n_2=0}^{n-n_1} \binom{n}{n_1} \binom{n-n_1}{n_2} \frac{(a_1)_{N_2} (-x_2)^{n_1} (-x_3)^{n_2}}{(c_2)_{n_1} (c_3)_{n_2}} \\ \times X_4(a_1 + N_2, c_2 + n_1, c_3 + n_2),$$

where $N_2 = n_1 + n_2$.

Theorem 5.3. *The following recursion formulas hold true for the denominator parameters $c_1, c_i; i = 2, 3$ of the Exton's triple hypergeometric function X_4 :*

$$(5.7) \quad X_4(c_1 - n) = X_4 + (a_1)_2 x_1 \sum_{n_1=1}^n \frac{1}{(c_1 - n_1)(c_1 + 1 - n_1)} X_4(a_1 + 2, c_1 + 2 - n_1),$$

$$(5.8) \quad X_4(c_1 - n) = \sum_{n_1=0}^n \binom{n}{n_1} \frac{x_1^{n_1} (a_1)_{2n_1}}{(c_1)_{n_1} (c_1 - n)_{n_1}} X_4(a_1 + 2n_1, c_1 + n_1),$$

$$(5.9) \quad X_4(c_i - n) \\ = X_4 + a_1 a_2 x_i \sum_{n_1=1}^n \frac{1}{(c_i - n_1)(c_i + 1 - n_1)} X_4(a_1 + 1, a_2 + 1, c_i + 2 - n_1),$$

$$(5.10) \quad X_4(c_i - n) = \sum_{n_1=0}^n \binom{n}{n_1} \frac{x_i^{n_1} (a_1)_{n_1} (a_2)_{n_1}}{(c_i)_{n_1} (c_i - n)_{n_1}} X_4(a_1 + n_1, a_2 + n_1, c_i + n_1).$$

6. Recursion Formulas for X_5

Theorem 6.1. *The following recursion formulas hold true for the numerator parameters $a_1, a_i; i = 2, 3$, of the Exton's triple hypergeometric function X_5 :*

$$X_5(a_1 + n) \\ = X_5 + \frac{2x_1}{c} \sum_{n_1=1}^n (a_1 + n_1) X_5(a_1 + 1 + n_1, c + 1)$$

(6.1)

$$+ \frac{a_2 x_2}{c} \sum_{n_1=1}^n X_5(a_1 + n_1, a_2 + 1, c + 1) + \frac{a_3 x_3}{c} \sum_{n_1=1}^n X_5(a_1 + n_1, a_3 + 1, c + 1),$$

$$X_5(a_1 - n) = X_5 - \frac{2x_1}{c} \sum_{n_1=1}^n (a_1 + 1 - n_1) X_5(a_1 + 2 - n_1, c + 1) \\ - \frac{a_2 x_2}{c} \sum_{n_1=1}^n X_5(a_1 + 1 - n_1, a_2 + 1, c + 1)$$

(6.2)

$$- \frac{a_3 x_3}{c} \sum_{n_1=1}^n X_5(a_1 + 1 - n_1, a_3 + 1, c + 1),$$

(6.3)

$$X_5(a_i + n) = X_5 + \frac{a_1 x_i}{c} \sum_{n_1=1}^n X_5(a_1 + 1, a_i + n_1, c + 1),$$

(6.4)

$$X_5(a_i - n) = X_5 - \frac{a_1 x_i}{c} \sum_{n_1=1}^n X_5(a_1 + 1, a_i + 1 - n_1, c + 1),$$

Theorem 6.2. *The following recursion formulas hold true for the numerator parameters a_i ; $i = 2, 3$, of the Exton's triple hypergeometric function X_5 :*

$$(6.5) \quad X_5(a_i + n) = \sum_{n_1=0}^n \binom{n}{n_1} \frac{(a_1)_{n_1} x_i^{n_1}}{(c)_{n_1}} X_5(a_1 + n_1, a_i + n_1, c + n_1),$$

$$(6.6) \quad X_5(a_i - n) = \sum_{n_1=0}^n \binom{n}{n_1} \frac{(a_1)_{n_1} (-x_i)^{n_1}}{(c)_{n_1}} X_5(a_1 + n_1, c + n_1).$$

Theorem 6.3. *The following recursion formulas hold true for the denominator parameter c of the Exton's triple hypergeometric function X_5 :*

$$X_5(c - n) = X_5 + (a_1)_2 x_1 \sum_{n_1=1}^n \frac{1}{(c - n_1)(c + 1 - n_1)} X_5(a_1 + 2, c + 2 - n_1) \\ + a_1 a_2 x_2 \sum_{n_1=1}^n \frac{1}{(c - n_1)(c + 1 - n_1)} X_5(a_1 + 1, a_2 + 1, c + 2 - n_1) \\ (6.7) \quad + a_1 a_3 x_3 \sum_{n_1=1}^n \frac{1}{(c - n_1)(c + 1 - n_1)} X_5(a_1 + 1, a_3 + 1, c + 2 - n_1).$$

7. Recursion Formulas for X_6

Theorem 7.1. *The following recursion formulas hold true for the numerator parameters $a_1, a_i; i = 2, 3$, of the Exton's triple hypergeometric function X_6 :*

$$(7.1) \quad \begin{aligned} X_6(a_1 + n) = & X_6 + \frac{2x_1}{c_1} \sum_{n_1=1}^n (a_1 + n_1) X_6(a_1 + 1 + n_1, c_1 + 1) \\ & + \frac{a_2 x_2}{c_1} \sum_{n_1=1}^n X_6(a_1 + n_1, a_2 + 1, c_1 + 1) \\ & + \frac{a_3 x_3}{c_2} \sum_{n_1=1}^n X_6(a_1 + n_1, a_3 + 1, c_2 + 1), \end{aligned}$$

$$(7.2) \quad \begin{aligned} X_6(a_1 - n) = & X_6 - \frac{2x_1}{c_1} \sum_{n_1=1}^n (a_1 + 1 - n_1) X_6(a_1 + 2 - n_1, c_1 + 1) \\ & - \frac{a_2 x_2}{c_1} \sum_{n_1=1}^n X_6(a_1 + 1 - n_1, a_2 + 1, c_1 + 1) \\ & - \frac{a_3 x_3}{c_2} \sum_{n_1=1}^n X_6(a_1 + 1 - n_1, a_3 + 1, c_2 + 1), \end{aligned}$$

$$(7.3) \quad X_6(a_i + n) = X_6 + \frac{a_1 x_i}{c_{i-1}} \sum_{n_1=1}^n X_6(a_1 + 1, a_i + n_1, c_{i-1} + 1),$$

$$(7.4) \quad X_6(a_i - n) = X_6 - \frac{a_1 x_i}{c_{i-1}} \sum_{n_1=1}^n X_6(a_1 + 1, a_i + 1 - n_1, c_{i-1} + 1).$$

Theorem 7.2. *The following recursion formulas hold true for the numerator parameters $a_i; i = 2, 3$, of the Exton's triple hypergeometric function X_6 :*

$$(7.5) \quad X_6(a_i + n) = \sum_{n_1=0}^n \binom{n}{n_1} \frac{(a_1)_{n_1} x_i^{n_1}}{(c_{i-1})_{n_1}} X_6(a_1 + n_1, a_i + n_1, c_{i-1} + n_1),$$

$$(7.6) \quad X_6(a_i - n) = \sum_{n_1=0}^n \binom{n}{n_1} \frac{(a_1)_{n_1} (-x_i)^{n_1}}{(c_{i-1})_{n_1}} X_6(a_1 + n_1, c_{i-1} + n_1).$$

Theorem 7.3. *The following recursion formulas hold true for the denominator parameters c_1, c_2 of the Exton's triple hypergeometric function X_6 :*

$$(7.7) \quad \begin{aligned} X_6(c_1 - n) = & X_6 + (a_1)_2 x_1 \sum_{n_1=1}^n \frac{1}{(c_1 - n_1)(c_1 + 1 - n_1)} X_6(a_1 + 2, c_1 + 2 - n_1) \\ & + a_1 a_2 x_2 \sum_{n_1=1}^n \frac{1}{(c_1 - n_1)(c_1 + 1 - n_1)} X_6(a_1 + 1, a_2 + 1, c_1 + 2 - n_1), \end{aligned}$$

$$\begin{aligned}
& X_6(c_2 - n) \\
(7.8) \quad & = X_6 + a_1 a_3 x_3 \sum_{n_1=1}^n \frac{1}{(c_2 - n_1)(c_2 + 1 - n_1)} X_6(a_1 + 1, a_3 + 1, c_2 + 2 - n_1), \\
(7.9) \quad & X_6(c_2 - n) = \sum_{n_1=0}^n \binom{n}{n_1} \frac{x_3^{n_1} (a_1)_{n_1} (a_3)_{n_1}}{(c_2)_{n_1} (c_2 - n)_{n_1}} X_6(a_1 + n_1, a_3 + n_1, c_2 + n_1).
\end{aligned}$$

8. Recursion Formulas for X_7

Theorem 8.1. *The following recursion formulas hold true for the numerator parameters a_1, a_i ; $i = 2, 3$, of the Exton's triple hypergeometric function X_7 :*

$$\begin{aligned}
(8.1) \quad X_7(a_1 + n) &= X_7 + \frac{2x_1}{c_2} \sum_{n_1=1}^n (a_1 + n_1) X_7(a_1 + 1 + n_1, c_2 + 1) \\
&\quad + \frac{a_2 x_2}{c_1} \sum_{n_1=1}^n X_7(a_1 + n_1, a_2 + 1, c_1 + 1) \\
&\quad + \frac{a_3 x_3}{c_1} \sum_{n_1=1}^n X_7(a_1 + n_1, a_3 + 1, c_1 + 1), \\
(8.2) \quad X_7(a_1 - n) &= X_7 - \frac{2x_1}{c_2} \sum_{n_1=1}^n (a_1 + 1 - n_1) X_7(a_1 + 2 - n_1, c_2 + 1) \\
&\quad - \frac{a_2 x_2}{c_1} \sum_{n_1=1}^n X_7(a_1 + 1 - n_1, a_2 + 1, c_1 + 1) \\
&\quad - \frac{a_3 x_3}{c_1} \sum_{n_1=1}^n X_7(a_1 + 1 - n_1, a_3 + 1, c_1 + 1), \\
(8.3) \quad X_7(a_i + n) &= X_7 + \frac{a_1 x_i}{c_1} \sum_{n_1=1}^n X_7(a_1 + 1, a_i + n_1, c_1 + 1), \\
(8.4) \quad X_7(a_i - n) &= X_7 - \frac{a_1 x_i}{c_1} \sum_{n_1=1}^n X_7(a_1 + 1, a_i + 1 - n_1, c_1 + 1).
\end{aligned}$$

Theorem 8.2. *The following recursion formulas hold true for the numerator parameters a_i ; $i = 2, 3$, of the Exton's triple hypergeometric function X_7 :*

$$(8.5) \quad X_7(a_i + n) = \sum_{n_1=0}^n \binom{n}{n_1} \frac{(a_1)_{n_1} x_i^{n_1}}{(c_1)_{n_1}} X_7(a_1 + n_1, a_i + n_1, c_1 + n_1),$$

$$(8.6) \quad X_7(a_i - n) = \sum_{n_1=0}^n \binom{n}{n_1} \frac{(a_1)_{n_1} (-x_i)^{n_1}}{(c_1)_{n_1}} X_7(a_1 + n_1, c_1 + n_1).$$

Theorem 8.3. *The following recursion formulas hold true for the denominator parameters c_1, c_2 of the Exton's triple hypergeometric function X_7 :*

$$(8.7) \quad \begin{aligned} & X_7(c_1 - n) \\ &= X_7 + a_1 a_2 x_2 \sum_{n_1=1}^n \frac{1}{(c_1 - n_1)(c_1 + 1 - n_1)} X_7(a_1 + 1, a_2 + 1, c_1 + 2 - n_1) \end{aligned}$$

$$(8.8) \quad + a_1 a_3 x_3 \sum_{n_1=1}^n \frac{1}{(c_1 - n_1)(c_1 + 1 - n_1)} X_7(a_1 + 1, a_3 + 1, c_1 + 2 - n_1),$$

$$(8.9) \quad X_7(c_2 - n) = X_7 + (a_1)_2 x_1 \sum_{n_1=1}^n \frac{1}{(c_2 - n_1)(c_2 + 1 - n_1)} X_7(a_1 + 2, c_2 + 2 - n_1),$$

$$(8.9) \quad X_7(c_2 - n) = \sum_{n_1=0}^n \binom{n}{n_1} \frac{x_1^{n_1} (a_1)_{2n_1}}{(c_2)_{n_1} (c_2 - n)_{n_1}} X_7(a_1 + 2n_1, c_2 + n_1).$$

9. Recursion Formulas for X_8

Theorem 9.1. *The following recursion formulas hold true for the numerator parameters $a_1, a_i; i = 2, 3$, of the Exton's triple hypergeometric function X_8 :*

$$(9.1) \quad \begin{aligned} X_8(a_1 + n) &= X_8 + \frac{2x_1}{c_1} \sum_{n_1=1}^n (a_1 + n_1) X_8(a_1 + 1 + n_1, c_1 + 1) \\ &\quad + \frac{a_2 x_2}{c_2} \sum_{n_1=1}^n X_8(a_1 + n_1, a_2 + 1, c_2 + 1) \\ &\quad + \frac{a_3 x_3}{c_3} \sum_{n_1=1}^n X_8(a_1 + n_1, a_3 + 1, c_3 + 1), \end{aligned}$$

$$(9.2) \quad \begin{aligned} X_8(a_1 - n) &= X_8 - \frac{2x_1}{c_1} \sum_{n_1=1}^n (a_1 + 1 - n_1) X_8(a_1 + 2 - n_1, c_1 + 1) \\ &\quad - \frac{a_2 x_2}{c_2} \sum_{n_1=1}^n X_8(a_1 + 1 - n_1, a_2 + 1, c_2 + 1) \\ &\quad - \frac{a_3 x_3}{c_3} \sum_{n_1=1}^n X_8(a_1 + 1 - n_1, a_3 + 1, c_3 + 1), \end{aligned}$$

$$(9.3) \quad X_8(a_i + n) = X_8 + \frac{a_1 x_i}{c_i} \sum_{n_1=1}^n X_8(a_1 + 1, a_i + n_1, c_i + 1),$$

$$(9.4) \quad X_8(a_i - n) = X_8 - \frac{a_1 x_i}{c_i} \sum_{n_1=1}^n X_8(a_1 + 1, a_i + 1 - n_1, c_i + 1).$$

Theorem 9.2. *The following recursion formulas hold true for the numerator parameters a_i ; $i = 2, 3$, of the Exton's triple hypergeometric function X_8 :*

$$(9.5) \quad X_8(a_i + n) = \sum_{n_1=0}^n \binom{n}{n_1} \frac{(a_1)_{n_1} x_i^{n_1}}{(c_i)_{n_1}} X_8(a_1 + n_1, a_i + n_1, c_i + n_1),$$

$$(9.6) \quad X_8(a_i - n) = \sum_{n_1=0}^n \binom{n}{n_1} \frac{(a_1)_{n_1} (-x_i)^{n_1}}{(c_i)_{n_1}} X_8(a_1 + n_1, c_i + n_1).$$

Theorem 9.3. *The following recursion formulas hold true for the denominator parameters c_1, c_i , $i = 2, 3$, of the Exton's triple hypergeometric function X_8 :*

$$(9.7) \quad X_8(c_1 - n) = X_8 + (a_1)_2 x_1 \sum_{n_1=1}^n \frac{1}{(c_1 - n_1)(c_1 + 1 - n_1)} X_8(a_1 + 2, c_1 + 2 - n_1),$$

$$(9.8) \quad X_8(c_1 - n) = \sum_{n_1=0}^n \binom{n}{n_1} \frac{x_1^{n_1} (a_1)_{2n_1}}{(c_1)_{n_1} (c_1 - n)_{n_1}} X_8(a_1 + 2n_1, c_1 + n_1),$$

$$(9.9) \quad \begin{aligned} & X_8(c_i - n) \\ &= X_8 + a_1 a_i x_i \sum_{n_1=1}^n \frac{1}{(c_i - n_1)(c_i + 1 - n_1)} X_8(a_1 + 1, a_i + 1, c_i + 2 - n_1), \end{aligned}$$

$$(9.10) \quad X_8(c_i - n) = \sum_{n_1=0}^n \binom{n}{n_1} \frac{(a_1)_{n_1} (a_i)_{n_1} x_i^{n_1}}{(c_i)_{n_1} (c_i - n)_{n_1}} X_8(a_1 + n_1, a_i + n_1, c_i + n_1).$$

10. Recursion Formulas for X_9

Theorem 10.1. *The following recursion formulas hold true for the numerator parameters a_1, a_2 of the Exton's triple hypergeometric function X_9 :*

$$X_9(a_1 + n) = X_9 + \frac{2x_1}{c} \sum_{n_1=1}^n (a_1 + n_1) X_9(a_1 + 1 + n_1, c + 1)$$

$$\begin{aligned}
(10.1) \quad & + \frac{a_2 x_2}{c} \sum_{n_1=1}^n X_9(a_1 + n_1, a_2 + 1, c + 1), \\
& X_9(a_1 - n) = X_9 - \frac{2x_1}{c} \sum_{n_1=1}^n (a_1 + 1 - n_1) X_9(a_1 + 2 - n_1, c + 1) \\
(10.2) \quad & - \frac{a_2 x_2}{c} \sum_{n_1=1}^n X_9(a_1 + 1 - n_1, a_2 + 1, c + 1), \\
& X_9(a_2 + n) = X_9 + \frac{a_1 x_2}{c} \sum_{n_1=1}^n X_9(a_2 + n_1, a_1 + 1, c + 1) \\
(10.3) \quad & + \frac{2x_3}{c} \sum_{n_1=1}^n (a_2 + n_1) X_9(a_2 + 1 + n_1, c + 1), \\
& X_9(a_2 - n) = X_9 - \frac{a_1 x_2}{c} \sum_{n_1=1}^n X_9(a_2 + 1 - n_1, a_1 + 1, c + 1) \\
(10.4) \quad & - \frac{2x_3}{c} \sum_{n_1=1}^n (a_2 + 1 - n_1) X_9(a_2 + 2 - n_1, c + 1).
\end{aligned}$$

Theorem 10.2. *The following recursion formulas hold true for the denominator parameter c of the Exton's triple hypergeometric function X_9 :*

$$\begin{aligned}
(10.5) \quad & X_9(c - n) = X_9 + (a_1)_2 x_1 \sum_{n_1=1}^n \frac{1}{(c - n_1)(c + 1 - n_1)} X_9(a_1 + 2, c + 2 - n_1) \\
& + a_1 a_2 x_2 \sum_{n_1=1}^n \frac{1}{(c - n_1)(c + 1 - n_1)} X_9(a_1 + 1, a_2 + 1, c + 2 - n_1) \\
& + (a_2)_2 x_3 \sum_{n_1=1}^n \frac{1}{(c - n_1)(c + 1 - n_1)} X_9(a_2 + 2, c + 2 - n_1).
\end{aligned}$$

11. Recursion Formulas for X_{10}

Theorem 11.1. *The following recursion formulas hold true for the numerator parameters a_1, a_2 of the Exton's triple hypergeometric function X_{10} :*

$$\begin{aligned}
(11.1) \quad & X_{10}(a_1 + n) = X_{10} + \frac{2x_1}{c_1} \sum_{n_1=1}^n (a_1 + n_1) X_{10}(a_1 + 1 + n_1, c_1 + 1) \\
& + \frac{a_2 x_2}{c_1} \sum_{n_1=1}^n X_{10}(a_1 + n_1, a_2 + 1, c_1 + 1),
\end{aligned}$$

$$\begin{aligned}
 X_{10}(a_1 - n) = & X_{10} - \frac{2x_1}{c_1} \sum_{n_1=1}^n (a_1 + 1 - n_1) X_{10}(a_1 + 2 - n_1, c_1 + 1) \\
 & - \frac{a_2 x_2}{c_1} \sum_{n_1=1}^n X_{10}(a_1 + 1 - n_1, a_2 + 1, c_1 + 1),
 \end{aligned}
 \tag{11.2}$$

$$\begin{aligned}
 X_{10}(a_2 + n) = & X_{10} + \frac{a_1 x_2}{c_1} \sum_{n_1=1}^n X_{10}(a_2 + n_1, a_1 + 1, c_1 + 1) \\
 & + \frac{2x_3}{c_2} \sum_{n_1=1}^n (a_2 + n_1) X_{10}(a_2 + 1 + n_1, c_2 + 1),
 \end{aligned}
 \tag{11.3}$$

$$\begin{aligned}
 X_{10}(a_2 - n) = & X_{10} - \frac{a_1 x_2}{c_1} \sum_{n_1=1}^n X_{10}(a_2 + 1 - n_1, a_1 + 1, c_1 + 1) \\
 & - \frac{2x_3}{c_2} \sum_{n_1=1}^n (a_2 + 1 - n_1) X_{10}(a_2 + 2 - n_1, c_2 + 1).
 \end{aligned}
 \tag{11.4}$$

Theorem 11.2. *The following recursion formulas hold true for the denominator parameters c_1, c_2 of the Exton's triple hypergeometric function X_{10} :*

$$\begin{aligned}
 X_{10}(c_1 - n) = & X_{10} + (a_1)_2 x_1 \sum_{n_1=1}^n \frac{1}{(c_1 - n_1)(c_1 + 1 - n_1)} X_{10}(a_1 + 2, c_1 + 2 - n_1) \\
 & + a_1 a_2 x_2 \sum_{n_1=1}^n \frac{1}{(c_1 - n_1)(c_1 + 1 - n_1)} X_{10}(a_1 + 1, a_2 + 1, c_1 + 2 - n_1),
 \end{aligned}
 \tag{11.5}$$

$$\begin{aligned}
 X_{10}(c_2 - n) = & X_{10} + (a_2)_2 x_3 \sum_{n_1=1}^n \frac{1}{(c_2 - n_1)(c_2 + 1 - n_1)} X_{10}(a_2 + 2, c_2 + 2 - n_1), \\
 & + a_1 a_2 x_2 \sum_{n_1=1}^n \frac{1}{(c_2 - n_1)(c_2 + 1 - n_1)} X_{10}(a_1 + 1, a_2 + 1, c_2 + 2 - n_1),
 \end{aligned}
 \tag{11.6}$$

$$\begin{aligned}
 X_{10}(c_2 - n) = & \sum_{n_1=0}^n \binom{n}{n_1} \frac{x_3^{n_1} (a_2)_{2n_1}}{(c_2)_{n_1} (c_2 - n)_{n_1}} X_{10}(a_2 + 2n_1, c_2 + n_1).
 \end{aligned}
 \tag{11.7}$$

12. Recursion Formulas for X_{11}

Theorem 12.1. *The following recursion formulas hold true for the numerator parameters a_1, a_2 of the Exton's triple hypergeometric function X_{11} :*

$$X_{11}(a_1 + n) = X_{11} + \frac{2x_1}{c_1} \sum_{n_1=1}^n (a_1 + n_1) X_{11}(a_1 + 1 + n_1, c_1 + 1)$$

$$\begin{aligned}
(12.1) \quad & + \frac{a_2 x_2}{c_2} \sum_{n_1=1}^n X_{11}(a_1 + n_1, a_2 + 1, c_2 + 1), \\
& X_{11}(a_1 - n) = X_{11} - \frac{2x_1}{c_1} \sum_{n_1=1}^n (a_1 + 1 - n_1) X_{11}(a_1 + 2 - n_1, c_1 + 1) \\
(12.2) \quad & - \frac{a_2 x_2}{c_2} \sum_{n_1=1}^n X_{11}(a_1 + 1 - n_1, a_2 + 1, c_2 + 1), \\
& X_{11}(a_2 + n) = X_{11} + \frac{a_1 x_2}{c_2} \sum_{n_1=1}^n X_{11}(a_2 + n_1, a_1 + 1, c_2 + 1) \\
(12.3) \quad & + \frac{2x_3}{c_1} \sum_{n_1=1}^n (a_2 + n_1) X_{11}(a_2 + 1 + n_1, c_1 + 1), \\
& X_{11}(a_2 - n) = X_{11} - \frac{a_1 x_2}{c_2} \sum_{n_1=1}^n X_{11}(a_2 + 1 - n_1, a_1 + 1, c_2 + 1) \\
(12.4) \quad & - \frac{2x_3}{c_1} \sum_{n_1=1}^n (a_2 + 1 - n_1) X_{11}(a_2 + 2 - n_1, c_1 + 1).
\end{aligned}$$

Theorem 12.2. *The following recursion formulas hold true for the denominator parameters c_1, c_2 of the Exton's triple hypergeometric function X_{11} :*

$$\begin{aligned}
(12.5) \quad & X_{11}(c_1 - n) = X_{11} + (a_1)_2 x_1 \sum_{n_1=1}^n \frac{1}{(c_1 - n_1)(c_1 + 1 - n_1)} X_{11}(a_1 + 2, c_1 + 2 - n_1) \\
& + (a_2)_2 x_3 \sum_{n_1=1}^n \frac{1}{(c_1 - n_1)(c_1 + 1 - n_1)} X_{11}(a_2 + 2, c_1 + 2 - n_1),
\end{aligned}$$

$$\begin{aligned}
(12.6) \quad & X_{11}(c_2 - n) \\
& = X_{11} + a_1 a_2 x_2 \sum_{n_1=1}^n \frac{1}{(c_2 - n_1)(c_2 + 1 - n_1)} X_{11}(a_1 + 1, a_2 + 1, c_2 + 2 - n_1),
\end{aligned}$$

$$(12.7) \quad X_{11}(c_2 - n) = \sum_{n_1=0}^n \binom{n}{n_1} \frac{x_2^{n_1} (a_1)_{n_1} (a_2)_{n_1}}{(c_2)_{n_1} (c_2 - n)_{n_1}} X_{11}(a_1 + n_1, a_2 + n_1, c_2 + n_1).$$

13. Recursion Formulas for X_{12}

Theorem 13.1. *The following recursion formulas hold true for the numerator pa-*

rameters a_1, a_2 of the Exton's triple hypergeometric function X_{12} :

$$(13.1) \quad \begin{aligned} X_{12}(a_1 + n) &= X_{12} + \frac{2x_1}{c_1} \sum_{n_1=1}^n (a_1 + n_1) X_{12}(a_1 + 1 + n_1, c_1 + 1) \\ &\quad + \frac{a_2 x_2}{c_2} \sum_{n_1=1}^n X_{12}(a_1 + n_1, a_2 + 1, c_2 + 1), \end{aligned}$$

$$(13.2) \quad \begin{aligned} X_{12}(a_1 - n) &= X_{12} - \frac{2x_1}{c_1} \sum_{n_1=1}^n (a_1 + 1 - n_1) X_{12}(a_1 + 2 - n_1, c_1 + 1) \\ &\quad - \frac{a_2 x_2}{c_2} \sum_{n_1=1}^n X_{12}(a_1 + 1 - n_1, a_2 + 1, c_2 + 1), \end{aligned}$$

$$(13.3) \quad \begin{aligned} X_{12}(a_2 + n) &= X_{12} + \frac{a_1 x_2}{c_2} \sum_{n_1=1}^n X_{12}(a_2 + n_1, a_1 + 1, c_2 + 1) \\ &\quad + \frac{2x_3}{c_3} \sum_{n_1=1}^n (a_2 + n_1) X_{12}(a_2 + 1 + n_1, c_3 + 1), \end{aligned}$$

$$(13.4) \quad \begin{aligned} X_{12}(a_2 - n) &= X_{12} - \frac{a_1 x_2}{c_2} \sum_{n_1=1}^n X_{12}(a_2 + 1 - n_1, a_1 + 1, c_2 + 1) \\ &\quad - \frac{2x_3}{c_3} \sum_{n_1=1}^n (a_2 + 1 - n_1) X_{12}(a_2 + 2 - n_1, c_3 + 1). \end{aligned}$$

Theorem 13.2. *The following recursion formulas hold true for the denominator parameters c_1, c_2 and c_3 of the Exton's triple hypergeometric function X_{12} :*

$$(13.5) \quad \begin{aligned} &X_{12}(c_1 - n) \\ &= X_{12} + (a_1)_2 x_1 \sum_{n_1=1}^n \frac{1}{(c_1 - n_1)(c_1 + 1 - n_1)} X_{12}(a_1 + 2, c_1 + 2 - n_1), \end{aligned}$$

$$(13.6) \quad \begin{aligned} &X_{12}(c_2 - n) \\ &= X_{12} + a_1 a_2 x_2 \sum_{n_1=1}^n \frac{1}{(c_2 - n_1)(c_2 + 1 - n_1)} X_{12}(a_1 + 1, a_2 + 1, c_2 + 2 - n_1), \end{aligned}$$

$$(13.7) \quad \begin{aligned} &X_{12}(c_3 - n) \\ &= X_{12} + (a_2)_2 x_3 \sum_{n_1=1}^n \frac{1}{(c_3 - n_1)(c_3 + 1 - n_1)} X_{12}(a_2 + 2, c_3 + 2 - n_1). \end{aligned}$$

Theorem 13.3. *The following recursion formulas hold true for the denominator parameters c_1 , c_2 and c_3 of the Exton's triple hypergeometric function X_{12} :*

$$(13.8) \quad X_{12}(c_1 - n) = \sum_{n_1=0}^n \binom{n}{n_1} \frac{x_1^{n_1} (a_1)_{2n_1}}{(c_1)_{n_1} (c_1 - n)_{n_1}} X_{12}(a_1 + 2n_1, c_1 + n_1),$$

$$(13.9) \quad X_{12}(c_2 - n) = \sum_{n_1=0}^n \binom{n}{n_1} \frac{x_2^{n_1} (a_1)_{n_1} (a_2)_{n_1}}{(c_2)_{n_1} (c_2 - n)_{n_1}} X_{12}(a_1 + n_1, a_2 + n_1, c_2 + n_1),$$

$$(13.10) \quad X_{12}(c_3 - n) = \sum_{n_1=0}^n \binom{n}{n_1} \frac{x_3^{n_1} (a_2)_{2n_1}}{(c_3)_{n_1} (c_3 - n)_{n_1}} X_{12}(a_2 + 2n_1, c_3 + n_1).$$

14. Recursion Formulas for X_{13}

Theorem 14.1. *The following recursion formulas hold true for the numerator parameters a_1, a_2, a_3 of the Exton's triple hypergeometric function X_{13} :*

$$(14.1) \quad \begin{aligned} X_{13}(a_1 + n) = & X_{13} + \frac{2x_1}{c} \sum_{n_1=1}^n (a_1 + n_1) X_{13}(a_1 + 1 + n_1, c + 1) \\ & + \frac{a_2 x_2}{c} \sum_{n_1=1}^n X_{13}(a_1 + n_1, a_2 + 1, c + 1), \end{aligned}$$

$$(14.2) \quad \begin{aligned} X_{13}(a_1 - n) = & X_{13} - \frac{2x_1}{c} \sum_{n_1=1}^n (a_1 + 1 - n_1) X_{13}(a_1 + 2 - n_1, c + 1) \\ & - \frac{a_2 x_2}{c} \sum_{n_1=1}^n X_{13}(a_1 + 1 - n_1, a_2 + 1, c + 1). \end{aligned}$$

$$(14.3) \quad \begin{aligned} X_{13}(a_2 + n) = & X_{13} + \frac{a_1 x_2}{c} \sum_{n_1=1}^n X_{13}(a_2 + n_1, a_1 + 1, c + 1) \\ & + \frac{a_3 x_3}{c} \sum_{n_1=1}^n X_{13}(a_2 + n_1, a_3 + 1, c + 1), \end{aligned}$$

$$(14.4) \quad \begin{aligned} X_{13}(a_2 - n) = & X_{13} - \frac{a_1 x_2}{c} \sum_{n_1=1}^n X_{13}(a_2 + 1 - n_1, a_1 + 1, c + 1) \\ & - \frac{a_3 x_3}{c} \sum_{n_1=1}^n X_{13}(a_2 + 1 - n_1, a_3 + 1, c + 1), \end{aligned}$$

$$(14.5) \quad X_{13}(a_3 + n) = X_{13} + \frac{a_2 x_3}{c} \sum_{n_1=1}^n X_{13}(a_3 + n_1, a_2 + 1, c + 1),$$

$$(14.6) \quad X_{13}(a_3 - n) = X_{13} - \frac{a_2 x_3}{c} \sum_{n_1=1}^n X_{13}(a_3 + 1 - n_1, a_2 + 1, c + 1).$$

Theorem 14.2. *The following recursion formulas hold true for the numerator parameters a_2, a_3 of the Exton's triple hypergeometric function X_{13} :*

$$(14.7) \quad \begin{aligned} X_{13}(a_2 + n) &= \sum_{n_1=0}^n \sum_{n_2=0}^{n-n_1} \binom{n}{n_1} \binom{n-n_1}{n_2} \frac{(a_1)_{n_1} (a_3)_{n_2} x_2^{n_1} x_3^{n_2}}{(c)_{N_2}} \\ &\times X_{13}(a_1 + n_1, a_2 + N_2, a_3 + n_2, c + N_2), \end{aligned}$$

$$(14.8) \quad \begin{aligned} X_{13}(a_2 - n) &= \sum_{n_1=0}^n \sum_{n_2=0}^{n-n_1} \binom{n}{n_1} \binom{n-n_1}{n_2} \frac{(a_1)_{n_1} (a_3)_{n_2} (-x_2)^{n_1} (-x_3)^{n_2}}{(c)_{N_2}} \\ &\times X_{13}(a_1 + n_1, a_3 + n_2, c + N_2), \end{aligned}$$

$$(14.9) \quad X_{13}(a_3 + n) = \sum_{n_1=0}^n \binom{n}{n_1} \frac{(a_2)_{n_1} x_3^{n_1}}{(c)_{n_1}} X_{13}(a_3 + n_1, a_2 + n_1, c + n_1),$$

$$(14.10) \quad X_{13}(a_3 - n) = \sum_{n_1=0}^n \binom{n}{n_1} \frac{(a_2)_{n_1} (-x_3)^{n_1}}{(c)_{n_1}} X_{13}(a_2 + n_1, c + n_1),$$

where $N_2 = n_1 + n_2$.

Theorem 14.3. *The following recursion formulas hold true for the denominator parameter c of the Exton's triple hypergeometric function X_{13} :*

$$(14.11) \quad \begin{aligned} X_{13}(c - n) &= X_{13} + (a_1)_2 x_1 \sum_{n_1=1}^n \frac{1}{(c - n_1)(c + 1 - n_1)} X_{13}(a_1 + 2, c + 2 - n_1) \\ &+ a_1 a_2 x_2 \sum_{n_1=1}^n \frac{1}{(c - n_1)(c + 1 - n_1)} X_{13}(a_1 + 1, a_2 + 1, c + 2 - n_1) \\ &+ a_2 a_3 x_3 \sum_{n_1=1}^n \frac{1}{(c - n_1)(c + 1 - n_1)} X_{13}(a_2 + 1, a_3 + 1, c + 2 - n_1). \end{aligned}$$

15. Recursion Formulas for X_{14}

Theorem 15.1. *The following recursion formulas hold true for the numerator parameters a_1, a_2, a_3 of the Exton's triple hypergeometric function X_{14} :*

$$(15.1) \quad \begin{aligned} X_{14}(a_1 + n) &= X_{14} + \frac{2x_1}{c_1} \sum_{n_1=1}^n (a_1 + n_1) X_{14}(a_1 + 1 + n_1, c_1 + 1) \\ &+ \frac{a_2 x_2}{c_1} \sum_{n_1=1}^n X_{14}(a_1 + n_1, a_2 + 1, c_1 + 1), \end{aligned}$$

$$\begin{aligned}
 X_{14}(a_1 - n) = & X_{14} - \frac{2x_1}{c_1} \sum_{n_1=1}^n (a_1 + 1 - n_1) X_{14}(a_1 + 2 - n_1, c_1 + 1) \\
 & - \frac{a_2 x_2}{c_1} \sum_{n_1=1}^n X_{14}(a_1 + 1 - n_1, a_2 + 1, c_1 + 1),
 \end{aligned}
 \tag{15.2}$$

$$\begin{aligned}
 X_{14}(a_2 + n) = & X_{14} + \frac{a_1 x_2}{c_1} \sum_{n_1=1}^n X_{14}(a_2 + n_1, a_1 + 1, c_1 + 1) \\
 & + \frac{a_3 x_3}{c_2} \sum_{n_1=1}^n X_{14}(a_2 + n_1, a_3 + 1, c_2 + 1),
 \end{aligned}
 \tag{15.3}$$

$$\begin{aligned}
 X_{14}(a_2 - n) = & X_{14} - \frac{a_1 x_2}{c_1} \sum_{n_1=1}^n X_{14}(a_2 + 1 - n_1, a_1 + 1, c_1 + 1) \\
 & - \frac{a_3 x_3}{c_2} \sum_{n_1=1}^n X_{14}(a_2 + 1 - n_1, a_3 + 1, c_2 + 1),
 \end{aligned}
 \tag{15.4}$$

$$X_{14}(a_3 + n) = X_{14} + \frac{a_2 x_3}{c_2} \sum_{n_1=1}^n X_{14}(a_3 + n_1, a_2 + 1, c_2 + 1),
 \tag{15.5}$$

$$X_{14}(a_3 - n) = X_{14} - \frac{a_2 x_3}{c_2} \sum_{n_1=1}^n X_{14}(a_3 + 1 - n_1, a_2 + 1, c_2 + 1).
 \tag{15.6}$$

Theorem 15.2. *The following recursion formulas hold true for the numerator parameters a_2, a_3 of the Exton's triple hypergeometric function X_{14} :*

$$\begin{aligned}
 X_{14}(a_2 + n) = & \sum_{n_1=0}^n \sum_{n_2=0}^{n-n_1} \binom{n}{n_1} \binom{n-n_1}{n_2} \frac{(a_1)_{n_1} (a_3)_{n_2} x_2^{n_1} x_3^{n_2}}{(c_1)_{n_1} (c_2)_{n_2}} \\
 & \times X_{14}(a_1 + n_1, a_2 + N_2, a_3 + n_2, c_1 + n_1, c_2 + n_2),
 \end{aligned}
 \tag{15.7}$$

$$\begin{aligned}
 X_{14}(a_2 - n) = & \sum_{n_1=0}^n \sum_{n_2=0}^{n-n_1} \binom{n}{n_1} \binom{n-n_1}{n_2} \frac{(a_1)_{n_1} (a_3)_{n_2} (-x_2)^{n_1} (-x_3)^{n_2}}{(c_1)_{n_1} (c_2)_{n_2}} \\
 & \times X_{14}(a_1 + n_1, a_3 + n_2, c_1 + n_1, c_2 + n_2),
 \end{aligned}
 \tag{15.8}$$

$$X_{14}(a_3 + n) = \sum_{n_1=0}^n \binom{n}{n_1} \frac{(a_2)_{n_1} x_3^{n_1}}{(c_2)_{n_1}} X_{14}(a_3 + n_1, a_2 + n_1, c_2 + n_1),
 \tag{15.9}$$

$$X_{14}(a_3 - n) = \sum_{n_1=0}^n \binom{n}{n_1} \frac{(a_2)_{n_1} (-x_3)^{n_1}}{(c_2)_{n_1}} X_{14}(a_2 + n_1, c_2 + n_1),
 \tag{15.10}$$

where $N_2 = n_1 + n_2$.

Theorem 15.3. *The following recursion formulas hold true for the denominator*

parameters c_1, c_2 of the Exton's triple hypergeometric function X_{14} :

$$\begin{aligned} X_{14}(c_1 - n) &= X_{14} + (a_1)_2 x_1 \sum_{n_1=1}^n \frac{1}{(c_1 - n_1)(c_1 + 1 - n_1)} X_{14}(a_1 + 2, c_1 + 2 - n_1) \\ (15.11) \quad &+ a_1 a_2 x_2 \sum_{n_1=1}^n \frac{1}{(c_1 - n_1)(c_1 + 1 - n_1)} X_{14}(a_1 + 1, a_2 + 1, c_1 + 2 - n_1), \end{aligned}$$

$$\begin{aligned} X_{14}(c_2 - n) &= X_{14} + a_2 a_3 x_3 \sum_{n_1=1}^n \frac{1}{(c_2 - n_1)(c_2 + 1 - n_1)} \\ (15.12) \quad &\times X_{14}(a_2 + 1, a_3 + 1, c_2 + 2 - n_1), \end{aligned}$$

$$\begin{aligned} X_{14}(c_2 - n) &= \sum_{n_1=0}^n \binom{n}{n_1} \frac{(a_2)_{n_1} (a_3)_{n_1} x_3^{n_1}}{(c_2)_{n_1} (c_2 - n)_{n_1}} \\ (15.13) \quad &\times X_{14}(a_2 + n_1, a_3 + n_1, c_2 + n_1). \end{aligned}$$

16. Recursion Formulas for X_{15}

Theorem 16.1. *The following recursion formulas hold true for the numerator parameters a_1, a_2, a_3 of the Exton's triple hypergeometric function X_{15} :*

$$\begin{aligned} X_{15}(a_1 + n) &= X_{15} + \frac{2x_1}{c_2} \sum_{n_1=1}^n (a_1 + n_1) X_{15}(a_1 + 1 + n_1, c_2 + 1) \\ (16.1) \quad &+ \frac{a_2 x_2}{c_1} \sum_{n_1=1}^n X_{15}(a_1 + n_1, a_2 + 1, c_1 + 1), \end{aligned}$$

$$\begin{aligned} X_{15}(a_1 - n) &= X_{15} - \frac{2x_1}{c_2} \sum_{n_1=1}^n (a_1 + 1 - n_1) X_{15}(a_1 + 2 - n_1, c_2 + 1) \\ (16.2) \quad &- \frac{a_2 x_2}{c_1} \sum_{n_1=1}^n X_{15}(a_1 + 1 - n_1, a_2 + 1, c_1 + 1), \end{aligned}$$

$$\begin{aligned} X_{15}(a_2 + n) &= X_{15} + \frac{a_1 x_2}{c_1} \sum_{n_1=1}^n X_{15}(a_2 + n_1, a_1 + 1, c_1 + 1) \\ (16.3) \quad &+ \frac{a_3 x_3}{c_1} \sum_{n_1=1}^n X_{15}(a_2 + n_1, a_3 + 1, c_1 + 1), \end{aligned}$$

$$X_{15}(a_2 - n) = X_{15} - \frac{a_1 x_2}{c_1} \sum_{n_1=1}^n X_{15}(a_2 + 1 - n_1, a_1 + 1, c_1 + 1)$$

$$(16.4) \quad -\frac{a_3 x_3}{c_1} \sum_{n_1=1}^n X_{15}(a_2 + 1 - n_1, a_3 + 1, c_1 + 1),$$

$$(16.5) \quad X_{15}(a_3 + n) = X_{15} + \frac{a_2 x_3}{c_1} \sum_{n_1=1}^n X_{15}(a_3 + n_1, a_2 + 1, c_1 + 1),$$

$$(16.6) \quad X_{15}(a_3 - n) = X_{15} - \frac{a_2 x_3}{c_1} \sum_{n_1=1}^n X_{15}(a_3 + 1 - n_1, a_2 + 1, c_1 + 1).$$

Theorem 16.2. *The following recursion formulas hold true for the numerator parameters a_2, a_3 of the Exton's triple hypergeometric function X_{15} :*

$$(16.7) \quad \begin{aligned} X_{15}(a_2 + n) &= \sum_{n_1=0}^n \sum_{n_2=0}^{n-n_1} \binom{n}{n_1} \binom{n-n_1}{n_2} \frac{(a_1)_{n_1} (a_3)_{n_2} x_2^{n_1} x_3^{n_2}}{(c_1)_{N_2}} \\ &\times X_{15}(a_1 + n_1, a_2 + N_2, a_3 + n_2, c_1 + N_2), \end{aligned}$$

$$(16.8) \quad \begin{aligned} X_{15}(a_2 - n) &= \sum_{n_1=0}^n \sum_{n_2=0}^{n-n_1} \binom{n}{n_1} \binom{n-n_1}{n_2} \frac{(a_1)_{n_1} (a_3)_{n_2} (-x_2)^{n_1} (-x_3)^{n_2}}{(c_1)_{N_2}} \\ &\times X_{15}(a_1 + n_1, a_3 + n_2, c_1 + N_2), \end{aligned}$$

$$(16.9) \quad X_{15}(a_3 + n) = \sum_{n_1=0}^n \binom{n}{n_1} \frac{(a_2)_{n_1} x_3^{n_1}}{(c_1)_{n_1}} X_{15}(a_3 + n_1, a_2 + n_1, c_1 + n_1),$$

$$(16.10) \quad X_{15}(a_3 - n) = \sum_{n_1=0}^n \binom{n}{n_1} \frac{(a_2)_{n_1} (-x_3)^{n_1}}{(c_1)_{n_1}} X_{15}(a_2 + n_1, c_1 + n_1),$$

where $N_2 = n_1 + n_2$.

Theorem 16.3. *The following recursion formulas hold true for the denominator parameters c_1, c_2 of the Exton's triple hypergeometric function X_{15} :*

$$(16.11) \quad \begin{aligned} &X_{15}(c_1 - n) \\ &= X_{15} + a_1 a_2 x_2 \sum_{n_1=1}^n \frac{1}{(c_1 - n_1)(c_1 + 1 - n_1)} X_{15}(a_1 + 1, a_2 + 1, c_1 + 2 - n_1) \\ &+ a_2 a_3 x_3 \sum_{n_1=1}^n \frac{1}{(c_1 - n_1)(c_1 + 1 - n_1)} X_{15}(a_2 + 1, a_3 + 1, c_1 + 2 - n_1), \end{aligned}$$

$$(16.12) \quad \begin{aligned} X_{15}(c_2 - n) &= X_{15} + (a_1)_2 x_1 \sum_{n_1=1}^n \frac{1}{(c_2 - n_1)(c_2 + 1 - n_1)} \\ &\times X_{15}(a_1 + 2, c_2 + 2 - n_1), \end{aligned}$$

$$(16.13) \quad X_{15}(c_2 - n) = \sum_{n=0}^n \binom{n}{n_1} \frac{(a_1)_{2n_1} x_1^{n_1}}{(c_2)_{n_1} (c_2 - n)_{n_1}} X_{15}(a_1 + 2n_1, c_2 + n_1).$$

17. Recursion Formulas for X_{16}

Theorem 17.1. *The following recursion formulas hold true for the numerator parameters a_1, a_2, a_3 of the Exton's triple hypergeometric function X_{16} :*

$$(17.1) \quad \begin{aligned} X_{16}(a_1 + n) = & X_{16} + \frac{2x_1}{c_1} \sum_{n_1=1}^n (a_1 + n_1) X_{16}(a_1 + 1 + n_1, c_1 + 1) \\ & + \frac{a_2 x_2}{c_2} \sum_{n_1=1}^n X_{16}(a_1 + n_1, a_2 + 1, c_2 + 1), \end{aligned}$$

$$(17.2) \quad \begin{aligned} X_{16}(a_1 - n) = & X_{16} - \frac{2x_1}{c_1} \sum_{n_1=1}^n (a_1 + 1 - n_1) X_{16}(a_1 + 2 - n_1, c_1 + 1) \\ & - \frac{a_2 x_2}{c_2} \sum_{n_1=1}^n X_{16}(a_1 + 1 - n_1, a_2 + 1, c_2 + 1), \end{aligned}$$

$$(17.3) \quad \begin{aligned} X_{16}(a_2 + n) = & X_{16} + \frac{a_1 x_2}{c_2} \sum_{n_1=1}^n X_{16}(a_2 + n_1, a_1 + 1, c_2 + 1) \\ & + \frac{a_3 x_3}{c_1} \sum_{n_1=1}^n X_{16}(a_2 + n_1, a_3 + 1, c_1 + 1), \end{aligned}$$

$$(17.4) \quad \begin{aligned} X_{16}(a_2 - n) = & X_{16} - \frac{a_1 x_2}{c_2} \sum_{n_1=1}^n X_{16}(a_2 + 1 - n_1, a_1 + 1, c_2 + 1) \\ & - \frac{a_3 x_3}{c_1} \sum_{n_1=1}^n X_{16}(a_2 + 1 - n_1, a_3 + 1, c_1 + 1), \end{aligned}$$

$$(17.5) \quad X_{16}(a_3 + n) = X_{16} + \frac{a_2 x_3}{c_1} \sum_{n_1=1}^n X_{16}(a_3 + n_1, a_2 + 1, c_1 + 1),$$

$$(17.6) \quad X_{16}(a_3 - n) = X_{16} - \frac{a_2 x_3}{c_1} \sum_{n_1=1}^n X_{16}(a_3 + 1 - n_1, a_2 + 1, c_1 + 1).$$

Theorem 17.2. *The following recursion formulas hold true for the numerator parameters a_2, a_3 of the Exton's triple hypergeometric function X_{16} :*

$$(17.7) \quad \begin{aligned} X_{16}(a_2 + n) = & \sum_{n_1=0}^n \sum_{n_2=0}^{n-n_1} \binom{n}{n_1} \binom{n-n_1}{n_2} \frac{(a_1)_{n_1} (a_3)_{n_2} x_2^{n_1} x_3^{n_2}}{(c_2)_{n_1} (c_1)_{n_2}} \\ & \times X_{16}(a_1 + n_1, a_2 + N_2, a_3 + n_2, c_2 + n_1, c_1 + n_2), \end{aligned}$$

$$(17.8) \quad \begin{aligned} X_{16}(a_2 - n) = & \sum_{n_1=0}^n \sum_{n_2=0}^{n-n_1} \binom{n}{n_1} \binom{n-n_1}{n_2} \frac{(a_1)_{n_1} (a_3)_{n_2} (-x_2)^{n_1} (-x_3)^{n_2}}{(c_2)_{n_1} (c_1)_{n_2}} \\ & \times X_{16}(a_1 + n_1, a_3 + n_2, c_2 + n_1, c_1 + n_2), \end{aligned}$$

$$(17.9) \quad X_{16}(a_3 + n) = \sum_{n_1=0}^n \binom{n}{n_1} \frac{(a_2)_{n_1} x_3^{n_1}}{(c_1)_{n_1}} X_{16}(a_2 + n_1, a_3 + n_1, c_1 + n_1),$$

$$(17.10) \quad X_{16}(a_3 - n) = \sum_{n_1=0}^n \binom{n}{n_1} \frac{(a_2)_{n_1} (-x_3)^{n_1}}{(c_1)_{n_1}} X_{16}(a_2 + n_1, c_1 + n_1),$$

where $N_2 = n_1 + n_2$.

Theorem 17.3. *The following recursion formulas hold true for the denominator parameters c_1, c_2 of the Exton's triple hypergeometric function X_{16} :*

$$(17.11) \quad \begin{aligned} X_{16}(c_1 - n) &= X_{16} + (a_1)_2 x_1 \sum_{n_1=1}^n \frac{1}{(c_1 - n_1)(c_1 + 1 - n_1)} X_{16}(a_1 + 2, c_1 + 2 - n_1) \\ &\quad + a_2 a_3 x_3 \sum_{n_1=1}^n \frac{1}{(c_1 - n_1)(c_1 + 1 - n_1)} X_{16}(a_2 + 1, a_3 + 1, c_1 + 2 - n_1), \end{aligned}$$

$$(17.12) \quad \begin{aligned} X_{16}(c_2 - n) &= X_{16} + a_1 a_2 x_2 \sum_{n_1=1}^n \frac{1}{(c_2 - n_1)(c_2 + 1 - n_1)} \\ &\quad \times X_{16}(a_1 + 1, a_2 + 1, c_2 + 2 - n_1), \end{aligned}$$

$$(17.13) \quad \begin{aligned} X_{16}(c_2 - n) &= \sum_{n_1=0}^n \binom{n}{n_1} \frac{(a_1)_{n_1} (a_2)_{n_1} x_2^{n_1}}{(c_2)_{n_1} (c_2 - n)_{n_1}} \\ &\quad \times X_{16}(a_1 + n_1, a_2 + n_1, c_2 + n_1). \end{aligned}$$

18. Recursion Formulas for X_{17}

Theorem 18.1. *The following recursion formulas hold true for the numerator parameters a_1, a_2 and a_3 of the Exton's triple hypergeometric function X_{17} :*

$$(18.1) \quad \begin{aligned} X_{17}(a_1 + n) &= X_{17} + \frac{2x_1}{c_1} \sum_{n_1=1}^n (a_1 + n_1) X_{17}(a_1 + 1 + n_1, c_1 + 1) \\ &\quad + \frac{a_2 x_2}{c_2} \sum_{n_1=1}^n X_{17}(a_1 + n_1, a_2 + 1, c_2 + 1), \end{aligned}$$

$$(18.2) \quad \begin{aligned} X_{17}(a_1 - n) &= X_{17} - \frac{2x_1}{c_1} \sum_{n_1=1}^n (a_1 + 1 - n_1) X_{17}(a_1 + 2 - n_1, c_1 + 1) \\ &\quad - \frac{a_2 x_2}{c_2} \sum_{n_1=1}^n X_{17}(a_1 + 1 - n_1, a_2 + 1, c_2 + 1), \end{aligned}$$

$$(18.3) \quad \begin{aligned} X_{17}(a_2 + n) &= X_{17} + \frac{a_1 x_2}{c_2} \sum_{n_1=1}^n X_{17}(a_2 + n_1, a_1 + 1, c_2 + 1) \\ &\quad + \frac{a_3 x_3}{c_3} \sum_{n_1=1}^n X_{17}(a_2 + n_1, a_3 + 1, c_3 + 1), \end{aligned}$$

$$(18.4) \quad \begin{aligned} X_{17}(a_2 - n) &= X_{17} - \frac{a_1 x_2}{c_2} \sum_{n_1=1}^n X_{17}(a_2 + 1 - n_1, a_1 + 1, c_2 + 1) \\ &\quad - \frac{a_3 x_3}{c_3} \sum_{n_1=1}^n X_{17}(a_2 + 1 - n_1, a_3 + 1, c_3 + 1), \end{aligned}$$

$$(18.5) \quad X_{17}(a_3 + n) = X_{17} + \frac{a_2 x_3}{c_3} \sum_{n_1=1}^n X_{17}(a_3 + n_1, a_2 + 1, c_3 + 1),$$

$$(18.6) \quad X_{17}(a_3 - n) = X_{17} - \frac{a_2 x_3}{c_3} \sum_{n_1=1}^n X_{17}(a_3 + 1 - n_1, a_2 + 1, c_3 + 1).$$

Theorem 18.2. *The following recursion formulas hold true for the numerator parameters a_2, a_3 of the Exton's triple hypergeometric function X_{17} :*

$$(18.7) \quad \begin{aligned} X_{17}(a_2 + n) &= \sum_{n_1=0}^n \sum_{n_2=0}^{n-n_1} \binom{n}{n_1} \binom{n-n_1}{n_2} \frac{(a_1)_{n_1} (a_3)_{n_2} x_2^{n_1} x_3^{n_2}}{(c_2)_{n_1} (c_3)_{n_2}} \\ &\quad \times X_{17}(a_1 + n_1, a_2 + N_2, a_3 + n_2, c_2 + n_1, c_3 + n_2), \end{aligned}$$

$$(18.8) \quad \begin{aligned} X_{17}(a_2 - n) &= \sum_{n_1=0}^n \sum_{n_2=0}^{n-n_1} \binom{n}{n_1} \binom{n-n_1}{n_2} \frac{(a_1)_{n_1} (a_3)_{n_2} (-x_2)^{n_1} (-x_3)^{n_2}}{(c_2)_{n_1} (c_3)_{n_2}} \\ &\quad \times X_{17}(a_1 + n_1, a_3 + n_2, c_2 + n_1, c_3 + n_2), \end{aligned}$$

$$(18.9) \quad X_{17}(a_3 + n) = \sum_{n_1=0}^n \binom{n}{n_1} \frac{(a_2)_{n_1} x_3^{n_1}}{(c_3)_{n_1}} X_{17}(a_2 + n_1, a_3 + n_1, c_3 + n_1),$$

$$(18.10) \quad X_{17}(a_3 - n) = \sum_{n_1=0}^n \binom{n}{n_1} \frac{(a_2)_{n_1} (-x_3)^{n_1}}{(c_3)_{n_1}} X_{17}(a_2 + n_1, c_3 + n_1),$$

where $N_2 = n_1 + n_2$.

Theorem 18.3. *The following recursion formulas hold true for the denominator parameters c_1, c_2 and c_3 of the Exton's triple hypergeometric function X_{17} :*

$$(18.11) \quad \begin{aligned} X_{17}(c_1 - n) &= X_{17} + (a_1)_2 x_1 \sum_{n_1=1}^n \frac{1}{(c_1 - n_1)(c_1 + 1 - n_1)} \\ &\quad \times X_{17}(a_1 + 2, c_1 + 2 - n_1), \end{aligned}$$

$$\begin{aligned}
 X_{17}(c_2 - n) &= X_{17} + a_1 a_2 x_2 \sum_{n_1=1}^n \frac{1}{(c_2 - n_1)(c_2 + 1 - n_1)} \\
 (18.12) \quad &\times X_{17}(a_1 + 1, a_2 + 1, c_2 + 2 - n_1),
 \end{aligned}$$

$$\begin{aligned}
 X_{17}(c_3 - n) &= X_{17} + a_2 a_3 x_3 \sum_{n_1=1}^n \frac{1}{(c_3 - n_1)(c_3 + 1 - n_1)} \\
 (18.13) \quad &\times X_{17}(a_2 + 1, a_3 + 1, c_3 + 2 - n_1).
 \end{aligned}$$

Theorem 18.4. *The following recursion formulas hold true for the denominator parameters c_1 , c_2 and c_3 of the Exton's triple hypergeometric function X_{17} :*

$$(18.14) \quad X_{17}(c_1 - n) = \sum_{n=0}^n \binom{n}{n_1} \frac{(a_1)_{2n_1} x_1^{n_1}}{(c_1)_{n_1} (c_1 - n)_{n_1}} X_{17}(a_1 + 2n_1, c_1 + n_1),$$

$$(18.15) \quad X_{17}(c_2 - n) = \sum_{n=0}^n \binom{n}{n_1} \frac{(a_1)_{n_1} (a_2)_{n_1} x_2^{n_1}}{(c_2)_{n_1} (c_2 - n)_{n_1}} X_{17}(a_1 + n_1, a_2 + n_1, c_2 + n_1),$$

$$(18.16) \quad X_{17}(c_3 - n) = \sum_{n=0}^n \binom{n}{n_1} \frac{(a_2)_{n_1} (a_3)_{n_1} x_3^{n_1}}{(c_3)_{n_1} (c_3 - n)_{n_1}} X_{17}(a_2 + n_1, a_3 + n_1, c_3 + n_1).$$

19. Recursion Formulas for X_{18}

Theorem 19.1. *The following recursion formulas hold true for the numerator parameters a_1, a_2, a_3 and a_4 of the Exton's triple hypergeometric function X_{18} :*

$$\begin{aligned}
 X_{18}(a_1 + n) &= X_{18} + \frac{2x_1}{c} \sum_{n_1=1}^n (a_1 + n_1) X_{18}(a_1 + 1 + n_1, c + 1) \\
 (19.1) \quad &+ \frac{a_2 x_2}{c} \sum_{n_1=1}^n X_{18}(a_1 + n_1, a_2 + 1, c + 1),
 \end{aligned}$$

$$\begin{aligned}
 X_{18}(a_1 - n) &= X_{18} - \frac{2x_1}{c} \sum_{n_1=1}^n (a_1 + 1 - n_1) X_{18}(a_1 + 2 - n_1, c + 1) \\
 (19.2) \quad &- \frac{a_2 x_2}{c} \sum_{n_1=1}^n X_{18}(a_1 + 1 - n_1, a_2 + 1, c + 1),
 \end{aligned}$$

$$(19.3) \quad X_{18}(a_2 + n) = X_{18} + \frac{a_1 x_2}{c} \sum_{n_1=1}^n X_{18}(a_2 + n_1, a_1 + 1, c + 1),$$

$$(19.4) \quad X_{18}(a_2 - n) = X_{18} - \frac{a_1 x_2}{c} \sum_{n_1=1}^n X_{18}(a_2 + 1 - n_1, a_1 + 1, c + 1),$$

$$(19.5) \quad X_{18}(a_3 + n) = X_{18} + \frac{a_4 x_3}{c} \sum_{n_1=1}^n X_{18}(a_3 + n_1, a_4 + 1, c + 1),$$

$$(19.6) \quad X_{18}(a_3 - n) = X_{18} - \frac{a_4 x_3}{c} \sum_{n_1=1}^n X_{18}(a_3 + 1 - n_1, a_4 + 1, c + 1),$$

$$(19.7) \quad X_{18}(a_4 + n) = X_{18} + \frac{a_3 x_3}{c} \sum_{n_1=1}^n X_{18}(a_3 + 1, a_4 + n_1, c + 1),$$

$$(19.8) \quad X_{18}(a_4 - n) = X_{18} - \frac{a_3 x_3}{c} \sum_{n_1=1}^n X_{18}(a_3 + 1, a_4 + 1 - n_1, c + 1).$$

Theorem 19.2. *The following recursion formulas hold true for the numerator parameters a_2, a_3 and a_4 of the Exton's triple hypergeometric function X_{18} :*

$$(19.9) \quad X_{18}(a_2 + n) = \sum_{n_1=0}^n \binom{n}{n_1} \frac{(a_1)_{n_1} x_2^{n_1}}{(c)_{n_1}} X_{18}(a_1 + n_1, a_2 + n_1, c + n_1),$$

$$(19.10) \quad X_{18}(a_2 - n) = \sum_{n_1=0}^n \binom{n}{n_1} \frac{(a_1)_{n_1} (-x_2)^{n_1}}{(c)_{n_1}} X_{18}(a_1 + n_1, c + n_1),$$

$$(19.11) \quad X_{18}(a_3 + n) = \sum_{n_1=0}^n \binom{n}{n_1} \frac{(a_4)_{n_1} x_3^{n_1}}{(c)_{n_1}} X_{18}(a_3 + n_1, a_4 + n_1, c + n_1),$$

$$(19.12) \quad X_{18}(a_3 - n) = \sum_{n_1=0}^n \binom{n}{n_1} \frac{(a_4)_{n_1} (-x_3)^{n_1}}{(c)_{n_1}} X_{18}(a_4 + n_1, c + n_1),$$

$$(19.13) \quad X_{18}(a_4 + n) = \sum_{n_1=0}^n \binom{n}{n_1} \frac{(a_3)_{n_1} x_3^{n_1}}{(c)_{n_1}} X_{18}(a_3 + n_1, a_4 + n_1, c + n_1),$$

$$(19.14) \quad X_{18}(a_4 - n) = \sum_{n_1=0}^n \binom{n}{n_1} \frac{(a_3)_{n_1} (-x_3)^{n_1}}{(c)_{n_1}} X_{18}(a_3 + n_1, c + n_1).$$

Theorem 19.3. *The following recursion formulas hold true for the denominator parameter c of the Exton's triple hypergeometric function X_{18} :*

$$(19.15) \quad \begin{aligned} X_{18}(c - n) &= X_{18} + (a_1)_2 x_1 \sum_{n_1=1}^n \frac{1}{(c - n_1)(c + 1 - n_1)} X_{18}(a_1 + 2, c + 2 - n_1) \\ &\quad + a_1 a_2 x_2 \sum_{n_1=1}^n \frac{1}{(c - n_1)(c + 1 - n_1)} X_{18}(a_1 + 1, a_2 + 1, c + 2 - n_1) \\ &\quad + a_3 a_4 x_3 \sum_{n_1=1}^n \frac{1}{(c - n_1)(c + 1 - n_1)} X_{18}(a_3 + 1, a_4 + 1, c + 2 - n_1). \end{aligned}$$

20. Recursion Formulas for X_{19}

Theorem 20.1. *The following recursion formulas hold true for the numerator parameters a_1, a_2, a_3 and a_4 of the Exton's triple hypergeometric function X_{19} :*

$$(20.1) \quad \begin{aligned} X_{19}(a_1 + n) = & X_{19} + \frac{2x_1}{c_2} \sum_{n_1=1}^n (a_1 + n_1) X_{19}(a_1 + 1 + n_1, c_2 + 1) \\ & + \frac{a_2 x_2}{c_1} \sum_{n_1=1}^n X_{19}(a_1 + n_1, a_2 + 1, c_1 + 1), \end{aligned}$$

$$(20.2) \quad \begin{aligned} X_{19}(a_1 - n) = & X_{19} - \frac{2x_1}{c_2} \sum_{n_1=1}^n (a_1 + 1 - n_1) X_{19}(a_1 + 2 - n_1, c_2 + 1) \\ & - \frac{a_2 x_2}{c_1} \sum_{n_1=1}^n X_{19}(a_1 + 1 - n_1, a_2 + 1, c_1 + 1), \end{aligned}$$

$$(20.3) \quad X_{19}(a_2 + n) = X_{19} + \frac{a_1 x_2}{c_1} \sum_{n_1=1}^n X_{19}(a_2 + n_1, a_1 + 1, c_1 + 1),$$

$$(20.4) \quad X_{19}(a_2 - n) = X_{19} - \frac{a_1 x_2}{c_1} \sum_{n_1=1}^n X_{19}(a_2 + 1 - n_1, a_1 + 1, c_1 + 1),$$

$$(20.5) \quad X_{19}(a_3 + n) = X_{19} + \frac{a_4 x_3}{c_1} \sum_{n_1=1}^n X_{19}(a_3 + n_1, a_4 + 1, c_1 + 1),$$

$$(20.6) \quad X_{19}(a_3 - n) = X_{19} - \frac{a_4 x_3}{c_1} \sum_{n_1=1}^n X_{19}(a_3 + 1 - n_1, a_4 + 1, c_1 + 1),$$

$$(20.7) \quad X_{19}(a_4 + n) = X_{19} + \frac{a_3 x_3}{c_1} \sum_{n_1=1}^n X_{19}(a_3 + 1, a_4 + n_1, c_1 + 1),$$

$$(20.8) \quad X_{19}(a_4 - n) = X_{19} - \frac{a_3 x_3}{c_1} \sum_{n_1=1}^n X_{19}(a_3 + 1, a_4 + 1 - n_1, c_1 + 1).$$

Theorem 20.2. *The following recursion formulas hold true for the numerator parameters a_2, a_3 and a_4 of the Exton's triple hypergeometric function X_{19} :*

$$(20.9) \quad X_{19}(a_2 + n) = \sum_{n_1=0}^n \binom{n}{n_1} \frac{(a_1)_{n_1} x_2^{n_1}}{(c_1)_{n_1}} X_{19}(a_1 + n_1, a_2 + n_1, c_1 + n_1),$$

$$(20.10) \quad X_{19}(a_2 - n) = \sum_{n_1=0}^n \binom{n}{n_1} \frac{(a_1)_{n_1} (-x_2)^{n_1}}{(c_1)_{n_1}} X_{19}(a_1 + n_1, c_1 + n_1),$$

$$(20.11) \quad X_{19}(a_3 + n) = \sum_{n_1=0}^n \binom{n}{n_1} \frac{(a_4)_{n_1} x_3^{n_1}}{(c_1)_{n_1}} X_{19}(a_3 + n_1, a_4 + n_1, c_1 + n_1),$$

$$(20.12) \quad X_{19}(a_3 - n) = \sum_{n_1=0}^n \binom{n}{n_1} \frac{(a_4)_{n_1} (-x_3)^{n_1}}{(c_1)_{n_1}} X_{19}(a_4 + n_1, c_1 + n_1),$$

$$(20.13) \quad X_{19}(a_4 + n) = \sum_{n_1=0}^n \binom{n}{n_1} \frac{(a_3)_{n_1} x_3^{n_1}}{(c_1)_{n_1}} X_{19}(a_3 + n_1, a_4 + n_1, c_1 + n_1),$$

$$(20.14) \quad X_{19}(a_4 - n) = \sum_{n_1=0}^n \binom{n}{n_1} \frac{(a_3)_{n_1} (-x_3)^{n_1}}{(c_1)_{n_1}} X_{19}(a_3 + n_1, c_1 + n_1).$$

Theorem 20.3. *The following recursion formulas hold true for the denominator parameters c_1, c_2 of the Exton's triple hypergeometric function X_{19} :*

$$(20.15) \quad \begin{aligned} X_{19}(c_1 - n) &= X_{19} \\ &+ a_1 a_2 x_2 \sum_{n_1=1}^n \frac{1}{(c_1 - n_1)(c_1 + 1 - n_1)} X_{19}(a_1 + 1, a_2 + 1, c_1 + 2 - n_1) \\ &+ a_3 a_4 x_3 \sum_{n_1=1}^n \frac{1}{(c_1 - n_1)(c_1 + 1 - n_1)} X_{19}(a_3 + 1, a_4 + 1, c_1 + 2 - n_1), \end{aligned}$$

$$(20.16) \quad \begin{aligned} X_{19}(c_2 - n) &= X_{19} + (a_1)_2 x_1 \sum_{n_1=1}^n \frac{1}{(c_2 - n_1)(c_2 + 1 - n_1)} \\ &\times X_{19}(a_1 + 2, c_2 + 2 - n_1), \end{aligned}$$

$$(20.17) \quad X_{19}(c_2 - n) = \sum_{n_1=0}^n \binom{n}{n_1} \frac{(a_1)_{2n_1} x_1^{n_1}}{(c_2)_{n_1} (c_2 - n)_{n_1}} X_{19}(a_1 + 2n_1, c_2 + n_1).$$

21. Recursion Formulas for X_{20}

Theorem 21.1. *The following recursion formulas hold true for the numerator parameters a_1, a_2, a_3 and a_4 of the Exton's triple hypergeometric function X_{20} :*

$$(21.1) \quad \begin{aligned} X_{20}(a_1 + n) &= X_{20} + \frac{2x_1}{c_1} \sum_{n_1=1}^n (a_1 + n_1) X_{20}(a_1 + 1 + n_1, c_1 + 1) \\ &+ \frac{a_2 x_2}{c_2} \sum_{n_1=1}^n X_{20}(a_1 + n_1, a_2 + 1, c_2 + 1), \end{aligned}$$

$$(21.2) \quad \begin{aligned} X_{20}(a_1 - n) &= X_{20} - \frac{2x_1}{c_1} \sum_{n_1=1}^n (a_1 + 1 - n_1) X_{20}(a_1 + 2 - n_1, c_1 + 1) \\ &- \frac{a_2 x_2}{c_2} \sum_{n_1=1}^n X_{20}(a_1 + 1 - n_1, a_2 + 1, c_2 + 1), \end{aligned}$$

$$(21.3) \quad X_{20}(a_2 + n) = X_{20} + \frac{a_1 x_2}{c_2} \sum_{n_1=1}^n X_{20}(a_2 + n_1, a_1 + 1, c_2 + 1),$$

$$(21.4) \quad X_{20}(a_2 - n) = X_{20} - \frac{a_1 x_2}{c_2} \sum_{n_1=1}^n X_{20}(a_2 + 1 - n_1, a_1 + 1, c_2 + 1),$$

$$(21.5) \quad X_{20}(a_3 + n) = X_{20} + \frac{a_4 x_3}{c_1} \sum_{n_1=1}^n X_{20}(a_3 + n_1, a_4 + 1, c_1 + 1),$$

$$(21.6) \quad X_{20}(a_3 - n) = X_{20} - \frac{a_4 x_3}{c_1} \sum_{n_1=1}^n X_{20}(a_3 + 1 - n_1, a_4 + 1, c_1 + 1),$$

$$(21.7) \quad X_{20}(a_4 + n) = X_{20} + \frac{a_3 x_3}{c_1} \sum_{n_1=1}^n X_{20}(a_3 + 1, a_4 + n_1, c_1 + 1),$$

$$(21.8) \quad X_{20}(a_4 - n) = X_{20} - \frac{a_3 x_3}{c_1} \sum_{n_1=1}^n X_{20}(a_3 + 1, a_4 + 1 - n_1, c_1 + 1).$$

Theorem 21.2. *The following recursion formulas hold true for the numerator parameters a_2, a_3 and a_4 of the Exton triple hypergeometric function X_{20} :*

$$(21.9) \quad X_{20}(a_2 + n) = \sum_{n_1=0}^n \binom{n}{n_1} \frac{(a_1)_{n_1} x_2^{n_1}}{(c_2)_{n_1}} X_{20}(a_1 + n_1, a_2 + n_1, c_2 + n_1),$$

$$(21.10) \quad X_{20}(a_2 - n) = \sum_{n_1=0}^n \binom{n}{n_1} \frac{(a_1)_{n_1} (-x_2)^{n_1}}{(c_2)_{n_1}} X_{20}(a_1 + n_1, c_2 + n_1),$$

$$(21.11) \quad X_{20}(a_3 + n) = \sum_{n_1=0}^n \binom{n}{n_1} \frac{(a_4)_{n_1} x_3^{n_1}}{(c_1)_{n_1}} X_{20}(a_3 + n_1, a_4 + n_1, c_1 + n_1),$$

$$(21.12) \quad X_{20}(a_3 - n) = \sum_{n_1=0}^n \binom{n}{n_1} \frac{(a_4)_{n_1} (-x_3)^{n_1}}{(c_1)_{n_1}} X_{20}(a_4 + n_1, c_1 + n_1),$$

$$(21.13) \quad X_{20}(a_4 + n) = \sum_{n_1=0}^n \binom{n}{n_1} \frac{(a_3)_{n_1} x_3^{n_1}}{(c_1)_{n_1}} X_{20}(a_3 + n_1, a_4 + n_1, c_1 + n_1),$$

$$(21.14) \quad X_{20}(a_4 - n) = \sum_{n_1=0}^n \binom{n}{n_1} \frac{(a_3)_{n_1} (-x_3)^{n_1}}{(c_1)_{n_1}} X_{20}(a_3 + n_1, c_1 + n_1).$$

Theorem 21.3. *The following recursion formulas hold true for the denominator parameters c_1, c_2 of the Exton's triple hypergeometric function X_{20} :*

$$(21.15) \quad \begin{aligned} X_{20}(c_1 - n) &= X_{20} + (a_1)_2 x_1 \sum_{n_1=1}^n \frac{1}{(c_1 - n_1)(c_1 + 1 - n_1)} X_{20}(a_1 + 2, c_1 + 2 - n_1) \\ &+ a_3 a_4 x_3 \sum_{n_1=1}^n \frac{1}{(c_1 - n_1)(c_1 + 1 - n_1)} X_{20}(a_3 + 1, a_4 + 1, c_1 + 2 - n_1), \end{aligned}$$

$$\begin{aligned}
 X_{20}(c_2 - n) &= X_{20} + a_1 a_2 x_2 \sum_{n_1=1}^n \frac{1}{(c_2 - n_1)(c_2 + 1 - n_1)} \\
 (21.16) \quad &\times X_{20}(a_1 + 1, a_2 + 1, c_2 + 2 - n_1),
 \end{aligned}$$

$$\begin{aligned}
 X_{20}(c_2 - n) &= \sum_{n_1=0}^n \binom{n}{n_1} \frac{(a_1)_{n_1} (a_2)_{n_1} x_2^{n_1}}{(c_2)_{n_1} (c_2 - n)_{n_1}} \\
 (21.17) \quad &\times X_{20}(a_1 + n_1, a_2 + n_1, c_2 + n_1).
 \end{aligned}$$

Acknowledgments. The authors thank the referee for valuable suggestions that led to a better presentation of the paper. The financial assistance provided to the second author in the form of a INSPIRE Fellowship (IF-110010) from Department of Science and Technology, New Delhi, India is gratefully acknowledged.

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